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S414
Advanced Mathematical Methods

Exercises

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Statistical Inference

Exercise 1 Error types and power of the test

When statistically testing hypotheses, we distinguish between the probability of making a type I (α) error, a type II (β) error and the power of the test ($1-\beta$). Explain the concepts and illustrate them graphically. Discuss the trade-off between the probabilities of either error type.
(compare A. Papoulis and S.U. Pillai (2002) *Probability, Random Variables and Stochastic Processes (4th ed.)*, McGraw-Hill, pp. 354-358)

Exercise 2 Interpretation of test results

Cases (**a**)-(**d**) cover different test statistics. Choose one interpretation from the 8 phrases provided on the following page. You may use an interpretation multiple times as some are implausible.

(a) A test statistic is standard normally distributed under the null hypothesis H_0. The problem requires a two-sided test (rejection of the null hypothesis for large positive and negative values of the test statistic) and a 5% significance level, α. Calculating the test statistic results in a value of -1.2. (Note: $1-F_X(x) = F_X(-x)$ where $F_X(x)$ is the cumulative distribution function of a normal distribution.)

(b) A test statistic is $\chi^2(1)$ distributed (chi-square with one degree of freedom) under H_0. The level of significance is 0.01 and the problem at hand requires a one-sided test (rejection of H_0 for large values of the test statistic). Calculation of the test statistic results in 7.1.

(c) Same test design as in (**b**): when calculating the test statistic, we obtain an empirical level of significance (p-value) of 0.015.

(d) A test statistic is Student t-distributed with 4 degrees of freedom under H_0. The problem requires a one-sided test, i.e. the null hypothesis is rejected for large values of the test statistic. The level of significance to be used is 0.01. The obtained test statistic is 1.9.

Possible interpretations

1. The null hypothesis cannot be rejected on a 5% significance level.
2. The null hypothesis is correct with a probability of 95% and is, therefore, accepted.
3. The null hypothesis cannot be rejected on a 1% significance level.
4. The null hypothesis is true and is, thus, accepted.

5. The null hypothesis is rejected on the 5% significance level.
6. The null hypothesis is only correct with a probability of 5% and is, thus, rejected.
7. The null hypothesis is incorrect and is, hence, rejected.
8. The null hypothesis is rejected on the 1% significance level.

Exercise 3 **Interpretation of test results**

Interpret the following test results.

(a) Using a Jarque-Bera test, we can test the null hypothesis whether a random variable is normally distributed. To construct the test statistic, we draw a random sample from the overall population. Under the null hypothesis, the Jarque-Bera test is χ^2 distributed with 2 degrees of freedom. The test is one-sided (we reject the null hypothesis for large values of the test statistic). The random sample consists of logarithmic wage data. We obtain a value of 10.9 for the test statistic. Interpret the outcome of the test.

(b) We evaluate the Jarque-Bera test statistic on the basis of an additional random sample of scores obtained by participants of an IQ test. For this data, we calculate a p-value of 0.12 (empirical significance level) for the JB test statistic. Interpret the outcome of the test.

(c) A Kruskal-Wallis test is used to test the null hypothesis that two or more random samples are drawn from populations in which the random variables of interest exhibit the same expected value. We could for example test whether the expected preference for a product (e.g. Weißbier) is identical in two different distribution regions (e.g. Schleswig-Hohlstein and Bavaria). To construct the test we need random samples from both distribution regions. Under the null hypothesis, the test statistic follows a distribution that is tabulated in many text books. The test is drafted as a one-sided test, i.e. we reject H_0 for large values of the test statistic.
In a marketing study, random samples asking for the preferences for the product are collected for both regions separately and a Kruskal-Wallis test is calculated. A p-value of 0.0001 is obtained. Interpret the outcome of the test.

(d) Lo and MacKinlay developed a test which can be used to test the null hypothesis that stock returns cannot be forecasted based on past returns. The alternative hypothesis states that we can obtain forecasts of stock returns using historic data. Under the null hypothesis, the Lo/MacKinlay test statistic is standard normally distributed. In order to construct the test statistic, we need stock return time series. The Lo/MacKinlay test is two-sided. Using stock market data retrieved from the New York Stock Exchange, we calculate return time series and obtain a value of -1.56 for the Lo/MacKinlay test statistic. Interpret this test statistic.

(e) By means of a KK-test, we can test the null hypothesis that the covariance of two random variables is equal to zero. The alternative hypothesis states that the two random variables display a non-zero covariance. The KK-test is two-sided. Under the null hypothesis, the test statistic is t-distributed with 4 degrees of freedom. In an implementation of the test, we obtain a value of 5.3 for the test statistic. Calculate the critical value for a significance level of $\alpha = 0.05$. Interpret the outcome of the test.

(f) In a second implementation of the test described in (**a**) (different sample) we obtain a p-value of 0.0003. Interpret the outcome of the test.

(g) A third implementation of the test in (**a**) (different sample) results in a p-value of 0.27. Interpret the outcome of the test.

(h) We can test competing distributional assumptions using a Likelihood-Ratio test. One of the assumed distributions is a nested (restricted) version of the more general assumption. For example, $X \sim N(0, \sigma^2)$ is more restrictive than $X \sim N(\mu, \sigma^2)$. The null hypothesis in a LR-test specifies that the restrictive distribution assumption is correct ($\mu = 0$ in the example). The alternative hypothesis states that the restriction does not hold (here $\mu \neq 0$). Under the null hypothesis, the LR-test is χ^2-distributed where the degrees of freedom equal the number of tested restrictions (here 1). The LR-test is a one-sided test, i.e. we reject the null hypothesis for large values of the test statistic.
In a specific application with one parameter restriction, we obtain a value of 2.051 for the "Likelihood-Ratio" test statistic. Calculate the p-value. Explain what the obtained p-value means. Interpret the outcome of the test using an appropriately chosen significance level α.

(i) A researcher conducts two two-sided t-tests on the significance of an estimated parameter $\hat{\theta}$. The parameter is estimated using Maximum Likelihood. She subsequently tests the null hypotheses that $H_0 : \theta = 0$ and $H_0 : \theta = 1$ where θ depicts the parameter in the population. For the first test ($H_0 : \theta = 0$) she obtains a value of the test statistic of 12.5 and in the second test ($H_0 : \theta = 1$) a value of -0.45. Interpret the outcome of the test using the rule of thumb $|t| > 2$.

(j) The marketing department of a company wants to test whether a new packaging augments product sales. They conduct a survey among two groups regarding their disposition to buy. The first group consists of 13 to 19 year olds, the second of 20 to 30 year olds. The department calculates a test statistic for each group. Under the null hypothesis that the new packaging does not enhance sales, the test statistic is t-distributed with 4 degrees of freedom and the test is two-sided. For the first group of test subjects (age group 13-19) they receive a value of the test statistic of 4.6, for the second group (age group 20-30) 1.53. Calculate the p-values and interpret the outcome of the test. Elaborate on the meaning of the p-value.

Exercise 4 **Wald Test**

You estimate a model

$$y = \beta_1 + \sum_{i=1}^{7} \beta_i x_i$$

$$\begin{aligned} H_0: \quad & \beta_1 = \beta_2 = \beta_3 \\ & \beta_1 + \beta_4 = 5 \\ & \beta_5 + \beta_6 = 1 \\ & \beta_2 + \beta_4 = 5 \end{aligned}$$

Write the Wald Test in matrix notation.

Exercise 5 **Confidence interval**

In an experiment the number of needles of $n = 12$ christmas trees is counted. The measured average number is $\bar{x} = 187,333$ needles. From former experiments we know that the standard deviation of the number of needles is $\sigma = 5,000$. Compute the $95\%-$ confidence interval for the number of needles under the assumption that the number of needles is normally distributed.

Solution Exercise 2:
False interpretations:
Phrases 2., 4., 6. and 7. are implausible.

Remaining interpretations:
1., 3. ,5. and 8.

(a) Phrase 1.

(b) Phrase 8.

(c) Phrase 3.

(d) Phrase 3.

Solution Exercise 3:

(a) The H_0 can be rejected on a significance level of $\alpha = 0.01$.

(b) The H_0 cannot be rejected up to a significance level of $\alpha = 0.12$.

(c) The H_0 can be rejected on every conventional significance level.

(d) The H_0 cannot be rejected on every conventional significance level.

(e) The H_0 can be rejected on a significance level of $\alpha = 0.05$.

(f) The H_0 can be rejected on a significance level of $\alpha = 0.01$.

(g) The H_0 cannot be rejected on a significance level of $\alpha = 0.01$.

(h) On every conventional significance level the H_0 is not rejected.

(i) $\rightarrow$ The H_0 can be rejected on a significance level of $\alpha = 0.05$
$\rightarrow$ The H_0 cannot be rejected on a significance level of $\alpha = 0.05$

(j) $\rightarrow$ The H_0 can be rejected on a significance level of $\alpha = 0.05$
$\rightarrow$ On every conventional significance level the H_0 is not rejected.

Solution Exercise 4:
The last equation is redundant.

Solution Exercise 5:
$\mu \in [184, 503.9836; 190, 162.0164]$