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Wirtschafts- und
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Fakultät

Chair of Statistics, Econometrics and Empirical Economics

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S414
Advanced Mathematical Methods

Exercises

WS 2018/19

Linear Algebra

Exercise 1 **Quadratic Form**

Given the matrix

$$\mathbf{A} = \begin{pmatrix} 2 & -2 \\ 1 & 2 \end{pmatrix} \quad .$$

a) Determine the definiteness of the quadratic form $Q = \boldsymbol{x}'\boldsymbol{A}\boldsymbol{x}$.

b) Explain in two sentences maximum what this means for the graph $\{(x_1, x_2, Q)|Q = (x_1; x_2)\boldsymbol{A}(x_1; x_2)'\}$.

Exercise 2 **Quadratic Form**

Write the quadratic form

$$Q = 4x_1^2 + 4x_1x_2 - x_2^2$$

in matrix notation and determine its definiteness.

Exercise 3 **Sign definiteness**

Express each quadratic form below as a matrix product involving a *symmetric* coefficient matrix:

a) $q = 3u^2 - 4uv + 7v^2$

b) $q = u^2 + 7uv + 3v^2$

c) $q = 8uv - u^2 - 31v^2$

d) $q = 6xy - 5y^2 - 2x^2$

e) $q = 3u_1^2 - 2u_1u_2 + 4u_1u_3 + 5u_2^2 + 4u_3^2 - 2u_2u_3$

f) $q = -u^2 + 4uv - 6uw - 4v^2 - 7w^2$

Exercise 4 **Sign definiteness**

Given a quadratic form $u'Du$. where D is 2×2, the characteristic equation of D can be written as:

$$\begin{vmatrix} d_{11} - r & d_{12} \\ d_{21} & d_{22} - r \end{vmatrix} = 0 \qquad (d_{12} = d_{21})$$

Expand the determinant; express the roots of this equation by use of the quadratic formula and deduce the following:

a) No imaginary number (a number involving $\sqrt{-1}$) can occur in r_1 and r_2.

b) To have repeated roots, the matrix D must be in the form of $\begin{pmatrix} c & 0 \\ 0 & c \end{pmatrix}$

c) To have either positive or negative smidefiniteness, the determinant of the matrix D must vanish, i.e. $|D| = 0$.

Solution Exercise 1:

a) positive definite

Solution Exercise 2:

$\boldsymbol{Q} = \boldsymbol{x}'\boldsymbol{A}\boldsymbol{x}$ with $A = \begin{pmatrix} 4 & 2 \\ 2 & -1 \end{pmatrix}$

$\boldsymbol{A}$ is indefinite

Solution Exercise 3:

Quadratic form: $q = \mathbf{x}'\mathbf{A}\mathbf{x}$

a)

$$q = \begin{pmatrix} -u \\ v \end{pmatrix}' \begin{pmatrix} 3 & 2 \\ 2 & 7 \end{pmatrix} \begin{pmatrix} -u \\ v \end{pmatrix}$$

b)

$$q = \begin{pmatrix} u \\ v \end{pmatrix}' \begin{pmatrix} 1 & 3.5 \\ 3.5 & 3 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}$$

c)

$$q = \begin{pmatrix} u \\ v \end{pmatrix}' \begin{pmatrix} -1 & 4 \\ 4 & -31 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}$$

d)

$$q = \begin{pmatrix} x \\ y \end{pmatrix}' \begin{pmatrix} -2 & 3 \\ 3 & -5 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

e)

$$q = \begin{pmatrix} u_1 \\ -u_2 \\ u_3 \end{pmatrix}' \begin{pmatrix} 3 & 1 & 2 \\ 1 & 5 & 1 \\ 2 & 1 & 4 \end{pmatrix} \begin{pmatrix} u_1 \\ -u_2 \\ u_3 \end{pmatrix}$$

f)

$$q = \begin{pmatrix} u \\ v \\ -w \end{pmatrix}' \begin{pmatrix} -1 & 2 & 3 \\ 2 & -4 & 0 \\ 3 & 0 & -7 \end{pmatrix} \begin{pmatrix} u \\ v \\ -w \end{pmatrix}$$