Advanced Mathematical Methods

WS 2020/21

4 Difference Equations

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Outline: Difference Equations

Readings

- J. D. Hamilton. *Time Series Analysis.*
  Princeton University Press, 1994, Chapter 1
- Knut Sydsaeter, Peter Hammond, Atle Seierstad, and Arne Strøm. *Further Mathematics for Economic Analysis.*
  Prentice Hall, 2008, Chapter 11

Online References

What is a Difference Equation? (Jonathan Mitchell)
https://www.youtube.com/watch?v=bfMjdvQoUYA
Introduction to Linear Difference Equations (Thomas Dimpfl)
https://youtu.be/Ir2QJ0rsUdM

4.1 First-Order Difference Equations

Linear first-order difference equation:

$$y_t = \phi y_{t-1} + w_t \tag{1}$$

- y_t is value at date t
- linear equation that relates y_t to y_{t-1}
- first-order since only first lag is included
- w_t: a variable coefficient

4.1 First-Order Difference Equations

homogeneous first order difference equation:

$$\Delta y_t + a y_{t-1} = 0$$

- with solution $y_t = (1-a)^t C^*$

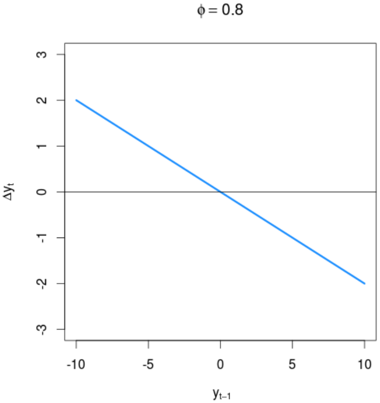
inhomogeneous first order difference equation:

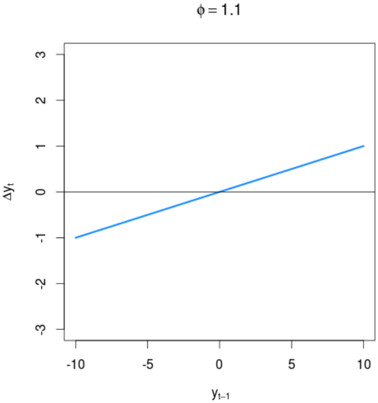
$$\Delta y_t + a y_{t-1} = b$$

- with solution $y_t = C^*(1-a)^t + \frac{b}{a}$

4.1 First-Order Difference Equations

Phase diagram





$$\Delta y_t = -0.2 y_{t-1}$$

$$\Delta y_t = 0.1 y_{t-1}$$

4.2 Dynamic First Order Difference Equation

$$y_t = \phi y_{t-1} + w_t$$

- inhomogenous case with $b = w_t$
  but: ω_t is dynamic
- Question: What are the effects on y_t of changes in w_t?

The dynamics described by the equation above govern the behaviour of y for all dates t

Date	Equation
0	$y_0 = \phi y_{-1} + w_0$
1	$y_1 = \phi y_0 + w_1$
2	$y_1 = \phi y_1 + w_2$
$\vdots$	$\vdots$
t	$y_t = \phi y_{t-1} + w_t$

4.2 Dynamic First Order Difference Equation

The following procedure is known as solving the difference equation above by *recursive substitution*:

$$y_t = \phi^{t+1} y_{-1} + \phi^t w_0 + \phi^{t-1} w_1 + \phi^{t-2} w_2 + \cdots + \phi w_{t-1} + w_t$$

4.3 Dynamic Multipliers

If w_0 were to change with y_{-1} and $w_1, w_2, \dots, w_t$ taken as unaffected, the effect on y_t would be given by

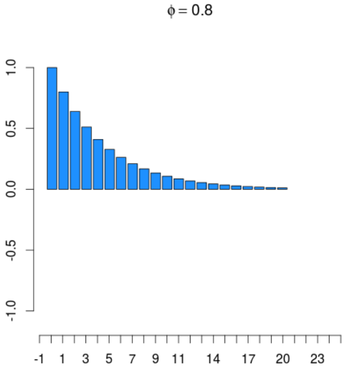
$$\frac{\partial y_t}{\partial w_0} = \phi^t$$

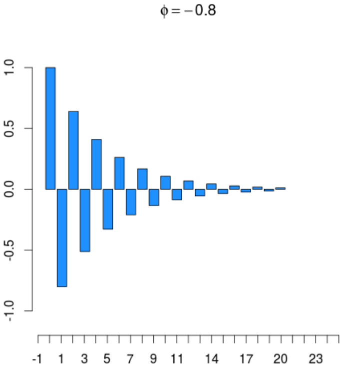
The effect of w_t on y_{t+j} is given by

$$\frac{\partial y_{t+j}}{\partial w_t} = \phi^j$$

4.3 Dynamic Multipliers

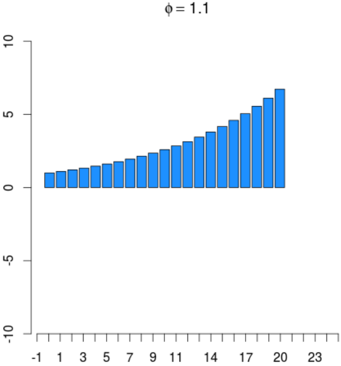
Dynamic Multiplier for the first-order difference equation for different values of ϕ (plot of $\frac{\partial y_{t+j}}{\partial w_t} = \phi^j$ as a function of the lag j)





4.3 Dynamic Multipliers

Dynamic Multiplier for the first-order difference equation for different values of ϕ (plot of $\frac{\partial y_{t+j}}{\partial w_t} = \phi^j$ as a function of the lag j)



4.3 Dynamic Multipliers

Consider a permanent change in w, i.e. all w_{t+j} increase by one unit. Then the effect on y_{t+j} of a permanent change in w beginning in period t is given by

$$\frac{\partial y_{t+j}}{\partial w_t} + \frac{\partial y_{t+j}}{\partial w_{t+1}} + \frac{\partial y_{t+j}}{\partial w_{t+2}} + \cdots + \frac{\partial y_{t+j}}{\partial w_{t+j}} = \phi^j + \phi^{j-1} + \phi^{j-2} + \phi + 1$$

When $|\phi| < 1$, the limit of this expression as j goes to infinity is sometimes described the **long-run** effect of w on y

$$\begin{aligned}
&\lim_{j \to \infty} \left(\frac{\partial y_{t+j}}{\partial w_t} + \frac{\partial y_{t+j}}{\partial w_{t+1}} + \frac{\partial y_{t+j}}{\partial w_{t+2}} + \cdots + \frac{\partial y_{t+j}}{\partial w_{t+j}} \right) \\
&= 1 + \phi + \phi^2 + \dots \\
&= \frac{1}{(1 - \phi)}
\end{aligned}$$

4.4 pth-Order Difference Equations

Generalize the dynamic system (1) by allowing the value of y at date t to depend on p of its own lags

Linear pth order difference equation:

$$y_t = \phi_1 y_{t-1} + \phi_2 y_{t-2} + \cdots + \phi_p y_{t-p} + w_t$$

Rewrite as first-order vector difference equation:
collect y_t and its lags in a $(p \times 1)$ vector

$$\boldsymbol{\xi_t} = \begin{pmatrix} y_t \\ y_{t-1} \\ y_{t-2} \\ \vdots \\ y_{t-p+1} \end{pmatrix}$$

4.4 *p*th-Order Difference Equations

Define the $(p \times p)$ matrix $\boldsymbol{F}$

$$\boldsymbol{F} = \begin{pmatrix} \phi_t & \phi_2 & \phi_3 & \dots & \phi_{p-1} & \phi_p \\ 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 & 0 \end{pmatrix}$$

and obtain the following first-order vector difference equation

$$\boldsymbol{\xi_t} = \boldsymbol{F}\boldsymbol{\xi_{t-1}} + \boldsymbol{v_t}$$

with $\boldsymbol{v_t} = (w_t, 0, 0, \dots, 0)$

4.4 pth-Order Difference Equations

recursive substitution of the first-order vector difference equation yields

$$\boldsymbol{\xi_t} = \boldsymbol{F}^{t+1}\boldsymbol{\xi_{-1}} + \boldsymbol{F}^{t}\boldsymbol{v_0} + \boldsymbol{F}^{t-1}\boldsymbol{v_1} + \boldsymbol{F}^{t-2}\boldsymbol{v_2} + \cdots + \boldsymbol{F}\boldsymbol{v_{t-1}} + \boldsymbol{v_t}$$

$$\begin{pmatrix} y_t \\ y_{t-1} \\ y_{t-2} \\ \vdots \\ y_{t-p+1} \end{pmatrix} = \boldsymbol{F}^{t+1} \begin{pmatrix} y_{-1} \\ y_{-2} \\ y_{-3} \\ \vdots \\ y_{-p} \end{pmatrix} + \boldsymbol{F}^{t} \begin{pmatrix} w_0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} + \boldsymbol{F}^{t-1} \begin{pmatrix} w_1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} + \dots$$

$$+\boldsymbol{F}^{1} \begin{pmatrix} w_{t-1} \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} + \begin{pmatrix} w_t \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

4.4 pth-Order Difference Equations

Let $f_{11}^{(t)}$ denote the $(1,1)$ element of $\boldsymbol{F}^{\boldsymbol{t}}$, $f_{12}^{(t)}$ the $(1,2)$ element of $\boldsymbol{F}^{\boldsymbol{t}}$, and so on.
Thus, for a pth-order difference equation, the dynamic multiplier is given by

$$\frac{\partial y_{t+j}}{\partial w_t} = f_{11}^{(j)}$$

4.4 pth-Order Difference Equations

This is the $(1,1)$ element of $\boldsymbol{F}^{\boldsymbol{j}}$ which can easily be obtained in terms of the eigenvalues of the matrix $\boldsymbol{F}$ via

$$|\boldsymbol{F} - \boldsymbol{\lambda} \boldsymbol{I}_{\boldsymbol{p}}| = 0$$

The eigenvalues of the matrix $\boldsymbol{F}$ are the values of $\boldsymbol{\lambda}$ that satisfy

$$\lambda^p - \phi_1 \lambda^{p-1} - \phi_2 \lambda^{p-2} - \cdots - \phi_{p-1} \lambda - \phi_p = 0$$