Advanced Mathematical Methods

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1 Linear Algebra

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Outline: Linear Algebra

Readings

- Chapters 15-16
- Chapter 1

Online Resources

MIT course on Linear Algebra (by Gilbert Strang)

- Lecture 1: Vectors, Matrices
  https://www.youtube.com/watch?v=ZK3O402wf1c
- Lecture 3: Multiplication and Inverse Matrices
  https://www.youtube.com/watch?v=QVKj3LADCnA
- Lecture 9: Independence, basis and dimension
  https://www.youtube.com/watch?v=yjBerM5jWsc
- Lecture 18: Properties of determinants
  https://www.youtube.com/watch?v=srxexLishgY

1.1 Vectors

Vector operations

multiplication of an n-dimensional vector $\boldsymbol{v}$ with a scalar $c \in \mathbb{R}$:

$$c \cdot \underset{(n\times1)}{\boldsymbol{v}} = \begin{pmatrix} c \cdot v_1 \\ \vdots \\ c \cdot v_n \end{pmatrix}$$

sum of two $n-$dimensional vectors $\boldsymbol{v}$ und $\boldsymbol{w}$:

$$\underset{(n\times1)}{\boldsymbol{v}} + \underset{(n\times1)}{\boldsymbol{w}} = \begin{pmatrix} v_1 + w_1 \\ \vdots \\ v_n + w_n \end{pmatrix}$$

The **difference** between two $n-$dimensional Vectors $\boldsymbol{v}$ and $\boldsymbol{w}$ is obtained by $\boldsymbol{v} - \boldsymbol{w} = \boldsymbol{v} + (-1)\boldsymbol{w}$.

1.1 Vectors

Vector operations

Inner product (Scalar product) $\boldsymbol{v}, \boldsymbol{w} \in \mathbb{R}^n$:

$$\underset{(1\times n)}{\boldsymbol{v}'} \cdot \underset{(n\times 1)}{\boldsymbol{w}} = \underset{(1\times 1)}{\sum_{i=1}^{n} v_i w_i}$$

1.2 Matrices

Matrix operations

Multiplication with a scalar:

$$\boldsymbol{C} = k \cdot \boldsymbol{A} \;\Leftrightarrow\; c_{ij} = k \cdot a_{ij} \quad \forall \quad i, j.$$

Addition (Subtraction) of matrices:

for two matrices $\boldsymbol{A}$ and $\boldsymbol{B}$ with the same dimensions

$$\boldsymbol{C} = \boldsymbol{A} \pm \boldsymbol{B} \;\Leftrightarrow\; c_{ij} = a_{ij} \pm b_{ij} \qquad \forall\ i, j.$$

1.2 Matrices

Matrix multiplication

$$C = A \cdot B$$

with

$$c_{kl} = \sum_{i=1}^{m} a_{ki} \cdot b_{il}$$

Note: Conformity and dimensionality.

$$\underset{(n \times p)}{C} = \underbrace{\underset{(n \times \underbrace{m)}}{A} \times \underset{(m \times p)}{B}}_{\text{dimensionality}}$$

$$\underbrace{m \quad m}_{\text{conformity}}$$

1.2 Matrices

Rules of matrix multiplication

Given conformity, it holds that:

- $(\boldsymbol{A} \cdot \boldsymbol{B}) \cdot \boldsymbol{C} = \boldsymbol{A} \cdot (\boldsymbol{B} \cdot \boldsymbol{C})$ (associative law)
- $(\boldsymbol{A} + \boldsymbol{B}) \cdot \boldsymbol{C} = \boldsymbol{A} \cdot \boldsymbol{C} + \boldsymbol{B} \cdot \boldsymbol{C}$ (distributive law from the right)
- $\boldsymbol{A} \cdot (\boldsymbol{B} + \boldsymbol{C}) = \boldsymbol{A} \cdot \boldsymbol{B} + \boldsymbol{A} \cdot \boldsymbol{C}$ (distributive law from the left)

Power of a matrix: For a quadratic matrix $\boldsymbol{A}$ we calculate the non-negative integer power as follows:

$$\boldsymbol{A}^n = \underbrace{\boldsymbol{A}\boldsymbol{A}\cdots\boldsymbol{A}}_{\text{n-mal}} \qquad \text{with} \quad n > 0$$

special case: $\boldsymbol{A}^0 = \boldsymbol{I}$.

1.2 Matrices

Kronecker product

$\boldsymbol{A}$ is $m \times n$ and $\boldsymbol{B}$ is $p \times q$, then the Kronecker product $\boldsymbol{A} \otimes \boldsymbol{B}$ is the $mp \times nq$ block matrix

$$\boldsymbol{A} \otimes \boldsymbol{B} = \begin{bmatrix} a_{11}\boldsymbol{B} & \dots & a_{1n}\boldsymbol{B} \\ a_{21}\boldsymbol{B} & \dots & a_{2n}\boldsymbol{B} \\ \vdots & \ddots & \vdots \\ a_{m1}\boldsymbol{B} & \dots & a_{mn}\boldsymbol{B} \end{bmatrix}$$

1.2 Matrices

Idempotent matrix:
A quadratic matrix $\boldsymbol{A}$ is idempotent if: $\boldsymbol{A}^2 \equiv \boldsymbol{A}\boldsymbol{A} = \boldsymbol{A}$.

Trace of a quadratic matrix:

$$\boldsymbol{tr}(\boldsymbol{A}) \equiv \sum_{i=1}^{n} a_{ii}$$

1.3 Inverse of a quadratic matrix

The inverse of a matrix $\boldsymbol{A}$, expressed by $\boldsymbol{A}^{-1}$, should have the following characteristics:

$$\boldsymbol{A} \cdot \boldsymbol{A}^{-1} = \boldsymbol{A}^{-1} \cdot \boldsymbol{A} = \boldsymbol{I}$$

Note:

1.) The matrix $\boldsymbol{A}$ has to be quadratic (due to conformity). Otherwise it is not invertible.
2.) The inverse doesn't have to exist for every single quadratic matrix
3.) If there is an inverse, we call the quadratic matrix *non-singular*, otherwise we call it *singular*.

1.3 Inverse of a quadratic matrix

4.) If there is an inverse, then it is unambiguous

Characteristics (for non-singular matrices $\boldsymbol{A}, \boldsymbol{B}$):

- $(\boldsymbol{A}^{-1})^{-1} = \boldsymbol{A}$
- $(\boldsymbol{A}\boldsymbol{B})^{-1} = \boldsymbol{B}^{-1}\boldsymbol{A}^{-1}$
- $(\boldsymbol{A}')^{-1} = (\boldsymbol{A}^{-1})'$

1.4 The determinant

Sarrus' Rule

For a 2×2 matrix

$$\boldsymbol{A} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

the determinant is defined as follows:

$$\det(\boldsymbol{A}) = |\, \boldsymbol{A} \,| = a_{11}\, a_{22} - a_{12}\, a_{21}$$

1.4 The determinant

An important application:

In general we can show that the determinant of a quadratic matrix with **linearly dependent columns (or rows)** has a zero determinant.

$\Longrightarrow$ The determinant criterion gives us information about the linear dependency (or independency) of the rows (or rather columns) of a matrix as well as about the existence of its inverse.

1.4 The determinant

The determinant of the (3×3)-matrix $\boldsymbol{A}$ is defined as

$$\det(\boldsymbol{A}) \; = \; a_{11} \cdot \mid \boldsymbol{A}_{11} \mid - \, a_{12} \cdot \mid \boldsymbol{A}_{12} \mid + \, a_{13} \cdot \mid \boldsymbol{A}_{13} \mid$$

(cofactor formula)

1.4 The determinant

Illustration:

$$\underset{(3\times 3)}{\boldsymbol{A}} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

Determining the **submatrices**:

Elimination of the 1^{st} row and the 1^{st} column of $\boldsymbol{A}$ yields the **submatrix** $\boldsymbol{A}_{11}$ of dimension (2×2):

$$\underset{(3\times 3)}{\boldsymbol{A}} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \Longrightarrow \underset{(2\times 2)}{\boldsymbol{A}_{11}} = \begin{pmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{pmatrix}$$

1.4 The determinant

Elimination of the 1^{st} row and the 2^{nd} column of $\boldsymbol{A}$ yields the **submatrix** $\boldsymbol{A}_{12}$ of dimension (2×2):

$$\underset{(3\times 3)}{\boldsymbol{A}} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \Longrightarrow \underset{(2\times 2)}{\boldsymbol{A}_{12}} = \begin{pmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{pmatrix}$$

Elimination of the 1^{st} row and the 3^{rd} column of $\boldsymbol{A}$ yields the **submatrix** $\boldsymbol{A}_{13}$ of dimension (2×2):

$$\underset{(3\times 3)}{\boldsymbol{A}} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \Longrightarrow \underset{(2\times 2)}{\boldsymbol{A}_{13}} = \begin{pmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{pmatrix}$$

The determinants $|\boldsymbol{A}_{ij}|$ of the submatrices $\boldsymbol{A}_{ij}$ are called **subdeterminants**; They can be calculated using the *Sarrus' Rule* (if of order of 3 or lower)

1.4 The determinant

Alternative: Extension of the (3×3)-matrix $\boldsymbol{A}$ for the application of the *Rule of Sarrus*:

$$\boldsymbol{A}^\star = \left(\begin{array}{ccc|cc} a_{11} & a_{12} & a_{13} & a_{11} & a_{12} \\ a_{21} & a_{22} & a_{23} & a_{21} & a_{22} \\ a_{31} & a_{32} & a_{33} & a_{31} & a_{32} \end{array}\right)$$

$$\begin{aligned} \det(\boldsymbol{A}) \;=\; & a_{11}\,a_{22}\,a_{33} + a_{12}\,a_{23}\,a_{31} + a_{13}\,a_{21}\,a_{32} \\ & - a_{13}\,a_{22}\,a_{31} - a_{11}\,a_{23}\,a_{32} - a_{12}\,a_{21}\,a_{33} \end{aligned}$$

1.4 The determinant

Cofactor expansion

Calculation of the determinant for general $n \times n$ matrices:
Cofactor expansion *across a row i:*

$$\det(\boldsymbol{A}) \;=\; \sum_{j=1}^{n} (-1)^{i+j}\, a_{ij} \mid \boldsymbol{A}_{ij} \mid$$

Alternatively: Cofactor expansion *down a column j:*

$$\det(\boldsymbol{A}) \;=\; \sum_{i=1}^{n} (-1)^{i+j} a_{ij} \mid \boldsymbol{A}_{ij} \mid$$

Note: The product $(-1)^{i+j} \mid \boldsymbol{A}_{ij} \mid$ is called **cofactor**.

1.4 The determinant

Properties of determinants

for $\boldsymbol{A}$ and $\boldsymbol{B}$ with dimension $n \times n$:

1.) The exchange of two rows or two columns of a matrix leads to a change in the sign of the determinant.

2.) The determinant doesn't change its value if we add to a row (column) within a matrix the multiple of another row (column).

3.) The determinants of a matrix and its transpose are equal:

$$\det(\boldsymbol{A}) = \det(\boldsymbol{A}')$$

4.) Multiplying all components of a $(n \times n)$ matrix with the same factor k leads to a change in the value of the determinant by the factor k^n:

$$\det(k\boldsymbol{A}) = k^n \det(\boldsymbol{A})$$

1.4 The determinant

Properties of determinants

5.) The determinant of every identity matrix is equal to 1; the determinant of every zero matrix is equal to 0.

6.) The determinant of the product of $\boldsymbol{A}$ and $\boldsymbol{B}$ equals the product of the determinants of $\boldsymbol{A}$ and $\boldsymbol{B}$:

$$\det(\boldsymbol{A} \cdot \boldsymbol{B}) = \det(\boldsymbol{A}) \cdot \det(\boldsymbol{B})$$

7.) From 6.) follows for a regular matrix $\boldsymbol{A}$ that:

$$\det(\boldsymbol{A}^{-1}) = \frac{1}{\det(\boldsymbol{A})}$$

8.) In general: $\det(\boldsymbol{A} + \boldsymbol{B}) \neq \det(\boldsymbol{A}) + \det(\boldsymbol{B})$.

1.5 Calculation of the inverse

We can determine regularity/non-singularity/invertibility of the *square matrix* $\boldsymbol{A}$ using the determinant. It holds that

$$\det(\boldsymbol{A}) \neq 0 \;\Leftrightarrow\; \boldsymbol{A}^{-1} \text{ exists.}$$

1.5 Calculation of the inverse

In general: The inverse of the $(n \times n)$-matrix $\boldsymbol{A}$ is denoted as

$$\boldsymbol{A}^{-1} = \boldsymbol{B} = \begin{pmatrix} b_{11} & \dots & b_{1n} \\ \vdots & & \vdots \\ b_{n1} & \dots & b_{nn} \end{pmatrix}$$

We get every single element of $\boldsymbol{B}$ by

$$b_{ij} = \frac{1}{|\boldsymbol{A}|}(-1)^{(i+j)} \; |\boldsymbol{A}_{ji}| \qquad \text{(note the index!)}$$

In order to get the element b_{ij}, you have to calculate the subdeterminant $\boldsymbol{A}_{ji}$ crossing out the $j-$th row and the $i-$th column of $\boldsymbol{A}$.

1.6 Linear independence and rank of a matrix

Linear combination of vectors

Definition: linear combination

For the vectors $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_k \in \mathbb{R}^n$ a n-dimensional vector $\boldsymbol{w}$ is called **linear combination** of vectors $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_k$, if there are real numbers $c_1, c_2, \ldots, c_k \in \mathbb{R}$, such that:

$$\boldsymbol{w} = c_1 \cdot \boldsymbol{v}_1 + c_2 \cdot \boldsymbol{v}_2 + \cdots + c_k \cdot \boldsymbol{v}_k = \sum_{i=1}^{k} c_i \cdot \boldsymbol{v}_i \ .$$

1.6 Linear independence and rank of a matrix

Linear independence

Definition: linear independence

The vectors $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_k \in \mathbb{R}^n$ are called **linearly independent**, if

$$c_1 \cdot \boldsymbol{v}_1 + c_2 \cdot \boldsymbol{v}_2 + \cdots + c_k \cdot \boldsymbol{v}_k = \mathbf{0} \qquad \text{with} \quad c_1, c_2, \ldots, c_k \in \mathbb{R}$$

is only attainable with $c_1 = c_2 = \cdots = c_k = 0$. Otherwise they are called **linearly dependent** and $\boldsymbol{v}_1 = d_2 \cdot \boldsymbol{v}_2 + \cdots + d_k \cdot \boldsymbol{v}_k$ (with $d_2, d_3, \ldots, d_k \in \mathbb{R}$) applies.

1.6 Linear independence and rank of a matrix

Rank

The **rank** of the $n \times m$-matrix $\boldsymbol{A}$ ($\text{rk}(\boldsymbol{A})$) is determined by the maximum number of linearly independent columns (rows) of the matrix $\boldsymbol{A}$.

$$\boxed{\text{rk}(\boldsymbol{A}) \leq \min(m, n)}$$

For every matrix the column rank equals the row rank.
The rank criterion allows to determine whether a quadratic $n \times n$ matrix $\boldsymbol{A}$ is regular/non-singular or not:

$$\begin{aligned} \text{rk}(\boldsymbol{A}) = n &\quad \Rightarrow \quad non-singular \\ \text{rk}(\boldsymbol{A}) < n &\quad \Rightarrow \quad singular \end{aligned}$$

1.6 Linear independence and rank of a matrix

Properties of the rank

1.) The rank of a matrix doesn't change if you exchange rows or columns among themselves.

2.) The rank of a matrix $\boldsymbol{A}$ is equal to the rank of the transpose $\boldsymbol{A}'$.

3.) For a $(m \times n)$ matrix $\boldsymbol{A}$ the following applies:
$\text{rk}(\boldsymbol{A}) = \text{rk}(\boldsymbol{A}'\boldsymbol{A})$, whereby $\boldsymbol{A}'\boldsymbol{A}$ is quadratic.

1.6 Linear independence and rank of a matrix

Determination of the rank of a matrix

1.) We consider all quadratic submatrices of a matrix of which the determinants are not 0. Then we search for the determinant of highest order. The order of this determinant is equal to the rank of the matrix.

2.) Using gaussian algorithm

3.) Using eigenvalues