Time Series Analysis

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An economic example: Using the Structural VAR (SVAR) model to analyse stock market returns from Tokyo, Singapore, Korea.

$$
\begin{aligned}
r_t^T &= k^T && +\beta_{12}^{(0)} r_t^S + && \beta_{13}^{(0)} r_t^K + && \beta_{11}^{(1)} r_{t-1}^T + \beta_{12}^{(1)} r_{t-1}^S + \beta_{13}^{(1)} r_{t-1}^K + u_t^T \\
r_t^S &= k^S && +\beta_{21}^{(0)} r_t^T + && \beta_{23}^{(0)} r_t^K + && \beta_{21}^{(1)} r_{t-1}^T + \beta_{22}^{(1)} r_{t-1}^S + \beta_{23}^{(1)} r_{t-1}^K + u_t^S \\
r_t^K &= k^K && +\beta_{31}^{(0)} r_t^T + \beta_{32}^{(0)} r_t^S + && && \beta_{31}^{(1)} r_{t-1}^T + \beta_{32}^{(1)} r_{t-1}^S + \beta_{33}^{(1)} r_{t-1}^K + u_t^K
\end{aligned}
$$

$$
\underset{(3\times1)}{\mathbf{y}_t} = \begin{bmatrix} r_t^T \\ r_t^S \\ r_t^K \end{bmatrix} \quad
\underset{(3\times1)}{\mathbf{k}} = \begin{bmatrix} k^T \\ k^S \\ k^K \end{bmatrix} \quad
\underset{(3\times3)}{\mathbf{B_0}} = \begin{bmatrix} 1 & -\beta_{12}^{(0)} & -\beta_{13}^{(0)} \\ -\beta_{21}^{(0)} & 1 & -\beta_{23}^{(0)} \\ -\beta_{31}^{(0)} & -\beta_{32}^{(0)} & 1 \end{bmatrix} \quad
\underset{(3\times1)}{\mathbf{u}_t} = \begin{bmatrix} u_t^T \\ u_t^S \\ u_t^K \end{bmatrix}
$$

$$
\mathbf{B}_0 \mathbf{y}_t = \mathbf{k} + \mathbf{B}_1 \mathbf{y}_{t-1} + \mathbf{u}_t
$$

A structural VAR in a primitive form

$$\mathbf{B}_0\mathbf{y}_t = \mathbf{k} + \mathbf{B}_1\mathbf{y}_{t-1} + \mathbf{B}_2\mathbf{y}_{t-2} + \ldots + \mathbf{B}_p\mathbf{y}_{t-p} + \mathbf{u}_t$$

$$\begin{aligned} \mathbb{E}(\mathbf{u}_t\mathbf{u}_t') &= \mathbf{D} \\ \mathbb{E}(\mathbf{u}_t\mathbf{u}_{t-\tau}') &= \mathbf{0} \quad \text{for} \quad t \neq \tau \end{aligned}$$

with $\mathbf{D}$ a diagonal matrix, ensuring that the elements of $\mathbf{u}_t$ are mutually uncorrelated. The (j, j) element of $\mathbf{D}$ gives the variance of u_{jt}.

pth-order vector autoregression VAR(p) in standard form

$$\mathbf{y}_t = \mathbf{c} + \mathbf{\Phi}_1 \mathbf{y}_{t-1} + \mathbf{\Phi}_2 \mathbf{y}_{t-2} + \ldots + \mathbf{\Phi}_p \mathbf{y}_{t-p} + \boldsymbol{\varepsilon}_t$$

$\mathbf{c} = \mathbf{B}_0^{-1}\mathbf{k}$ $(n \times 1)$ vector of constants

$\mathbf{\Phi}_s = \mathbf{B}_0^{-1}\mathbf{B}_s$ $(n \times n)$ matrix of AR coefficients for $s = 1, ..., p$

$\boldsymbol{\varepsilon}_t = \mathbf{B}_0^{-1}\mathbf{u}_t$ $(n \times 1)$ vector generalization of white noise.

$$\begin{aligned} \mathbb{E}(\boldsymbol{\varepsilon}_t) &= \mathbf{0} \\ \mathbb{E}(\boldsymbol{\varepsilon}_t \boldsymbol{\varepsilon}_\tau') &= \begin{cases} \mathbf{\Omega} & \text{for} \quad t = \tau \\ \mathbf{0} & \text{otherwise.} \end{cases} \end{aligned}$$

since $\mathbb{E}(\boldsymbol{\varepsilon}_t \boldsymbol{\varepsilon}_{t-\tau}') = \mathbf{0}$ follows $\mathbb{E}(\mathbf{u}_t \mathbf{u}_{t-\tau}') = \mathbf{0}$ for $t \neq \tau$.

The VMA(∞) Representation

$$\mathbf{y}_t \ = \ \boldsymbol{\mu} + \boldsymbol{\varepsilon}_t + \boldsymbol{\Psi}_1 \boldsymbol{\varepsilon}_{t-1} + \boldsymbol{\Psi}_2 \boldsymbol{\varepsilon}_{t-2} + \boldsymbol{\Psi}_3 \boldsymbol{\varepsilon}_{t-3} + \ldots$$

The interpretation of $\boldsymbol{\Psi}_{\mathbf{s}}$ is

$$\frac{\partial \mathbf{y}_{t+s}}{\partial \boldsymbol{\varepsilon}_t'} = \boldsymbol{\Psi}_{\mathbf{s}}$$

i.e. row i, column j element of $\boldsymbol{\Psi}_s$ identifies the consequences of a one-unit increase in the jth variable's innovation at date t (ε_{jt}) for the value of the ith variable at time $(t+s)(\mathbf{y}_{i,t+s})$, holding all other innovations at all dates constant.

Impulse Responses using the MA coefficients. A plot of row i, column j element of $\mathbf{\Psi}_s$ as a function of s is called impulse response function: response of $y_{i,t+s}$ to a one-time impulse in $y_{j,t}$ with all other variables dated t or earlier held constant. $\frac{\partial y_{i,t+s}}{\partial \boldsymbol{\varepsilon}_{jt}}$