Mass in Newtonian gravity and general relativity

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Colloquium @ Monash

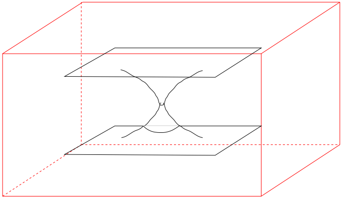
August 28, 2014

Setting

We work in the setting of isolated gravitating systems

modeled in

- Newton's theory of gravity (NG)
- Einstein's general relativity (GR)



These model stars, galaxies, or (in GR) black holes.

Aim

Better understand their:

- definition
- local and total mass
- local and total center of mass
- Newtonian limit $c \to \infty$

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Mass and center of mass in NG

- A Newtonian gravitating system is described by its matter density

$$\rho : \mathbb{R}^3 \to [0, \infty) .$$

- It is isolated if ρ decays "fast enough" for $r \to \infty$.
- Its *(total) mass* is defined as

$$m := \int_{\mathbb{R}^3} \rho \, dV .$$

- Its *(total) center of mass* (CoM) is defined as

$$\vec{z} := \frac{1}{m} \int_{\mathbb{R}^3} \rho \, \vec{x} \, dV$$

w. r. t. Euclidean coordinates (x^i) or position vector $\vec{x}$.

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Sufficient fall-off

By theorem on dominated convergence (Lebesgue integration):

- Sufficient fall-off for convergence of mass $m = \int_{\mathbb{R}^3} \rho \, dV$:

$$\rho = \mathcal{O}\left(\frac{1}{r^{3+\varepsilon}}\right), \varepsilon > 0.$$

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Sufficient fall-off ctd.

Alternatively, use indefinite Riemann integrals in spherical polars:

- Split $\rho =: \rho_{symm} + \rho_{anti}$.
- Observe that ρ_{symm} does not contribute to $\vec{z}$:

$$\lim_{R\to\infty} \int_{B_R(0)} \rho_{symm}(\vec{x})\,\vec{x}\,dV(\vec{x}) = \lim_{R\to\infty} \int_0^R \int_{\mathbb{S}^2} \rho_{symm}(r\vec{\eta})\,\vec{\eta}\,d\sigma(\vec{\eta})\,r^3\,dr = \vec{0}.$$

- Only ρ_{anti} contributes to $\vec{z}$.
- Thus, ρ should be asymptotically symmetric.
- Corresponds to “Regge-Teitelboim conditions” in general relativity.

Critical fall-off: What happens when $\rho = \mathcal{O}(\frac{1}{r^4})$?

- If $\rho = \frac{1}{r^4}$ then $\vec{z} = \vec{0}$ converges.
- If $\rho = \frac{1}{r^4}\left(|\vec{a}| + \frac{\vec{a}\cdot\vec{z}}{r}\right) \geq 0$ for $\vec{a} \neq \vec{0}$ then

$$\int_{\mathbb{R}^3} \rho\, \vec{x}\, dV \approx \lim_{R\to\infty} \int^R \frac{1}{r}\, dr \cdot \frac{4\pi}{3}\vec{a}$$ diverges logarithmically.

- If $\rho = \frac{1}{r^4}\left(|\vec{a}| + \sin(r)\,\frac{\vec{a}\cdot\vec{z}}{r}\right) \geq 0$ for $\vec{a} \neq \vec{0}$ then

$$\vec{z} = \frac{1}{m}\int_{\mathbb{R}^3} \rho\, \vec{x}\, dV = \frac{2\pi^2}{3m}\vec{a}$$ gives a prescribed center of mass.

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Reminder: the Newtonian potential

Newtonian gravity (NG) is governed by

Newton's equation

$$\triangle U = 4\pi G\rho \qquad \text{in } \mathbb{R}^3$$

where

- U is the Newtonian potential
- ρ is the matter density
- G is Newton's gravitational constant

Asymptotic behavior

It is well-known that

Theorem

Suitably isolated systems (solutions of Newton's equation) satisfy

$$U = -\frac{mG}{r} - \frac{mG\,\vec{z}\cdot\vec{x}}{r^3} + \mathcal{O}_2\left(\frac{1}{r^3}\right)$$

in canonical coordinates, where m is the Newtonian mass and $\vec{z} \in \mathbb{R}^3$ is the Newtonian center of mass (CoM).

The critical order $\rho = \mathcal{O}\left(\frac{1}{r^4}\right)$ corresponds to the critical order

$$U = -\frac{mG}{r} + \mathcal{O}_2\left(\frac{1}{r^2}\right).$$

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Formal structure of GR

A relativistic gravitating system/spacetime consists of

- a spacetime 4-manifold M^4
- a symmetric $(0,2)$-matter tensor field T
- a Lorentzian metric ${}^4g$

satisfying the Einstein equations

$$ {}^4\mathrm{Ric} - \frac{1}{2}\,{}^4\mathrm{R}\,{}^4g = \frac{8\pi G}{c^4}\,T $$

with gravitational constant G, speed of light c.

Most important example: Schwarzschild spacetime

Lorentzian metric:

$$ {}^{4}g = -\,{}^{S}N^{2}dt^{2} + {}^{S}g $$

Timeslice metric:

$$ {}^{S}g_{ij} = \left(1 + \frac{mG}{2rc^{2}}\right)^{4} \delta_{ij} $$

Lapse function:

$$ {}^{S}N = \frac{1 - \frac{mG}{2rc^{2}}}{1 + \frac{mG}{2rc^{2}}} $$

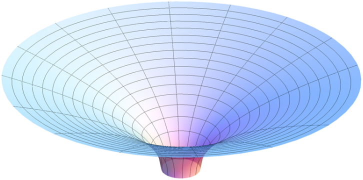
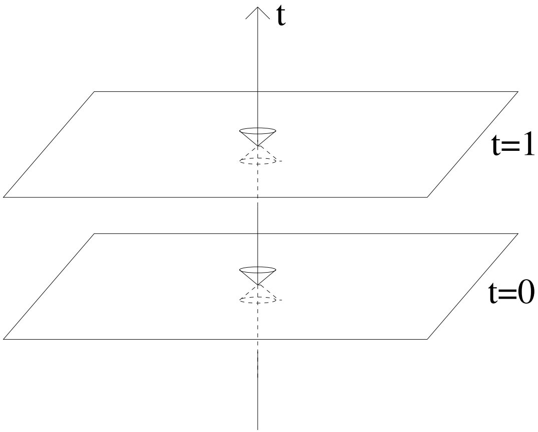


Figure : Timeslice of Schwarzschild source: AllenMcC, wikipedia.org/wiki/Schwarzschild_metric

3+1 decomposition: making it physical



3+1 decomposition

Theorem (Choquet-Bruhat et al. 1952)

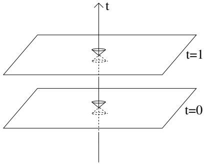
The Einstein equations can be reformulated as a well-posed hyperbolic initial value problem (for suitable matter models).

Remarks:

- involves (non-canonical) 3+1 decomposition, constraint equations
- involves choice of coordinates (lapse and shift)
- gives rise to phenomena like gravitational waves
- IVP approach is used in numerical simulations

Timeslices

- The initial data for the Choquet-Bruhat initial value problem is a timeslice $\{t = 0\}$.
- It is represented by a 3-dimensional Riemannian manifold (M^3, g), together with a second fundamental form $(0, 2)$-tensor K on M^3.
- g, K satisfy constraint equations induced by Einstein's equation.
- For simplicity, we will ignore K from now on.



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Asymptotically Schwarzschildean ends

An asymptotically Schwarzschildean (AS) end is

- a three-dimensional Riemannian manifold (M^3, g)
- diffeomorphic to $\mathbb{R}^3 \setminus B_1(0)$
- satisfying fall-off conditions as $|\vec{x}| =: r \to \infty$
- formulated as deviation from the (spatial) Schwarzschild metric

$$^S g_{ij} = \left(1 + \frac{mG}{2rc^2}\right)^4 \delta_{ij},$$

with m the mass parameter.

Summary: the role of AS ends in GR

In GR, asymptotically Schwarzschildean ends appear as models of

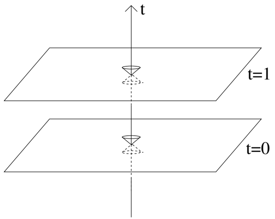

Figure : timeslices

- (spacelike) timeslices
- the exterior regions of isolated gravitating systems, e.g.
  - stars or
  - black holes.

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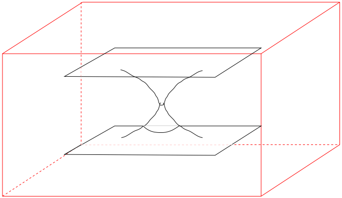

Figure : isolated system

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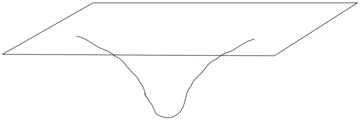

Figure : a star

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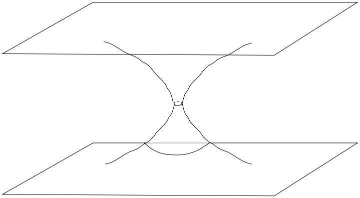

Figure : a black hole

- (spacelike) timeslices
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Asymptotic charts: other representation

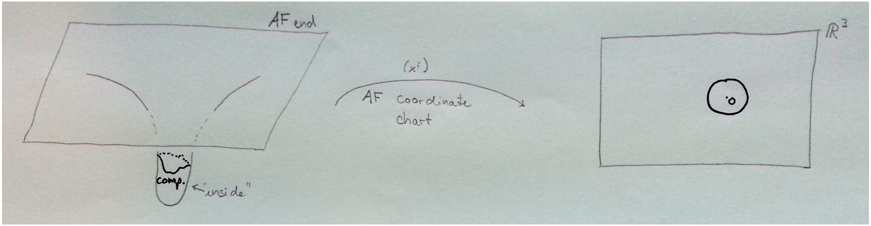

Figure : AS coordinate chart

Mass in GR

- The (total) mass of an asymptotically Schwarzschildean end (timeslice) is given by the parameter m.
- In GR, there is a more general definition of mass by Arnowitt-Deser-Minner '68 called m_{ADM}, using surface integrals w. r. t. the asymptotic coordinates.
- Here, $m_{ADM} = m$.

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Definitions of center of mass in GR

CoM is a difficult concept:

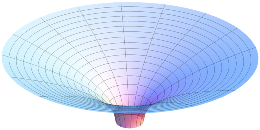


Figure : Timeslice of Schwarzschild

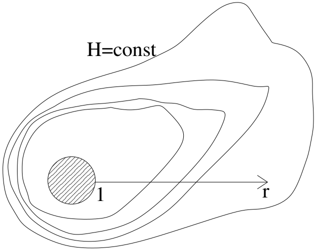
Several definitions of CoM in the literature:

- Definition à la ADM: Regge-Teitelboim ’74, Beig-Ó Murchadha ’86.
- Geometric definition by Huisken-Yau ’96.
- Several others (Schoen, Corvino-Wu, Wang-Yau,. . .).

Huisken-Yau definition of CoM

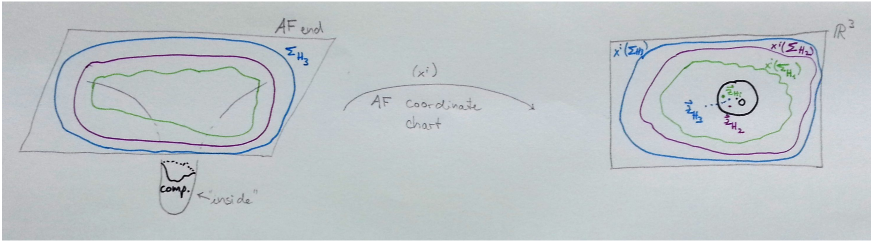
Theorem (Huisken-Yau 1996)

In any asymptotically Schwarzschildean Riemannian end with mass $m > 0$, there exists a unique foliation near infinity by stable spheres of constant mean curvature (CMC).



- "Precise" asymptotic condition: $g_{ij} = {}^{S}g_{ij} + \mathcal{O}_4(\frac{1}{r^2})$.
- Assumptions improved by Metzger, Huang, Nerz, ...
- C.-Nerz: *abstract center of mass*

Huisken-Yau definition of CoM ctd.



Theorem (Huisken-Yau 1996 ctd.)

Euclidean center $\vec{z}_H$ of Σ_H and (total) center $\vec{z}_{Huisken-Yau}$ are defined as

$$\vec{z}_H := \oint\limits_{x^i(\Sigma_H)} \vec{x}\, d\sigma, \quad \vec{z}_{Huisken-Yau} := \lim_{H\to 0} \vec{z}_H.$$

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- C.-Nerz: *coordinatization of abstract CoM w. r. t. chosen chart near infinity*
- coincides with other notions of CoM by Regge-Teitelboim, Beig-Ó Murchadha, Schoen, Wang-Yau, ...
  [Huang, Eichmair-Metzger, Nerz, ...]

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However:

Theorem (C.-Nerz 2014)

The total CoM limit

$$\vec{z}_{Huisken-Yau} := \lim_{H\to 0} \vec{z}_H$$

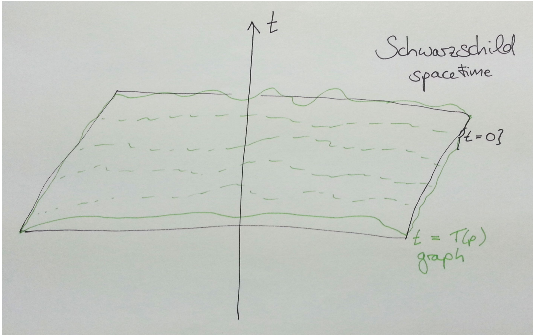
does not always converge under the assumptions of Huisken-Yau. It does however converge under the stronger assumption

$$g_{ij} = {}^S g_{ij} + \mathcal{O}\left(\frac{1}{r^{2+\varepsilon}}\right)$$

for any $\varepsilon > 0$.

- Different explicit counter-examples to Huisken-Yau assertion.
- All coinciding center of mass definitions also diverge.

C.-Nerz counter-example



Pick graph function

$$t = T(\vec{x}) = \sin(\ln r) + \frac{\vec{a} \cdot \vec{x}}{r}, \quad \vec{a} \neq \vec{0}.$$

Remarks

- Counter-example satisfies the (vacuum) constraint equations.
- Other counter-example related to motion in spacetime,
- simplified by Chan-Tam to conformally flat example

$$g_{ij} = \left(1 + \frac{4mG}{rc^2} + \sin(\ln r)\frac{mG\,\vec{a}\cdot\vec{x}}{r^3c^2}\right)\delta_{ij}, \quad \vec{a} \neq \vec{0}.$$

- Same critical order of decay as (critical) Newtonian example!
- Can also prescribe a freely chosen CoM at the critical order.

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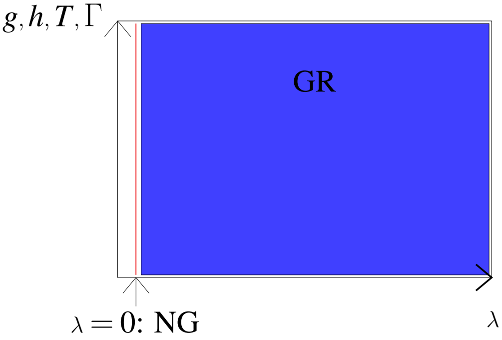
Newtonian limit

Ehlers constructed Frame Theory (FT) encompassing GR and NG

→ Frame Theory allows rigorous definition of Newtonian Limit $c \to \infty$.

→ Rendall, Oliynyk, ...showed existence of converging families.



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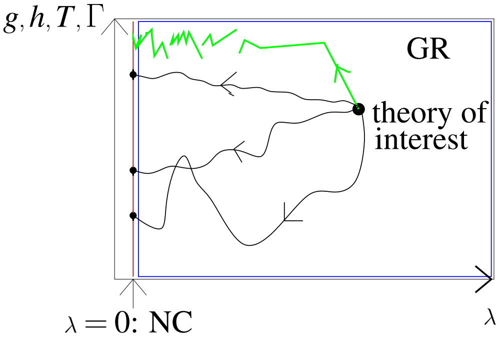

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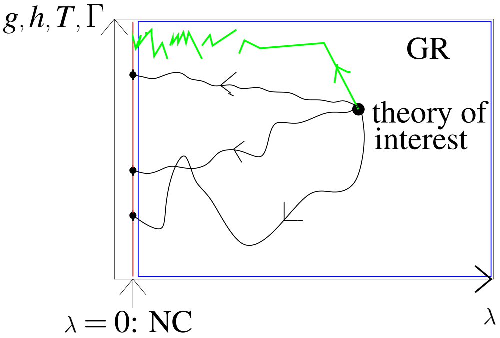

Static Newtonian limit

In the static (non-dynamical) setting:

Theorem (C. 2011)

- *One can define "staticity" and asymptotic decay in FT.*
- *One can define local and total mass in static FT coinciding with ADM and Newtonian mass, respectively.*
- *One can define local and total CoM in static FT coinciding with Huisken-Yau and Newtonian CoM, respectively.*
- *Static AS spacetimes converge to static isolated Newtonian systems in the Newtonian limit as $c \to \infty$.*
- *Then mass and CoM are continuous w. r. t. the Newtonian limit:*

$$\lim_{c\to\infty} m_{ADM}(c) \to m_{Newtonian},$$

$$\lim_{c\to\infty} \vec{z}_{Huisken-Yau}(c) \to \vec{z}_{Newtonian}.$$

Strategy of proof

- Rewrite static GR in "pseudo-Newtonian variables".
- Rewrite Newtonian mass as surface integral by divergence thm:

$$m = \int_{\mathbb{R}^3} \rho \, dV = \int_C \rho \, dV = \frac{1}{4\pi G} \int_C \triangle U \, dV = \frac{1}{4\pi G} \int_{\partial C} \frac{\partial U}{\partial \nu} \, d\sigma$$

  where C is any compact domain with ∂C smooth and $\operatorname{supp} \rho \subset C$.
- Mimic this in pseudo-Newtonian gravity.
- Repeat with center of mass.
- Define Killing vector fields etc. consistently in frame theory.
- Analyze continuity in the right function spaces.

Remark: Theorem is consistent with counter-examples of C.-Nerz.

Thank you for your attention!