Time Series Analysis

First set of assignments

1. The stochastic process $\{\varepsilon_t\}(t = 1, 2, \cdots)$ consists of independent random variables $\varepsilon_t \sim N(0, 1)$. Compute the probability $P(\varepsilon_t \leq 0 \cap \varepsilon_{t+1} > 1.96 \cap \varepsilon_{t+2} \leq -1.96)$.

2. Write the joint density $f_{\varepsilon_t\varepsilon_{t+1}}(\varepsilon_t, \varepsilon_{t+1})$. Interpret your result.

3. Write the conditional density $f_{\varepsilon_{t+1}|\varepsilon_t}(\varepsilon_{t+1}|\varepsilon_t)$.

4. Denote a realisation of the stochastic process $\{\varepsilon_t\}$ as $\{x_1, x_2, \cdot\cdot, x_T\}$.
   Write down the joint density function of the random vector $\underline{\varepsilon} = \{\varepsilon_1, \varepsilon_2, \cdot\cdot, \varepsilon_T\}$ evaluated at $\{x_1, x_2, \cdot\cdot, x_T\}$.

   Since the random vector $\underline{\varepsilon} = \{\varepsilon_1, \varepsilon_2, \cdot\cdot, \varepsilon_T\}$ is jointly normally distributed you can use the multivariate normal density which is generally written as

   $$f_{\underline{X}} = 2\pi^{-n/2}|\Omega|^{-0.5}\, exp\left[\frac{(\underline{x} - \underline{\mu})'\Omega^{-1}(\underline{x} - \underline{\mu})}{-2}\right]$$

   What is in our example $n, \underline{x}, \underline{\mu}$ and Ω ?

5. Is the process $\{\varepsilon_t\}$ weakly stationary?

6. Is the process $\{\varepsilon_t\}$ strictly stationary?

7. A new stochastic process $\{Y_t\}$ is generated as $Y_t = a + b \cdot \varepsilon_t$
   The joint distribution of $\underline{Y} = (Y_1, Y_2, \cdot\cdot, Y_T)$ is still the multivariate normal (see 4.)
   What is $\underline{\mu}$ and Ω now?

8. $\{X_t\}$ denotes a stochastic process. We have $E(X_t) = E(X_{t+1}) = 2$
   $cov(X_t, X_{t+1}) = 2$ and $var(X_t) = var(X_{t+1}) = 1$
   using $A = \begin{bmatrix} 0.3 & 0.7 \\ 0.5 & 0.5 \end{bmatrix}$ we generate two new random variables Z_1, Z_2 by
   $\underline{Z} = \begin{bmatrix} Z_1 \\ Z_2 \end{bmatrix} = A \cdot \begin{bmatrix} X_t \\ X_{t+1} \end{bmatrix}$
   compute $E(\underline{Z})$ and $cov(\underline{Z}) = \begin{bmatrix} var\,(Z_1)\, cov\,(Z_1, Z_2) \\ cov\,(Z_1, Z_2)\, var\,(Z_2) \end{bmatrix}$

Solutions to the *first set* of assignments:

1. $P(\varepsilon_t \leq 0) \cdot P(\varepsilon_{t+1} > 1.96) \cdot P(\varepsilon_t \leq -1.96) = 0.5 \cdot 0.025 \cdot 0.025 = 0.0003125$

8. $E(\underline{Z}) = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$

   $cov(\underline{Z}) = \begin{bmatrix} 1.42 & 1.5 \\ 1.5 & 1.5 \end{bmatrix}$

Second set of assignments

1. Are the following stochastic processes $\{y_t\}$ stationary and ergodic?

   $$\left[\begin{array}{ll} \{\varepsilon_t\} & \text{denotes a Gaussian white noise process} \\ \text{i.e.} & \mathbb{E}(\varepsilon_t)=0, \quad \mathbb{E}(\varepsilon_t^2)=Var(\varepsilon_t)=\sigma^2, \quad \mathbb{E}(\varepsilon_t\cdot\varepsilon_\tau)=0 \quad t\neq\tau \end{array}\right]$$

   $$\begin{array}{lllll} \text{a)} & y_t & = & \varepsilon_t & \\ \text{b)} & y_t & = & y_{t-1}+\varepsilon_t & \text{with} \quad y_1=\varepsilon_1 \\ \text{c)} & y_t & = & y_{t-1}-y_{t-2}+\varepsilon_t & \text{with} \quad y_1=\varepsilon_1 \\ \text{d)} & y_t & = & a\cdot t+\varepsilon_t & \text{with} \quad a \quad \text{a real number} \end{array}$$

2. Compute $\mathbb{E}(y_t-\mu)(y_{t-j}-\mu)$ [i.e. $cov(y_t, y_{t-j})$] for the stochastic processes b) and d).

3. - Check, by writing $\mathbb{E}(y_t)$, $Var(y_t)$ and $cov(y_t, y_{t-j})$ $j \geq 1$, whether a MA(2) process

   $y_t = \mu + \theta_1\varepsilon_{t-1} + \theta_2\varepsilon_{t-2} + \varepsilon_t$

   is stationary and ergodic.

   - Plot the autocorrelation function for a MA(2) where $\theta_1 = 0.5$ and $\theta_2 = -0.3$.

4. Write $\mathbb{E}(y_t)$ and $Var(y_t)$ for a MA(q) process.
   $y_t = \mu + \theta_1\varepsilon_{t-1} + \theta_2\varepsilon_{t-2} + ... + \theta_q\varepsilon_{t-q} + \varepsilon_t$

5. The sequence of autocovariances $\{\gamma_j\}_{j=0}^{\infty}$ of a Gaussian process $\{y_t\}$ evolves as $\gamma_j = \theta^j$ where $|\theta| < 1$.
   Is the process ergodic?

6. What do we mean by a Gaussian process?

7. Why is ergodic stationarity such an important property for the purpose of estimating the moments $\mathbb{E}(y_t)$, $Var(y_t)$, $cov(y_t, y_{t-j})$,... of a stochastic process $\{y_t\}$?

   Hint: refer to the ergodic theorem (Hayashi, *Econometrics*, p. 101) and note that if $\{y_t\}$ is stationary and ergodic, so is $\{f(y_t)\}$ where $f(\cdot)$ is a measurable function like $\ln(y_t)$, y_t^2 i.e. a function that produces a new random variable.

8. A MA(∞) is given by
   $y_t = \mu + \theta^2\varepsilon_{t-1} + \theta^4\varepsilon_{t-2} + \theta^6\varepsilon_{t-3} + ...$
   where $|\theta| < 1$.
   Compute $\mathbb{E}(y_t)$ and $Var(y_t)$.

Third set of assignments

1. An AR(1) process is given by

   $Y_t = 0.5 + 0.9\,Y_{t-1} + \varepsilon_t$ where $\{\varepsilon_t\}$ is Gaussian White Noise $\varepsilon_t \sim N(0,9)$

   Compute $E(Y_t)$ and $Var(Y_t)$. Compute the first 5 auto covariances $\gamma_1, \gamma_2, \cdot\cdot, \gamma_5$ and plot the corresponding autocorrelations $\rho_1, \rho_2, \cdot\cdot, \rho_5$.

   Hint $\rho_j = \frac{Cov(Y_t, Yt-j)}{\sqrt{Var(Y_t)}\sqrt{Var(Y_{t-j})}} = \frac{\gamma_j}{\gamma_0}$

2. Show by applying the "brute force" method that the sequence of autocovariances for an AR(1) process

   $Y_t = c + \phi Y_{t-1} + \varepsilon_t$

   can be written as

   $\gamma_j = \frac{\phi^j}{1-\phi^2}\,\sigma^2$

3. Express the stochastic process in 1) in an alternative representation that has the change of Y_t(i.e. $Y_t - Y_{t-1}$) on the left hand side and the difference of the lagged value of Y_t(i.e. Y_{t-1}) and $E(Y_t)$ on the right hand side ("Ornstein-Uhlenbeck-representation" ⇒ lecture notes)
   Using this representation: What is the expected change $E(Y_t - Y_{t-1})$ given a deviation of $Y_{t-1} - E(Y_t) = 10$ in the previous period?
   What is the variance of $Y_t - Y_{t-1}$ given $Y_{t-1} - E(Y_t) = 10$?

4. From the course page you can download the EVIEWS workfile svar.wf1. The file contains macroeconomic variables at a quarterly frequency. The series ZS3MLIBQ contains an interest rate series, the 3-month Swiss France LIBOR (1974-2002). The series BIPNSA contains the nominal gross domestic product (seasonally adjusted)of Switzerland (1974-2002). The series WKUSDQ contains the Swiss France/US dollar exchange rate (1974-2002).
   Select and estimate an ARMA(p,q) model for
   a) the series ZS3MLIBQ
   b) the log-difference(natural logs) of the BIPNSA series
   c) the log-difference of the WKUSDQ series.

Let the significance of the parameter estimates, the Akaike and Schwartz Information criteria and the sample autocorrelations guide your specification search.

Solutions to the *third set* of assignments:

1. $E(y_t) = 5$; $var(y_t) = 47.368$; $\gamma_1 = 42.632$; $\gamma_2 = 38.368$; $\gamma_3 = 34.532$; $\gamma_4 = 31.078$; $\gamma_5 = 27.971$

3. $E(y_t - y_{t-1}) = -1$; $var(y_t - y_{t-1}) = 81$

Fourth set of assignments

1. Identify the following ARMA processes (e.g. ARMA(0,1),..)?

$$
\begin{array}{ll}
(a) & (1-\phi L)(1-L)Y_t = (1+\theta L)\varepsilon_t \\
(b) & Y_t = (1+0.4L+0.3L^2)\varepsilon_t \\
(c) & (1-0.9L)(1-L)Y_t = (1+0.3L)\varepsilon_t \\
(d) & (1-0.3L)(1-0.2L^{12})Y_t = (1+0.2L)(1+0.3L^{12})\varepsilon_t \\
(e) & (1-\phi L)(1-L)Y_t = (1+\theta L)\varepsilon_t \\
(f) & (1-\phi_1 L)(1-\phi_{12}L^{12})Y_t = (1+\theta_1 L)(1+\theta_{12}L^{12})\varepsilon_t
\end{array}
$$

2. Use the eigenvalues of $\mathbf{F}$, to check whether the following AR processes are stationary

$$
(1)\ \mathbf{F} = \begin{pmatrix} 0.6 & -0.4 \\ 1 & 0 \end{pmatrix}, \qquad (2)\ \mathbf{F} = \begin{pmatrix} 0.4 & 0.8 & -0.3 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad (3)\ \mathbf{F} = \begin{pmatrix} 1.2 & -0.1 \\ 1 & 0 \end{pmatrix}
$$

where

$\lambda_1 = 0.30 + 0.55677644i$
$\lambda_2 = 0.30 - 0.55677644i$

where

$\lambda_1 = 0.91584462$
$\lambda_2 = -0.88568851$
$\lambda_3 = 0.36984389$

where

$\lambda_1 = 1.1099020$
$\lambda_2 = 0.090098049$

3. In the following, $\{\varepsilon_t\}$ denotes a Gaussian White Noise process. Which of the following processes $\{Y_t\}$ is a stationary and ergodic process? Give a brief explanatory statement and describe each process as a special case of an ARMA(p,q) process. For example 'This is a stationary AR(2) process...' et cetera.

$$
\begin{array}{ll}
(a) & (1-0.5L-0.7L^2)Y_t = \varepsilon_t \\
(b) & (1-0.9L-0.1L^2)Y_t = (1+0.3L)\varepsilon_t \\
(c) & Y_t = (1-L)\varepsilon_t \\
(d) & Y_t = (1+0.9L^2)\varepsilon_t \\
(e) & Y_t = c + 0.5Y_{t-1} + 0.3Y_{t-2} + 1.2\varepsilon_{t-1} + \varepsilon_t \\
(f) & Y_t = \dfrac{(1-1.3L^2)}{1-0.8L-0.1L^2}\varepsilon_t \\
(g) & (1-0.9L)Y_t = \varepsilon_t \\
(h) & (1-0.8L-0.1L^2)Y_t = \varepsilon_t \\
(i) & Y_t = (1+0.4L+0.3L^2)\varepsilon_t
\end{array}
$$

4. Give your opinion to the following statements. Answer "Correct, since..." or "Incorrect, rather..."

   (a) Any MA process is a stationary process .

   (b) Any finite Gaussian AR(p) process is stationary .

   (c) Whether an ARMA(p,q) is stationary is solely determined by its MA part.

   (f) A White Noise process is an ergodic process

   (g) Any finite MA(q) is ergodic.

Solutions to the *fourth set* of assignments:

1. (a), (c), (e) ARIMA(1,1,1) (b) MA(2) (d),(f) ARIMA$(0,1,1)(0,1,1)_{12}$
2. (1) stationary (2) stationary (3) not stationary
3. (a) $\lambda_1 = 1.123$ $\lambda_2 = -0.623 \to$ not stationary;
   (b) finite MA(q) stationary, Check AR part: $\lambda_1 = 1$ $\lambda_2 = -0.1 \to$ not stationary;
   (c),(d),(i) finite MA(q) stationary
   (e) $\lambda_1 = -0.352$ $\lambda_2 = 0.852 \to$ stationary;
   (f),(h) $\lambda_1 = 0.910$ $\lambda_2 = -0.110 \to$ stationary
   (g) stationary

Fifth set of assignments

1. You want to construct the exact likelihood function of an AR(2) process

$$Y_t = c + \phi_1 Y_{t-1} + \phi_2 Y_{t-2} + \varepsilon_t \qquad \varepsilon_t \sim N(0, \sigma^2) \text{ and i.i.d.}$$

a) Write down the joint density of the first two observations $f_{Y_1,Y_2}(y_1, y_2)$

b) Using the conditional density of the third observation $f_{Y_3|Y_2,Y_1}(y_3|y_2, y_1)$ write down the joint density of the first three observations $f_{Y_1,Y_2,Y_3}(y_1, y_2, y_3)$

2. Are the following $MA(q)$ processes invertible?

$$Y_t = c - 0.9\varepsilon_{t-1} + \varepsilon_t$$
$$Y_t = c + \varepsilon_{t-1} + \varepsilon_t$$
$$Y_t = c + 1.2\varepsilon_{t-1} + \varepsilon_t$$
$$Y_t = c + (1 + 0.7L + 0.4L^2)\varepsilon_t$$
$$Y_t = c + (1 + 0.2L + 0.4L^2)\varepsilon_t$$

3. a) Write down the joint density of the first three observations of the $MA(3)$ process

$$Y_t = c + 0.3\varepsilon_{t-1} + 0.2\varepsilon_{t-2} - 0.1\varepsilon_{t-1} + \varepsilon_t \qquad \varepsilon_t \sim N(0, \sigma^2) \text{ and } \varepsilon_t \text{ i.i.d. } N(0, \sigma^2)$$

b) Suppose you want to set up the conditional likelihood function of this process. You condition on pre-sample values $\varepsilon_0, \varepsilon_{-1}, \varepsilon_{-2}$. Write down the first three elements of the conditional likelihood function.

$$f_{Y_1|\varepsilon_0=0,\varepsilon_{-1}=0,\varepsilon_{-2}=0} =$$
$$f_{Y_2|Y_1,\varepsilon_0=0,\varepsilon_{-1}=0,\varepsilon_{-2}=0} =$$
$$f_{Y_3|Y_1,Y_2,\varepsilon_0=0,\varepsilon_{-1}=0,\varepsilon_{-2}=0} =$$

c) Which condition has to hold in order to make the Conditional Maximum-Likelihood work?

4. You have succeeded in providing Maximum-Likelihood estimates of the parameters of an $ARMA(2, 2)$ process.

$$(1 - L\phi_1 - L\phi_2)Y_t = c + (1 + \theta_1 L + \theta_2 L^2)\varepsilon_t \qquad \varepsilon_t \sim \ i.i.d.N(0, \sigma^2)$$

The (conditioned) Maximum-Likelihood estimates are

$$\begin{array}{llll} \hat{c} & = 0.2 & \hat{\theta}_1 & = 0.2 \\ \hat{\phi}_1 & = 0.6 & \hat{\theta}_2 & = -0.1 \\ \hat{\phi}_2 & = 0.1 & \hat{\sigma}^2 & = 0.8 \end{array}$$

The value of the log likelihood function evaluated at these estimates is -1432.6.

Suppose you want to test the null hypothesis

$$\begin{array}{ll} & H_0 : \theta_1 = 0.5 \text{ against } H_A : \theta_1 \neq 0.5 \\ \text{and} & H_0 : \theta_1 = 0 \text{ against } H_A : \theta_1 \neq 0 \end{array}$$

Perform and interpret the appropriate tests.

An estimate of the variance-covariance matrix of the estimates $\hat{\theta} = (\hat{c}, \hat{\phi}_1, \hat{\phi}_2, \hat{\theta}_1, \hat{\theta}_2, \hat{\sigma}^2)$ is given by

$$\widehat{Var}(\hat{\theta}) = \left[-\frac{\partial^2 lnL(\theta)}{\partial\theta\partial\theta'}\bigg|_{\hat{\theta}} \right]^{-1} = \begin{bmatrix} 0.007 & \cdots & & & & \vdots \\ 0.001 & 0.005 & & & & \\ 0.002 & 0.001 & 0.003 & & & \\ 0.003 & 0.002 & 0.001 & 0.01 & & \\ 0.001 & 0.003 & 0.004 & 0.001 & 0.002 & \\ 0.001 & 0.0001 & 0.0001 & 0.0001 & 0.00002 & 0.0001 \end{bmatrix}$$

$\theta = (c, \phi_1, \phi_2, \theta_1, \theta_2, \sigma^2)'$

You have also estimated an $ARMA(2, 0)$ i.e. an $AR(2)$ model. The estimation of this restricted model yields a log likelihood value equal to -1434.3.

Compute and interpret a likelihood ratio statistic to test the hypothesis that the restrictions implied by the $ARMA(2, 0)$ specification are correct. Here the $ARMA(2, 2)$ specification is the unrestricted model, the $ARMA(2, 0)$ is the restricted model.

As another alternative you have estimated an $MA(2)$ model. The log likelihood evaluated at the maximum likelihood estimates is -1442.2. Perform a test of the $ARMA(2, 2)$ specification against the MA(2) model.

Solutions to the *fifth set* of assignments:

2. (1) invertible (2) not invertible (3) not invertible (4) invertible (5) invertible

4. test statistic: first null hypothesis $t_1 = \frac{0.2-0.5}{\sqrt{0.01}} = -3$
   second null hypothesis $t_2 = \frac{0.2}{\sqrt{0.01}} = 2$

   Likelihood ratio test statistic for ARMA(2,2) vs. AR(2): $LR_1 = 3.4$
   Likelihood ratio test statistic for ARMA(2,2) vs. MA(2): $LR_2 = 19.2$
   critical value: $\chi^2(2) = 5.99$

Sixth set of assignments

1. Estimate a suitable ARIMA(p,d,q) model for the seasonally adjusted consumer price index (variable `pcqsa` in the dataset `svar.wf1`).

   - First, take the logarithm and conduct a unit root test to check if differencing the series is necessary.
   - Choose your specification of the ARMA(p,q) model by looking at the correlogram and checking the information criteria for different orders of p and q (up to ARMA(2,2)).
   - For estimation, use the sample up to the first quarter of the year 2000.
   - Then, forecast the consumer price index (in levels) from 2000:2 to 2002:1.
   - Save the forecast values and their standard errors in order to compute a 95% confidence interval.
   - Finally, plot your result.

2. Use an ARIMA(0,1,1)(0,1,1)$_4$ model to estimate and forecast the nominal GDP (variable `bipn` in the dataset `svar.wf1`). The above notation reads as follows (see Enders (1995) pp. 111-118 for details):

   ARIMA(p,d,q)(P,D,Q)$_s$, where

   $$\begin{aligned} p \text{ and } q &= \text{the nonseasonal ARMA coefficients} \\ d &= \text{number of nonseasonal differences} \\ P &= \text{number of multiplicative autoregressive coefficients} \\ D &= \text{number of seasonal differences} \\ Q &= \text{number of multiplicative moving average coefficients} \\ s &= \text{seasonal period} \end{aligned}$$

   In algebraic terms, the model looks like:

   $$(1 - L)(1 - L^4)Y_t = (1 - \theta_1 L)(1 - \theta_4 L^4)\varepsilon_t$$

   - For estimation, use the sample up to the first quarter of the year 1997
   - In order to forecast the level of the series, use `d(log(bipn)-log(bipn(-4)))` as the dependent variable in the equation (forecast horizon: 1997:2-2002:1).
   - Save the forecast values and their standard errors in order to compute a 95% confidence interval.
   - Finally, plot your result.