Advanced Mathematical Methods

WS 2020/21

5 Mathematical Statistics

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Outline: Mathematical Statistics

Readings

- A. Papoulis and A. U. Pillai. *Probability, Random Variables and Stochastic Processes.*
  Mc Graw Hill, fourth edition, 2002, Chapters 1-4

Online References

MIT Course on Probabilistic Systems Analysis and Applied Probability (by John Tsitsiklis)

- Discrete RVs I: Concept of random variables, probability mass function, expected value, variance
  https://www.youtube.com/watch?v=3MOahpLxj6A
- Continuous RVs: probability density function, cumulative distribution function, expected value, variance
  https://www.youtube.com/watch?v=mHfn_7ym6to

5.1 Measure spaces

Notation: Ω

- fundamental measure (or probability, or sample) space
- consists of all points (singletons) ω possible as the outcome to an experiment

Definition: Event

An Event A is a subset of Ω. The empty event $\emptyset$ and the whole space Ω are also events.

5.1 Measure spaces

Definition: Topological space

A topological space $(\Omega, \mathcal{F})$ is a space Ω together with a class $\mathcal{F}$ of subsets of Ω. The members of the set $\mathcal{F}$ are called open sets. $\mathcal{F}$ has the property that unions of any number of the sets in $\mathcal{F}$ (finite or infinite, countable or uncountable) remain in $\mathcal{F}$, and intersections of finite numbers of sets in $\mathcal{F}$ also remain in $\mathcal{F}$. The closed sets are those whose complements are in $\mathcal{F}$.

5.1 Measure spaces

Definition: Sigma-Algebra

$\mathcal{F}$ is a sigma algebra if

(i) $A_k \in \mathcal{F}$ for all k implies $\cup_{k=1}^{\infty} A_k \in \mathcal{F}$,

(ii) $A \in \mathcal{F}$ implies $\bar{A} \in \mathcal{F}$,

(iii) $\emptyset \in \mathcal{F}$.

Theorem: Properties of a Sigma-Algebra

If $\mathcal{F}$ is a sigma algebra, then

(iv) $\Omega \in \mathcal{F}$,

(v) $A_k \in \mathcal{F}$ for all k implies $\cap_{k=1}^{\infty} A_k \in \mathcal{F}$.

5.1 Measure spaces

Definition: Measurable space

A pair $(\Omega, \mathcal{F})$ where the former is a set and the latter a sigma-algebra of subsets of Ω is called a measurable space.

Definition: Probability measure

A probability measure is a measure P in the measurable space $(\Omega, \mathcal{F})$ which satisfies the following properties:

(i) $P(A) \geq 0$ for all A

(ii) $P(\Omega) = 1$

(iii) $P(\emptyset) = 0$

(iv) $P(\bar{A}) = 1 - P(A)$

(v) monotonicity, subadditivity

5.1 Measure spaces

Definition: Probability space

The triple $(\Omega, \mathcal{F}, \mathcal{P})$ is called a probability space.

Theorem: Conditional probability

For $B \in \mathcal{F}$ with $P(B) > 0$, $Q(A) = P(A \mid B) = P(A \cap B)/P(B)$ is a probability measure on the same space $(\Omega, \mathcal{F})$

5.2 Random Variables

Definition: Measurable function

Let f be a function from a measurable space $(\Omega, \mathcal{F})$ into the real numbers. The function f is measurable if for each Borel set $B \in \mathcal{B}$, the set $\{\omega; f(\omega) \in B\} \in \mathcal{F}$.

Definition: Random variable

A random variable X is a measurable function from a probability space $(\Omega, \mathcal{F}, \mathcal{P})$ into the real numbers $\mathbb{R}$.

4.3 Cumulative Distribution Functions

Probability distribution function: discrete case

$$f_X(x) = P(X = x)$$

requirements:

- $0 \leq P(X = x) \leq 1$
- $\sum_{x} f_X(x) = 1$

4.3 Cumulative Distribution Functions

(Probability) Density function: continuous case

it holds that $P(X = x) = 0$

requirements:

- $P(a \leq X \leq b) = \int_a^b f_X(x)\,\mathrm{d}x \geq 0$
- $\int_{-\infty}^{\infty} f_X(x)\,\mathrm{d}x = 1$

4.3 Cumulative Distribution Functions

Definition: Cumulative distribution function

The cumulative distribution function (cdf) of a random variable X is defined to be the function $F_X(x) = P(X \leq x)$, for $x \in \mathbb{R}$.

to get the cdf:

discrete:

$$F_X(x) = \sum_{X \leq x} f_X(x) = P(X \leq x)$$

continuous:

$$F_X(x) = \int_{-\infty}^{x} f_X(t)\,\mathrm{d}t$$

4.3 Cumulative Distribution Functions

Properties

(i) $F_X(+\infty) = 1$; $F_X(-\infty) = 0$

(ii) $F_X(x)$ is a nondecreasing function of x:
if $x_1 < x_2$, $F_X(x_1) \leq F_X(x_2)$
note: the event $\{X \leq x_1\}$ is a subset of $\{X \leq x_2\}$

(iii) if $F_X(x_0) = 0$, then $F_X(x) = 0 \quad \forall \quad x \leq x_0$

(iv) $P(X > x) = 1 - F_X(x)$
events $\{X \leq x\}$ and $\{X > x\}$ are mutually exclusive and
$\{X \leq x\} \cup \{X > x\} = \Omega$

(v) $F_X(x)$ is continuous from the right:
$\lim_{x \to a^+} F_X(x) = F_X(a)$

(vi) $P(x_1 \leq X \leq x_2) = F_X(x_2) - F_X(x_1)$

5.4 Expectation, Variance and Moments

Expectations of a random variable

$$E[X] = \begin{cases} \sum_{x_i} x f_X(x_i) & \text{if } x \text{ is discrete} \\ \int_{-\infty}^{\infty} x f_X(x) \mathrm{d}x & \text{if } x \text{ is continuous} \end{cases}$$

$g(X)$ a measurable function of x, then:

$$E[g(X)] = \begin{cases} \sum_{x} g(x_i) f_X(x_i) & \text{if } x \text{ is discrete} \\ \int_{-\infty}^{\infty} g(x) f_X(x) \mathrm{d}x & \text{if } x \text{ is continuous} \end{cases}$$

4.4 Expectation, Variance and Moments

Calculation rules

- $E[a] = a$
- $E[bX] = b \cdot E[X]$
- linear transformation $E[a + bX] = a + bE[X]$
- $E[g_1(X) + g_2(X)] = E[g_1(X)] + E[g_2(X)]$

4.4 Expectation, Variance and Moments

Variance of a random variable

let $g(X) = (X - E[X])^2$

$$Var[X] = \sigma^2 = E[(X - E[X])^2]$$

$$= \begin{cases} \sum_x (x_i - E[X])^2 f_X(x_i) & \text{if } x \text{ is discrete} \\ \int_{-\infty}^{\infty} (x - E[X])^2 f_X(x) \mathrm{d}x & \text{if } x \text{ is continuous} \end{cases}$$

4.4 Expectation, Variance and Moments

Calculation rules

- $Var[a] = 0$
- $Var[X + a] = Var[X]$
- $Var[bX] = b^2 Var[X]$
- $Var[a + bX] = b^2 Var[X]$

important result:

$Var[X] = E[X^2] - E[X]^2$

4.4 Expectation, Variance and Moments

Standardization

an important transformation: standardization of a random variable X

$$\text{let } g(X) = \frac{X-\mu}{\sigma} = Z$$

$$Z = \frac{X-\mu}{\sigma} = \frac{-\mu}{\sigma} + \frac{1}{\sigma}X$$

$$\Rightarrow E[Z] = 0$$

$$\Rightarrow Var[Z] = 1$$

4.4 Expectation, Variance and Moments

Chebychev Inequality

for any random variable X with finite expected value μ and finite variance $\sigma^2 > 0$ and a positive constant k

$$P(\mu - k\sigma \leq X \leq \mu + k\sigma) \geq 1 - \frac{1}{k^2}$$

4.4 Expectation, Variance and Moments

Skewness and Kurtosis

central moments of a random variable:

$$\mu_r = E[(X-\mu)^r]$$

as r grows, μ_r tends to explode

solution: normalization

- skewness coefficient: $\gamma = \dfrac{E[(X-\mu)^3]}{\sigma^3}$
- kurtosis: $\kappa = \dfrac{E[(X-\mu)^4]}{\sigma^4}$

  often reported as excess kurtosis $\kappa - 3$