Advanced Financial Econometrics

Modules I - III

Topics in financial econometrics include

- **Estimation and testing of asset pricing models** Cochrane (2001)
  Campbell et al. 1997 Ch. 5 and 6

- **Modelling dynamics of financial market processes using statistical models**
  Tsay (2002), Brooks (2002)

- **Estimation of Value at Risk**
  Tsay (2002)

- **Estimation of Continuous Time Finance Models**
  Tsay (2002), Campbell et al. (1997) Ch. 11

- **Predictability of asset returns**
  Campbell et al. (1997) Ch. 2

- **Empirical Market Microstructure**
  Price formation processes in real markets
  Campbell et al. (1997) Ch. 3
  Bauwens and Giot (2001)
  Gourieroux and Jasiak (2001)

- **Event Studies**
  Measure effect of an economic event on value of firm
  Campbell et al. (1997) Ch. 4

What is financial econometrics?

Financial Economics

Deals with: valuation of assets, portfolio choice

Time dimension
(payoff in future)

Risk (payoff uncertain)

Economic agents: time preferences & risk aversion

proposes:

- Economic models explaining behaviour of asset prices/returns
- models contain unknown parameters
- models imply time series and cross sectional properties of asset prices/returns

Data

Prices/returns of financial assets
(stocks, bonds, options)

Time series

Cross section

other micro- and macro-economic data

statistical features of data (stylized facts)

Financial Econometrics

- Estimate unknown model parameters
- Test hypotheses about parameters
- Develop statistical models that account for stylized facts (more or less close link to theory)

Topics in financial econometrics include

- **Estimation and testing of asset pricing models** Cochrane (2001)
  Campbell et al. 1997 Ch. 5 and 6
- **Modelling dynamics of financial market processes using statistical models**
  Tsay (2002), Brooks (2002)
- **Estimation of Value at Risk**
  Tsay (2002)
- **Estimation of Continuous Time Finance Models**
  Tsay (2002), Campbell et al. (1997) Ch. 11
- **Predictability of asset returns**
  Campbell et al. (1997) Ch. 2
- **Empirical Market Microstructure**
  Price formation processes in real markets
  Campbell et al. (1997) Ch. 3
  Bauwens and Giot (2001)
  Gourieroux and Jasiak (2001)
- **Event Studies**
  Measure effect of an economic event on value of firm
  Campbell et al. (1997) Ch. 4

Textbooks on Financial Econometrics

- Gourieroux and Jasiak (2001) *Financial Econometrics*, Princeton University Press
  Covers rather specialized topics

- Tsay (2002) *Analysis of Financial Time Series*, Wiley
  Time series oriented, some specialities like VaR and continuous time finance and transaction data

- Brooks (2002) *Introductory Econometrics for finance*, Cambridge University Press
  Useful beginners econometrics book with many financial applications (of Brooks)

- Bauwens and Giot (2001) *Econometric Modelling of Stock Market Intraday Activity,* Kluwer
  Focusses on econometrics of high frequency data in finance. Specialized topics

- Cochrane (2001) *Asset Pricing*, Princeton (revised edition 2005)
  One of the best economics/finance textbooks and synopsis of the recent years. Theory and Econometrics

- Campbell, Lo, MacKinlay (1997) *The Econometrics of Financial Markets*, Princeton University Press
  The classic. Very broad topics, comprehensive chapter on event study methodology

- Boehmer, Broussard, Kallunki (2002) Using SAS in Financial Research, SAS Institute.
  Hands on financial econometrics, uses SAS with applications

- Hasbrouck (2004) forthcoming textbook on Econometrics of Market Microstructure. Download preview:
  `http://pages.stern.nyu.edu/~jhasbrou/Empirical%20Market%20Microstructure/Microstructure%20Notes%2002%20Full.pdf`
  Will close a gap in textbooks, reviews comprehensively accomplishments of past two decades

I. Empirical Asset Pricing

Readings:

Cochrane (2001), Ch. 1 (without 1.5), 3 (3.1 and 3.2), 4 (4.1 and 4.2), 7, 10, 11
Hamilton (1994), Ch. 14
Hayashi (2000), Ch. 3
Lettau and Ludvigson (2001)
Garcia, Renault and Semenov (2002)

Empirical asset pricing - Introduction (1)

Asset pricing (Valuation of financial assets)

delay of payoff } account for { risk of payoff

⇒ risk correction

50 years US stocks: 9% average return (real) p.a.
1% real interest rate p.a. (treasury bills)

8% premium earned for holding risk
What is the risk that is priced?

Asset pricing

normative

how should the world work?
are the prices "wrong"?
- trading opportunities?
- cost of capital
- non traded assets: "fair" price

positive

how does the world work?

Empirical asset pricing - Introduction (2)

Basic : Prices equal discounted $\underbrace{\text{expected}}_{\text{What probability measure?}}$ payoff

Absolute Asset Pricing

↓

exposure to "fundamental" macroeconomic risk

Asset priced given other asset prices (e.g. option pricing)

↑

Relative Asset Pricing

e.g. CAPM:

$$\mathbb{E}\left(R^i\right) = R^f + \beta_i \left(\underbrace{\mathbb{E}(R^m) - R^f}\right)$$

$$\beta_i = \frac{cov\left(R^i, R^m\right)}{var(R^m)}$$

↓

Market price of risk (factor)risk premium not explained

Empirical asset pricing - Introduction (3)

Basic pricing equation $p_t = \mathbb{E}_t(m_{t+1}x_{t+1})$

asset price at t

stochastic discount factor (r.v.)

payoff (r.v.)

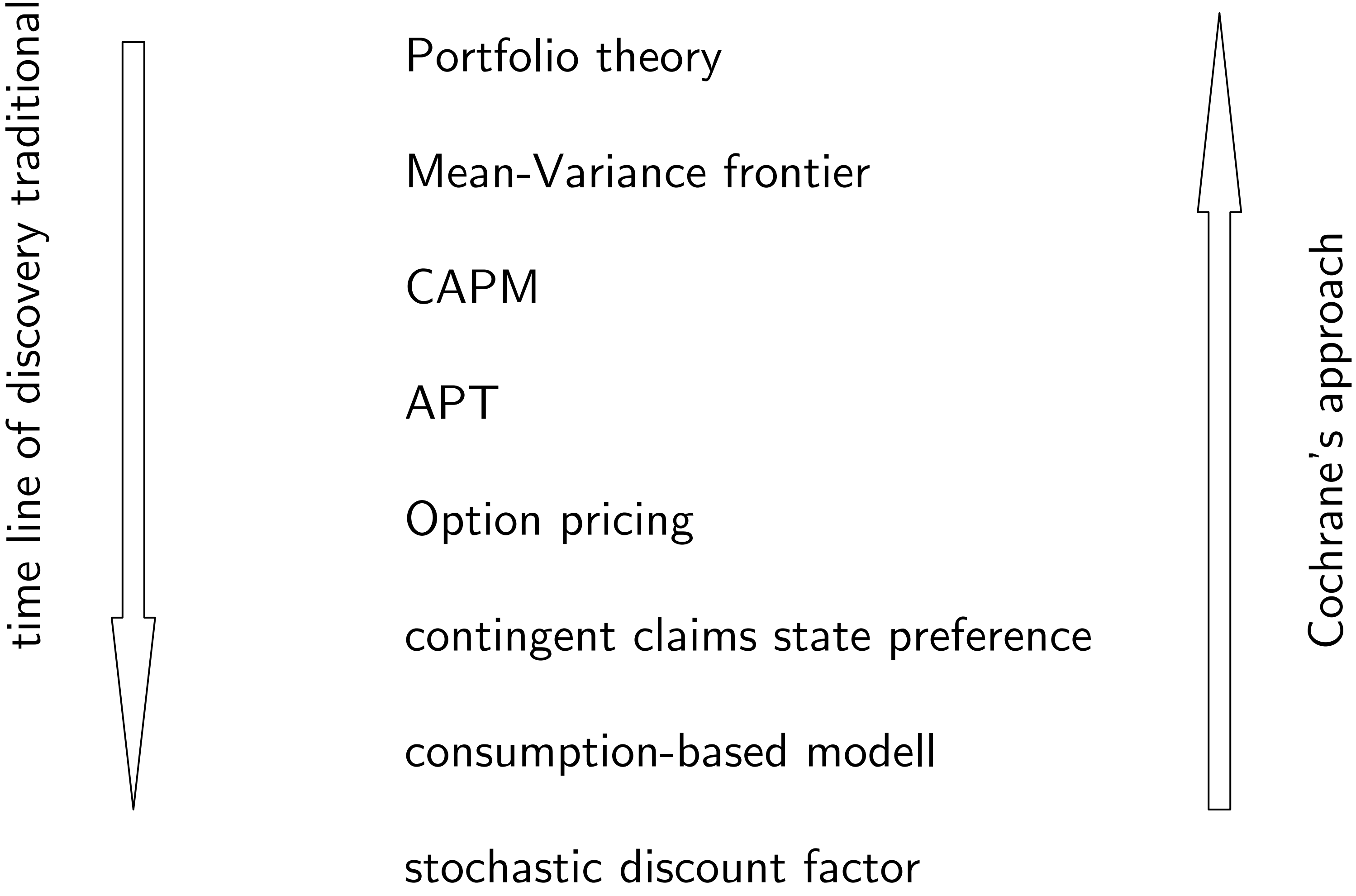
$$m_{t+1} = f(\underbrace{\text{data , parameters}}_{\text{the model}})$$

Moment condition: $\mathbb{E}_t(m_{t+1}x_{t+1}) - p_t \quad = 0$

use $\frac{1}{n}\sum \quad \rightarrow \mathbb{E}()$ WLLN

Generalized Method of Moments (GMM) to estimate parameters

Empirical asset pricing - Introduction (4)



time line of discovery traditional

Portfolio theory

Mean-Variance frontier

CAPM

APT

Option pricing

contingent claims state preference

consumption-based modell

stochastic discount factor

Cochrane's approach

From an utility maximising investor`s first order conditions we obtain the basic asset pricing formula (1)

Basic objective: find p_t, the present value of stream of uncertain payoff x_{t+1}

dividend

$$x_{t+1} = p_{t+1} + d_{t+1}$$

price of asset in t+1

Utility function

period utility function

$$U\left(c_t, c_{t+1}\right) = u\left(c_t\right) + \beta \mathbb{E}_t\left[u\left(c_{t+1}\right)\right]$$

expected utility

consumption

subjective discount factor

consumption level without asset purchase (other income)

$$\begin{aligned} c_t &= e_t - p_t \xi \\ c_{t+1} &= e_{t+1} + x_{t+1} \xi \end{aligned}$$

quantity of asset bought/sold

Random variables: $p_{t+1},\ d_{t+1},\ x_{t+1},\ e_{t+1},\ c_{t+1},\ u\left(c_{t+1}\right)$ $\qquad \mathbb{E}_t\left[\cdot\right] \triangleq \mathbb{E}\left[\cdot \mid \mathcal{F}_t\right]$

From an utility maximising investor`s first order conditions we obtain the basic asset pricing formula (2)

$$\max_{(\xi)} \left[U\left(c_t, c_{t+1}\right)\right] \text{ s.t.}$$

$$c_t = e_t - p_t\xi;\ c_{t+1} = e_{t+1} + x_{t+1}\xi$$

$$\max_{(\xi)} \left\{ u\left(e_t - p_t\xi\right) + \beta\mathbb{E}_t\left[u\left(e_{t+1} + x_{t+1}\xi\right)\right]\right\}$$

$$-p_t \cdot u'\left(c_t\right) + \beta \cdot \mathbb{E}_t\left[u'\left(c_{t+1}\right) \cdot x_{t+1}\right] = 0$$

utility loss if investor buys another unit of the asset

discounted expected utility increase from extra payoff

$$\begin{aligned} p_t u'\left(c_t\right) &= \mathbb{E}_t\left[\beta u'\left(c_{t+1}\right) x_{t+1}\right] \\ p_t &= \mathbb{E}_t\left[\beta \frac{u'\left(c_{t+1}\right)}{u'\left(c_t\right)} x_{t+1}\right] \end{aligned}$$

Investor continues to buy or sell the asset until marginal loss equals marginal gain.

No complete solution: endogenous variables

Turning off uncertainty we are in the standard two-goods case (1)

$$\max\left[u\left(c_t\right)+\beta u\left(c_{t+1}\right)\right] \text{ s.t. } c_t=e_t-p_t\cdot\xi,\, c_{t+1}=e_{t+1}+x_{t+1}\cdot\xi$$

$$\frac{\partial U\left(c_t,c_{t+1}\right)}{\partial\xi}=-p_t\cdot\frac{\partial u\left(c_t\right)}{\partial c_t}+\beta\cdot x_{t+1}\cdot\frac{\partial u\left(c_{t+1}\right)}{\partial c_{t+1}}=0$$

$$\begin{aligned}p_t\cdot u'\left(c_t\right) &= x_{t+1}\cdot\beta u'\left(c_{t+1}\right)\\ p_t &= x_{t+1}\cdot\frac{\beta u'\left(c_{t+1}\right)}{u'\left(c_t\right)}\end{aligned}$$

marginal valuation of consumption in t+1 in terms of consumption in t ⟶

$$\frac{dc_t}{dc_{t+1}}=\frac{\beta\cdot u'\left(c_{t+1}\right)}{u'\left(c_t\right)}=\frac{p_t}{x_{t+1}}$$

⟵ opportunity cost to transfer consumption from t to t+1

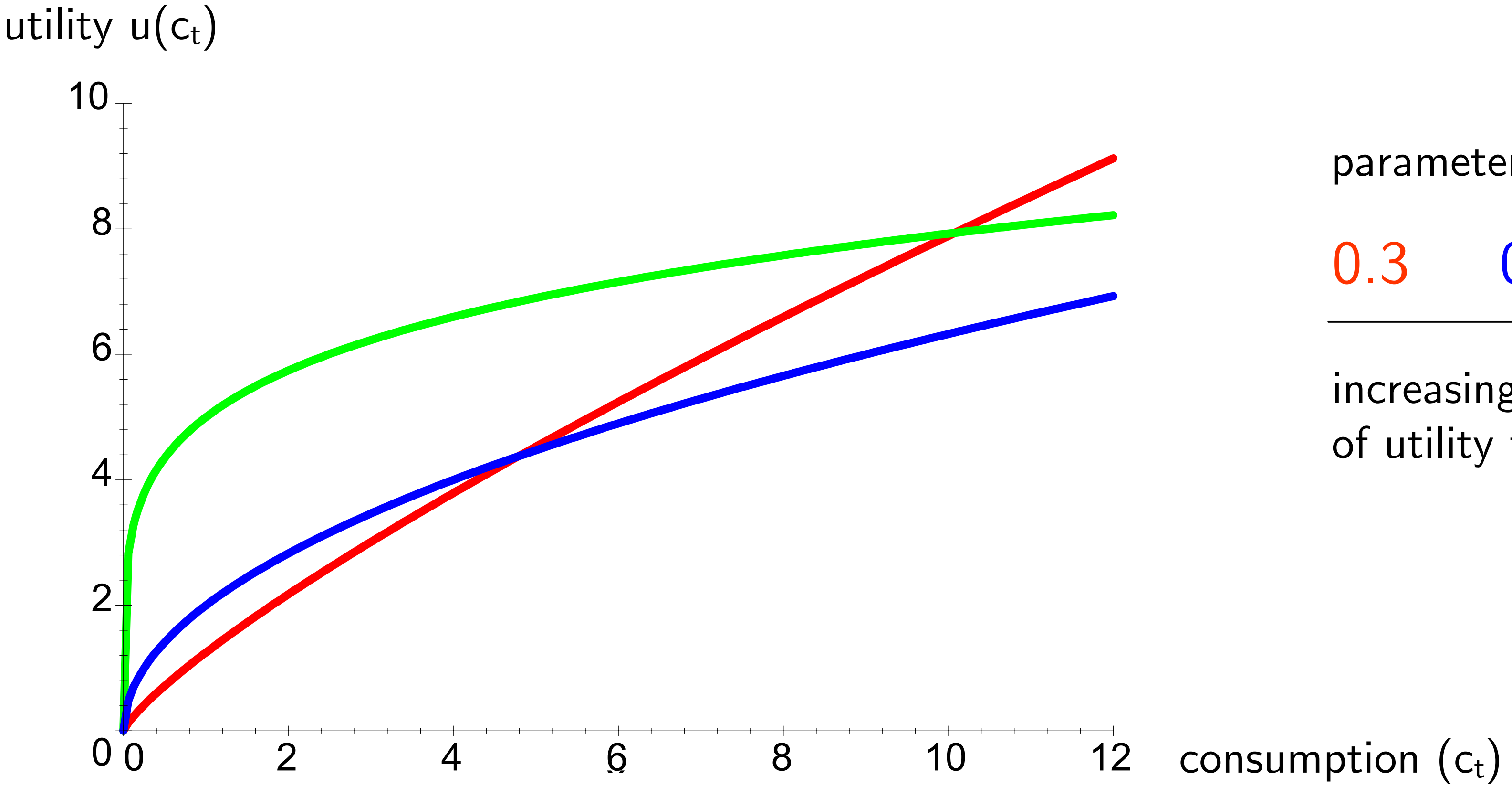
$$\begin{aligned}p_t u'\left(c_t\right) &= \mathbb{E}_t\left[\beta u'\left(c_{t+1}\right)x_{t+1}\right]\\ p_t &= \mathbb{E}_t\left[\beta\frac{u'\left(c_{t+1}\right)}{u'\left(c_t\right)}x_{t+1}\right]\end{aligned}$$

We often use a convenient power utility function (1)

$$u\left(c_t\right) = \frac{1}{1-\gamma} c_t^{1-\gamma} \qquad \lim_{\gamma\to1}\left(\frac{1}{1-\gamma} c_t^{1-\gamma}\right) = \ln\left(c_t\right)$$

$$u'\left(c_t\right) = c_t^{-\gamma} \qquad \frac{dc_t}{dc_{t+1}} = \frac{\beta u'\left(c_{t+1}\right)}{u'\left(c_t\right)} = \beta\left(\frac{c_{t+1}}{c_t}\right)^{-\gamma}$$

← marginal rate of substitution



parameter γ:

0.3 0.5 0.8

→

increasing concavity of utility function

Prices, payoffs, excess returns

	Price p_t	Payoff x_{t+1}
stock	p_t	$p_{t+1} + d_{t+1}$
return	1	R_{t+1}
excess return	0	$R^e_{t+1} = R^a_{t+1} - R^b_{t+1}$
one $ one period discount bond	p_t	1
risk-free rate	1	R^f

Payoff x_{t+1} divided by price $p_t \Rightarrow$ gross return $R_{t+1} = \frac{x_{t+1}}{p_t}$
Return: payoff with price one

$$1 = \mathbb{E}_t \left(m_{t+1} \cdot R_{t+1} \right)$$

Zero-cost portfolio:
Short selling one stock, investing proceeds in another stock
$\Rightarrow$excess return R^e

Example: Borrow $1\$$ at R^f, invest it in risky asset with return R.
Pay no money out of the pocket today → get payoff $R^e = R - R^f$.

Zero price does not imply zero payoff.

The *covariance* of the payoff with the discount factor rather than its *variance* determines the risk-adjustment

$$\begin{aligned} cov\left(m_{t+1}, x_{t+1}\right) &= \mathbb{E}\left(m_{t+1} \cdot x_{t+1}\right) - \mathbb{E}\left(m_{t+1}\right)\mathbb{E}\left(x_{t+1}\right) \\ p_t &= \mathbb{E}\left(m_{t+1} \cdot x_{t+1}\right) \\ &= \mathbb{E}\left(m_{t+1}\right)\mathbb{E}\left(x_{t+1}\right) + cov\left(m_{t+1}, x_{t+1}\right) \end{aligned}$$

$$\begin{aligned} R^f &= \frac{1}{\mathbb{E}\left(m_{t+1}\right)} \\ p_t &= \frac{\mathbb{E}\left(x_{t+1}\right)}{R^f} + cov\left(m_{t+1}, x_{t+1}\right) \end{aligned}$$

$$\begin{aligned} p_t &= \frac{\mathbb{E}\left(x_{t+1}\right)}{R^f} + cov\left(\beta\frac{u'\left(c_{t+1}\right)}{u'\left(c_t\right)}, x_{t+1}\right) \\ p_t &= \underbrace{\frac{\mathbb{E}\left(x_{t+1}\right)}{R^f}}_{\text{price in risk-neutral world}} + \underbrace{\beta\frac{cov\left(u'\left(c_{t+1}\right), x_{t+1}\right)}{u'\left(c_t\right)}}_{\text{risk adjustment}} \end{aligned}$$

Marginal utility declines as consumption rises.

Price is lowered if payoff covaries positively with consumption. (makes consumption stream more volatile)

Price is increased if payoff covaries negatively with consumption. (smoothens consumption) Insurance !

Investor does not care about volatility of an individual asset, if he can keep a steady consumption.

All assets have an expected return equal to the risk-free rate, plus risk adjustment

$$
\begin{aligned}
1 &= \mathbb{E}\left(m_{t+1} \cdot R^i_{t+1}\right) \\
1 &= \mathbb{E}\left(m_{t+1}\right)\mathbb{E}\left(R^i_{t+1}\right) + cov\left(m_{t+1}, R^i_{t+1}\right) \\
R^f &= \frac{1}{\mathbb{E}\left(m_{t+1}\right)};\ 1 - \frac{1}{R^f}\mathbb{E}\left(R^i_{t+1}\right) = cov\Big(m_{t+1}, R^i_{t+1} \\
\mathbb{E}\left(R^i_{t+1}\right) - R^f &= -R^f \cdot cov\left(m_{t+1}, R^i_{t+1}\right) \\
\mathbb{E}\left(R^i_{t+1}\right) - R^f &= -\frac{1}{\mathbb{E}\left(\beta\frac{u'(c_{t+1})}{u'(c_t)}\right)} \cdot cov\left(\beta\frac{u'\left(c_{t+1}\right)}{u'\left(c_t\right)}, R^i_{t+1}\right)
\end{aligned}
$$

$$
\overbrace{\mathbb{E}\left(R^i_{t+1}\right) - R^f}^{\text{excess return}} = -\frac{cov\left(u'\left(c_{t+1}\right), R^i_{t+1}\right)}{\mathbb{E}\left(u'\left(c_{t+1}\right)\right)}
$$

Investors demand higher excess returns for assets that covary positively with consumption.
Investors may accept expected returns below the risk-free rate. Insurance !

The basic pricing equation has an expected return-beta representation

$$\begin{aligned}\mathbb{E}\left(R^i_{t+1}\right) - R^f &= -R^f \cdot cov\left(R^i_{t+1}, m_{t+1}\right) \\ \mathbb{E}\left(R^i_{t+1}\right) - R^f &= -\frac{cov\left(R^i_{t+1}, m_{t+1}\right)}{Var\left(m_{t+1}\right)} \frac{Var\left(m_{t+1}\right)}{\mathbb{E}\left(m_{t+1}\right)}\end{aligned}$$

$$\mathbb{E}\left(R^i_{t+1}\right) = R^f - \left(\frac{cov\left(R^i_{t+1}, m_{t+1}\right)}{Var\left(m_{t+1}\right)}\right) \cdot \left(\frac{Var\left(m_{t+1}\right)}{\mathbb{E}\left(m_{t+1}\right)}\right)$$

asset specific quantity of risk

price of risk for all assets

Beta-pricing model:

$$\mathbb{E}\left(R^i\right) = R^f + \beta_{R^i,m} \cdot \lambda_m$$

With $m = \beta\left(\frac{c_{t+1}}{c_t}\right)^{-\gamma}$ and lognormal consumption growth $\frac{c_{t+1}}{c_t}$

$$\begin{aligned}\mathbb{E}\left(R^i\right) &= R^f + \beta_{R^i,\Delta c} \cdot \lambda_{\Delta c} \\ \lambda_{\Delta c} &\approx \gamma \cdot Var\left(\Delta \ln c\right)\end{aligned}$$

The more risk averse the investors or the riskier the environment, the larger the expected return premium for risky (high-beta) assets.

Marginal utility weighted prices follow martingales (1)

Basic first order condition:

$$p_t u'(c_t) = \mathbb{E}_t \left(\beta \left(u'(c_{t+1})\right) \overbrace{(p_{t+1} + d_t}^{x_{t+1}})\right)$$

Market efficiency ⇔ Prices follow martingales (random walks)? **NO!**

Required:

- Risk neutral investors u'( )=const. or no variation in consumption
- $\beta = 1$ ⇐ OK short time horizon
- no dividends

Then: $p_t = \mathbb{E}(p_{t+1})$

$$p_{t+1} = p_t + \varepsilon_{t+1}$$

if $\sigma^2(\varepsilon_{t+1}) = \sigma^2$ = Random Walk

⇒ Returns are not predictable $\mathbb{E}\left(\frac{p_{t+1}}{p_t}\right) = 1$

Marginal utility weighted prices follow martingales (2)

With risk aversion (but no dividends) and β=1

$$
\begin{aligned}
\tilde{p}_t &= \mathbb{E}(\tilde{p}_{t+1}) \\
\tilde{p}_t &= \tilde{p}_t \cdot u'(c_t)
\end{aligned}
$$

Scale prices by marginal utility, correct for dividends and apply risk neutral valuation formulas

Predictability in the short horizon?

$$\left.\begin{array}{l}\text{consumption}\\ \text{risk aversion}\end{array}\right\} \text{does not change day by day}$$

⇒ Random Walks successful ⇒ Predictability of asset returns (day by day)?

Technical analysis, media reports...

Some popular linear factor models

Factor pricing models

return on wealth portfolio

CAPM : $$\underbrace{m_{t+1} = \underset{\text{Free parameters}}{a + b}R^w_{t+1}}_{\text{Compatible with utility maximisation ?}}$$

ICAPM : $$\underbrace{m_{t+1} = a + \underset{\text{parameter vector}}{b'}\underset{\text{factors}}{f_{t+1}}}$$

factors (macro, term spread, price-earnings ratio help forecast conditional distribution of future asset returns)

APT : $$\underbrace{\text{similar,}}$$

but factors determined by principal component analysis of payoff covariance matrix

Practice : just test $m = b'f$ and don't worry about derivations

The basic pricing equation implies a set of CONDTIONAL moment restrictions

$$\begin{aligned} p_t &= \mathbb{E}_t\left(m_{t+1}x_{t+1}\right) \\ &= \mathbb{E}\left(m_{t+1}x_{t+1} \mid I_t\right) \end{aligned}$$

$\{m_t\}$ and
$\{x_t\}$ non i.i.d. $\Rightarrow$
$\mathbb{E}_t(\cdot) \neq \mathbb{E}(\cdot)$

Information set (partially) not observed,
conditional density not known, conditional expectation cannot be computed

Conditioning down to coarser
information set

$$\begin{aligned} p_t &= \mathbb{E}_t\left(m_{t+1}x_{t+1}\right) \\ \mathbb{E}(p_t) &= \mathbb{E}\left(\mathbb{E}_t\left(m_{t+1}x_{t+1}\right)\right) \\ &= \mathbb{E}\left(m_{t+1}x_{t+1}\right) \end{aligned}$$

I.i.e.

Estimation and evaluation of asset pricing models (Basics)

Models contain **free parameters**

$$p_t = \mathbb{E}_t \left(\beta \left(\frac{c_{t+1}}{c_t} \right)^{-\gamma} x_{t+1} \right)$$

- Estimation from data
- Testing hypotheses about parameters
- How good is the model?

Estimation and evaluation of asset pricing models (CBM)

$$p_t = \mathbb{E}_t(m_{t+1}\, x_{t+1}) \quad \text{or} \quad 1 = \mathbb{E}_t(m_{t+1}\, R_{t+1})$$

$$\uparrow f(data, parameters)$$

e.g. CBM with $u(c) = \frac{1}{1-\gamma}\, c^{1-\gamma} \Rightarrow m_{t+1} = \beta(\frac{c_{t+1}}{c_t})^{-\gamma}$

$\frac{c_{t+1}}{c_t}$: data (random variables)
$b = (\beta, \gamma)'$:free parameters

Assume model correct: "Best" choice for β, γ?
Best "fit", smallest (average) pricing errors

Estimation and evaluation of asset pricing models. The basic idea.

Estimates $\widehat{b}$ from data, distribution of $\widehat{b}$?

Average pricing errors:

$$\text{sample mean } \underbrace{\text{(observed price - predicted price)}}_{\text{should be close to zero}} = \alpha$$

$$p_t = \mathbb{E}_t\left(m_{t+1}(b) \cdot x_{t+1}\right) = \mathbb{E}\left(m_{t+1}(b) \cdot x_{t+1} | I_t\right)$$

$$\mathbb{E}(p_t) = \mathbb{E}[\mathbb{E}_t\left(m_{t+1}(b) \cdot x_{t+1}\right)] = \mathbb{E}[m_{t+1}(b) \cdot x_{t+1}]$$

Unconditional expectation: $\mathbb{E}[m_{t+1}(b)x_{t+1} - p_t] = 0$

Equivalently using returns:

$$1 = \mathbb{E}_t\left(m_{t+1}(b)R_{t+1}\right) \Rightarrow 0 = \mathbb{E}\left(m_{t+1}(b)R_{t+1} - 1\right)$$

Generalized Methods of Moments estimation is based on the WLLN

$$WLLN : \frac{1}{N}\sum_{i=1}^{N} y_i \quad \overrightarrow{p} \quad \mathbb{E}(Y)$$

sample average consistent estimate for population moment

$$\underbrace{\frac{1}{T}\sum_{t=1}^{T} p_t - \frac{1}{T}\sum_{i=1}^{T} m_{t+1}(b)x_{t+1}}_{\alpha} \approx 0$$

GMM basic idea(first step):
choose $\hat{b}$ to minimize α^2 (squared average pricing error) among set of test assets.

The two asset, two parameter case

$$\begin{aligned}
\mathbb{E}\left(m_{t+1}(\beta,\gamma)\, x^1_{t+1} - p^1_t\right) &= 0 \\
\mathbb{E}\left(m_{t+1}(\beta,\gamma)\, x^2_{t+1} - p^2_t\right) &= 0
\end{aligned}$$

$$\begin{aligned}
\mathbb{E}\left(m_{t+1}(\beta,\gamma)\, R^1_{t+1} - 1\right) &= 0 \\
\mathbb{E}\left(m_{t+1}(\beta,\gamma)\, R^2_{t+1} - 1\right) &= 0
\end{aligned}$$

$$\begin{aligned}
\frac{1}{T}\sum_{t=1}^{T} m_{t+1}(\beta,\gamma)\, R^1_{t+1} - 1 &= 0 \\
\frac{1}{T}\sum_{t=1}^{T} m_{t+1}(\beta,\gamma)\, R^2_{t+1} - 1 &= 0
\end{aligned}$$

solve equations for $\beta,\ \gamma \ \Rightarrow\ \widehat{\beta},\ \widehat{\gamma} \Rightarrow$

To apply GMM data have to be generated by stationary (and ergodic) processes (not necessarily i.i.d.)

Problem: WLLN works for **stationary data**:

(Weakly) stationary process: $\{Y_t\}_{t=-\infty}^{\infty}$

$\{\ldots, \mathrm{y}_0, y_1, \ldots, y_5, \ldots\}$

$\mathbb{E}(Y_t) = u$

$var(Y_t) = \sigma^2$

$cov(Y_t, Y_{t-j}) = \gamma_j$

Solution: $\Rightarrow$ We use:

$1 = \mathbb{E}\Big(m_{t+1}(b) \cdot R_{t+1}\Big)$ instead of $\mathbb{E}(p_t) = \mathbb{E}\Big(m_{t+1}(b) \cdot x_{t+1}\Big)$

$0 = \mathbb{E}\Big(m_{t+1}(b) \cdot R_{t+1} - 1\Big)$

We define the GMM residual or “pricing error“

Define GMM residual: object whose mean should be zero

$$u_{t+1}(b) = m_{t+1}(b)R_{t+1} - 1$$

$$\mathbb{E}(u_{t+1}(b)) = 0$$

$$\mathbb{E}_T[u_t(b)] = \frac{1}{T}\sum_{t=1}^{T} u_t(b) \approx 0$$

Notational convenience (Hansen’s notation, sometimes causing confusion)

$$\mathbb{E}_T(\cdot) = \frac{1}{T}\sum_{t=1}^{T}(\cdot)$$

We have more assets than unknown model parameters

For GMM parameter estimation: Select N test assets
${R_t}^1, {R_t}^2, \cdots, {R_t}^N \quad t = 1, \cdots, T$

$$\begin{bmatrix} \mathbb{E}_T[u_t^1(b)] \\ \mathbb{E}_T[u_t^2(b)] \\ \vdots \\ \vdots \\ \mathbb{E}_T[u_t^N(b)] \end{bmatrix} = g_T(b) \quad N \times 1 \quad \text{vector}$$

If $\sharp$ assets $=$ $\sharp$ parameters b can be chosen such that average pricing errors are zero usually $\sharp$ assets $>$ $\sharp$ parameters.

The GMM objective function

$$\widehat{b} = \underset{\{b\}}{\text{argmin}} \quad g_T'(b) \cdot I_N \cdot g_T(b) \qquad \text{first step } \textbf{GMM estimate}$$

$$= \underset{\{b\}}{\text{argmin}} \quad \Big[\mathbb{E}_T[u_{t+1}^1(b)]\Big]^2 + \Big[\mathbb{E}_T[u_{t+1}^2(b)]\Big]^2 + \ldots + \Big[\mathbb{E}_T[u_{t+1}^N(b)]\Big]^2$$

$\Rightarrow$ minimize sum of squared average (pricing)errors

equal weight for all test assets $1, \ldots, N$

Alternatively other weight matrix

$$\widehat{b} = \underset{\{b\}}{\text{argmin}} \quad g_T'(b)\; W\; g_T(b) \quad \text{e. g. } W = \begin{bmatrix} 1 & 0 & & \\ 0 & 2 & & \\ & & 100 & \ddots \\ 0 & & & \end{bmatrix}$$

GMM estimators have desirable properties

GMM estimators consistent:
Bias and variance of estimator go to zero asymptotically $\widehat{b} \underset{p}{\rightarrow} b$

GMM estimators asymptotically normal. Required for inference:

$$var(\widehat{b}) = \begin{pmatrix} var(\widehat{b}_1) & \cdots & \\ cov(\widehat{b}_1, \widehat{b}_2) & var(\widehat{b}_2) & \\ \vdots & \vdots & \\ cov(\widehat{b}_1, \widehat{b}_k) & \cdots & var(\widehat{b}_k) \end{pmatrix}$$

To conduct $t-$test: $\frac{\widehat{b}_k}{\widehat{\sigma} k} \overset{a}{\sim} N(0,1)$

There exists an optimal weighting matrix

Optimal weighting matrix
(and GMM parameter standard errors): use consistent estimate $\widehat{S}$ of S in minimization:

$$\widehat{b} = \underset{\{b\}}{argmin} \quad g_T(b)' \, \widehat{S}^{-1} \, g_T(b)$$

write $u_t(b) = \begin{pmatrix} u_t^1(b) \\ \vdots \\ u_t^N(b) \end{pmatrix} \qquad \Big(\underset{i=assets}{u_t^i(b) = m_{t+1}(b) x_{t+1}^i - p_t^i} \Big)$

Recall: $\mathbb{E}(u_t^i) = 0 \quad \Rightarrow \quad \mathbb{E}(u_t(b)) = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$

The optimal weighing matrix takes into account variances and covariances of pricing errors across assets

$$S = \mathbb{E}\left[u_t(b) \cdot u_t'(b)\right] = \begin{bmatrix} \mathbb{E}\left([u_t^1(b)]^2\right) \cdots & & \\ & \ddots & \\ \mathbb{E}\left[u_t^1(b) u_t^2(b)\right] & & \\ \vdots & & \mathbb{E}\left([u_t^N(b)]^2\right) \end{bmatrix}$$

$S=$ variance covariance matrix of pricing errors

$$= \begin{bmatrix} var\left(u_t^1(b)\right) \cdots & & \\ cov\left(u_t^1(b) u_t^2(b)\right) var\left(u_t^2(b)\right) & \ddots & \\ \vdots & & \\ & & var\left(u_t^N(b)\right) \end{bmatrix}$$

Estimate $\widehat{S}$: Replace $\mathbb{E}$ by $\frac{1}{N}\sum$ using $\widehat{b}$ obtained with weighting matrix $I_N \Rightarrow \widehat{S}$.

Steps of GMM estimation

1) $\widehat{b}^1 = \underset{\{b\}}{argmin} \quad g_T(b)' \, I_N \, g_T(b) \Rightarrow$

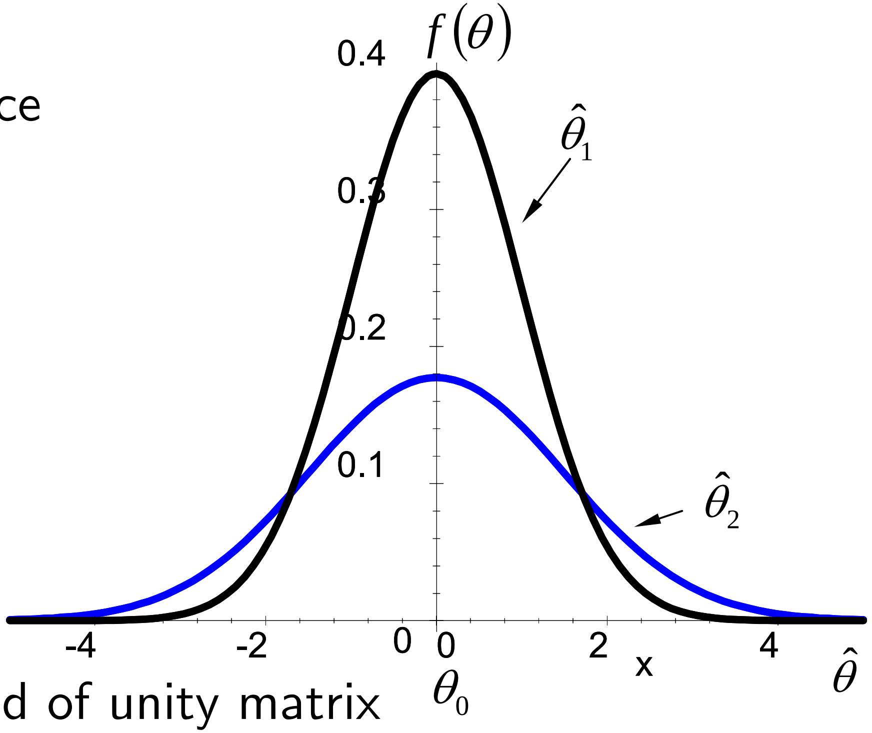
2) $\widehat{S} \Rightarrow$

3) $\widehat{b}^2 = \underset{\{b\}}{argmin} \quad \mathsf{g}_T(b)' \widehat{S}^{-1} \, g_T(b)$

...repeat... ...

Another look at the optimal weighting matrix

Efficiency: Smallest asymptotic variance
among GMM esimators



Efficient estimator: employ S^{-1} instead of unity matrix

$$S = \mathbb{E}\left[u_t(b) \cdot u_t'(b)\right] \text{ resp. } \underbrace{\sum_{j=-\infty}^{\infty} \mathbb{E}\left[u_t(b) \cdot u_{t-j}'(b)\right]}$$

variance-covariance
matrix of moments
conditions!

when no serial correlation
in moment conditions

with serial correlation
in moment conditions

Some intuition behind optimal weighting matrix (1)

Intuition behind GMM weighting matrix

Example

$N = 2$, $cov(u_t^1(b),\ u_t^2(b)) = 0$ [zero covariance of pricing errors]

$$S = \begin{bmatrix} var[u_t^1(b)] & 0 \\ 0 & var[u_t^2(b)] \end{bmatrix}$$

$$S^{-1} = \begin{bmatrix} \frac{1}{var[u_t^1(b)]} & 0 \\ 0 & \frac{1}{var[u_t^2(b)]} \end{bmatrix} = \begin{bmatrix} W_1 & 0 \\ 0 & W_2 \end{bmatrix}$$

Example $S = \begin{pmatrix} 10 & 0 \\ 0 & 0.1 \end{pmatrix}$

Some intuition behind optimal weighting matrix (2)

GMM objective $g_T(b)'S^{-1}\, g_T(b)$ becomes

$$\underset{\{b\}}{argmin}\, \mathbb{E}_T \left[u_t^1(b)\right]^2 \cdot W_1 + \mathbb{E}_T \left[u^2(b)\right]^2 \cdot W_2$$

Example

W_1 : 0.1 $\Rightarrow var\left(u_t^1(b)\right) = 10$

W_2 : 10 $\Rightarrow var\left(u_t^2(b)\right) = 0.1$

$\Rightarrow$ Asset (1) gets less weight in minimization

"Model imprecise" for asset 1, more precise for asset 2.

Some more intuition behind optimal weighting matrix: Correlations across pricing errors (1)

Another example: Correlations between asset returns: Two "similar" assets (high correlation of pricing errors) are downweighted. Count more like **one** asset.

Example $S = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0.999 \\ 0 & 0.999 & 1 \end{pmatrix}$ $\quad cov(u_t^2, u_t^3) = 0.999$

$$corr(u_t^2, u_t^3) \approx 1 = \frac{0.999}{\sqrt{1}\sqrt{1}}$$

$$\underset{\{b\}}{argmin} \left[\mathbb{E}_T(u_t^1(b)),\ \mathbb{E}_T(u_t^2(b)), \mathbb{E}_T(u_t^3(b)) \right] \times \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0.99 \\ 0 & 0.99 & 1 \end{bmatrix}^{-1} \times$$

$$\begin{bmatrix} \mathbb{E}_T(u_t^1(b)) \\ \mathbb{E}_T(u_t^2(b)) \\ \mathbb{E}_T(u_t^3(b)) \end{bmatrix}$$

Some more intuition behind optimal weighting matrix: Correlations across pricing errors (2)

$$S^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 500.25 & -499.75 \\ 0 & -499.75 & 500.25 \end{bmatrix}$$

$$\underset{\{b\}}{argmin}\; g_T(b)'\, S^{-1} g_T(b) =$$

$$\Big[\mathbb{E}_T\left(u_t^1(b)\right), \mathbb{E}_T\left(u_t^2(b)\right)\cdot 500.25 - \mathbb{E}_T\left(u_t^3(b)\right)\cdot 499.75,$$

$$\mathbb{E}_T\left(u_t^3(b)\right)\cdot 500.75 - \mathbb{E}_T\left(u_t^2(b)\right)\cdot 499.75\Big] \times \begin{bmatrix} \mathbb{E}_T(u_t^1(b)) \\ \mathbb{E}_T(u_t^2(b)) \\ \mathbb{E}_T(u_t^3(b)) \end{bmatrix}$$

Some more intuition behind optimal weighting matrix: Correlations of pricing errors (3)

$$\underset{\{b\}}{argmin}\; g_T(b)'\, S^{-1} g_T(b) =$$

$$\mathbb{E}_T\left(u_t^1(b)\right)^2 + \mathbb{E}_T\left(u_t^2(b)\right)^2 \cdot 500.25 + \mathbb{E}_T\left(u_t^3(b)\right)^2 \cdot 500.25 - 2 \cdot \mathbb{E}_T\left(u_t^2(b)\right)\mathbb{E}_T\left(u_t^3(b)\right) \cdot 499.75$$

$$\approx \mathbb{E}_T\left(u_t^1(b)\right)^2 + 0.5\,\mathbb{E}_T\left(u_t^2(b)\right)^2 + 0.5\,\mathbb{E}_T\left(u_t^3(b)\right)^2$$

since

$$\mathrm{E}_T\left(u_t^2(b)\right) \approx \mathbb{E}_T\left(u_t^3(b)\right)$$

To test hypotheses about our models we need the distribution of the GMM estimates

Standard errors of GMM estimates

We want:

$$var(\widehat{b}) = \begin{pmatrix} var(\widehat{b}_1) & cov(\widehat{b}_1, \widehat{b}_2) \cdots & cov(\widehat{b}_1, \widehat{b}_k) \\ cov(\widehat{b}_1, \widehat{b}_2) & var(\widehat{b}_2) & \\ & & \ddots \\ cov(\widehat{b}_1, \widehat{b}_k) & \cdots & var(\widehat{b}_k) \\ & & \end{pmatrix} (K \times K)$$

$$b = (b_0, b_1, \cdots, b_k)$$

$t = \frac{\widehat{b}_k - 0}{\sqrt{var(\widehat{b}_k)}} \overset{a}{\sim} N(0,1)$ under $H_0 :\ b_k = 0$

The central limit theorem plus an application of the delta method gives the asymptotic variance covariance matrix of estimated parameters

Application of Delta-Method

C.L.T. + delta method gives:

$$\sqrt{T} \cdot (\widehat{b} - b) \overset{a}{\backsim} N\Big(0, (d'S^{-1}d)^{-1}\Big)$$

$$\underbrace{\widehat{var}(\widehat{b})}_{\text{asymptotic VC matrix}} = \frac{1}{T}(d'S^{-1}d)^{-1} \quad d = \frac{\partial g_T(b)}{\partial b}\bigg|_{\widehat{b}}$$

(Note: asymptotic variances $T \rightarrow \infty$)

Some details of the asymptotic variance covariance matrix (1)

Some more details:

a) In application: replace S^{-1} by consistent estimate $\widehat{S}^{-1}$
b) Recall

$$g_T(b) = \begin{bmatrix} \frac{1}{T}\sum u_t^1(b) \\ \vdots \\ \frac{1}{T}\sum u_t^N(b) \end{bmatrix} = \begin{bmatrix} \frac{1}{T}\sum m_t(b)R_t^1 - 1 \\ \vdots \\ \frac{1}{T}\sum m_t(b)R_t^N - 1 \end{bmatrix}$$

$$\frac{\partial g_T(b)}{\partial b} = \underset{[N \times k]}{\begin{bmatrix} \frac{1}{T}\sum \frac{\partial u_t^1(b)}{\partial b_1} \; \frac{1}{T}\sum \frac{\partial u_t^1(b)}{\partial b_2} \cdots \frac{1}{T}\sum \frac{\partial u_t^1(b)}{\partial b_k} \\ \vdots \\ \frac{1}{T}\sum \frac{\partial u_t^N(b)}{\partial b_1} \; \frac{1}{T}\sum \frac{\partial u_t^N(b)}{\partial b_2} \cdots \frac{1}{T}\sum \frac{\partial u_t^N(b)}{\partial b_k} \end{bmatrix}}$$

Some details of the asymptotic variance covariance matrix (2)

$$\frac{\partial g_T(b)}{\partial b} = \left[\begin{array}{ll} \frac{1}{T}\sum_{t=1}^{T} & \frac{\partial m_t(b)}{\partial b_1} R_t, \frac{\cdots}{\partial b_2} \cdots \\ \downarrow & \longrightarrow \\ & \text{Parameters} \\ N & \end{array} \right]$$

For power utility

$$m_{t+1}(b) = \beta \left(\frac{c_{t+1}}{c_t}\right)^{-\gamma}$$

$b = \beta, \gamma$

Linear factor models $m_{t+1} = b' f_{t+1}$ $b \neq 0$?

Risk factor?

$$\frac{\partial m_{t+1}(b)}{\partial b_1} = ?$$

We employ the estimated variance covariance matrix to test hypotheses about the model

$var(\widehat{b})$ used for testing hypotheses:

$H_0 : \quad b_k = 0$

t-statistic: $\frac{\widehat{b}_k - 0}{\sqrt{var(\widehat{b}_k)}} \overset{a}{\sim} N(0,1) \mathrel{\widehat{=}}$ Standard t-test.

joint significance:

$$H_0 : \underbrace{(b_{j1} = b_{j2} = \quad = b_{jN}}_{\text{some subset of } b} = \ 0) \text{ or } \underset{J \times 1}{b_J} = 0$$

$$\widehat{b}_j' \left[\underbrace{var(\widehat{b})_J}_{\text{appropriate subset of } var(\widehat{b})} \right]^{-1} \widehat{b}_j \overset{a}{\sim} \chi^2(J) \Bigg] \mathrel{\widehat{=}} \text{Standard } F\text{-test}$$

One can test the validity of the model (the moment conditions) using the J-test

$\{R_t, \Delta c_t, \ldots\}$ data is a random sample $\Rightarrow \widehat{b}$ is a random variable $\Rightarrow$

$u_t(b)$ is a random variable $\Rightarrow \mathbb{E}_T\left(u_t(b)\right) = \frac{1}{N}\sum\cdots$ is a random variable

pricing errors too large to be explained by random sampling?

$\Leftrightarrow$ Is the model in correct?

is a random vector

$$T \cdot J_T = T \cdot \underbrace{\left[g_T(\widehat{b})'\, \widehat{S}^{-1}\, g_T(\widehat{b})\right]}_{\text{objective function at minimum}} \overset{a}{\sim} \chi^2\left(\begin{array}{l}\text{no. moment conditions}\\ \text{- no. of parameters.}\end{array}\right)$$

is a random variable, too

$\Rightarrow$ Reject or accept model (resp. moment conditions) at given specificance level

Example: no. of moment conditions: 10, no. parameters: 2,

$TJ_T = 7.9,$

Some important remarks

Inference is different if other weighting matrix than optimal weighting matrix is used

- different formula for parameter standard errors

- different formula for J-statistic

When comparing alternative models (e.g. parameter restrictions) use the same weighting matrix (weighting matrix depends on unknown parameters)

Performance comparison (1)

Problems using J-statistic

Popular measure

Compare observed average return with $\mathbb{E}(R)$ predicted by model

From

$$\begin{aligned} 1 = &\ \mathbb{E}(mR) \\ 1 = &\ \mathbb{E}(m)\mathbb{E}(R) + cov(m, R) \\ \mathbb{E}(R) = &\ \tfrac{1}{\mathbb{E}(m)} - \tfrac{cov(m,R)}{\mathbb{E}(m)} \end{aligned}$$

Use as predictor

$$\widehat{\mathbb{E}(R)} = \frac{1}{\frac{1}{T}\sum_{t=1}^{T} m_t} - \frac{\frac{1}{T}\sum_{t=1}^{T} m_t R_t - \frac{1}{T}\sum_{t=1}^{T} m_t \frac{1}{T}\sum_{t=1}^{T} R_t}{\frac{1}{T}\sum_{t=1}^{T} m_t}$$

Performance comparison (2)

Plot $\widehat{\mathbb{E}(R)}$ vs. $\frac{1}{T}\sum_{t=1}^{T} R_t = \bar{R}$

Similarly using excess returns as test assets

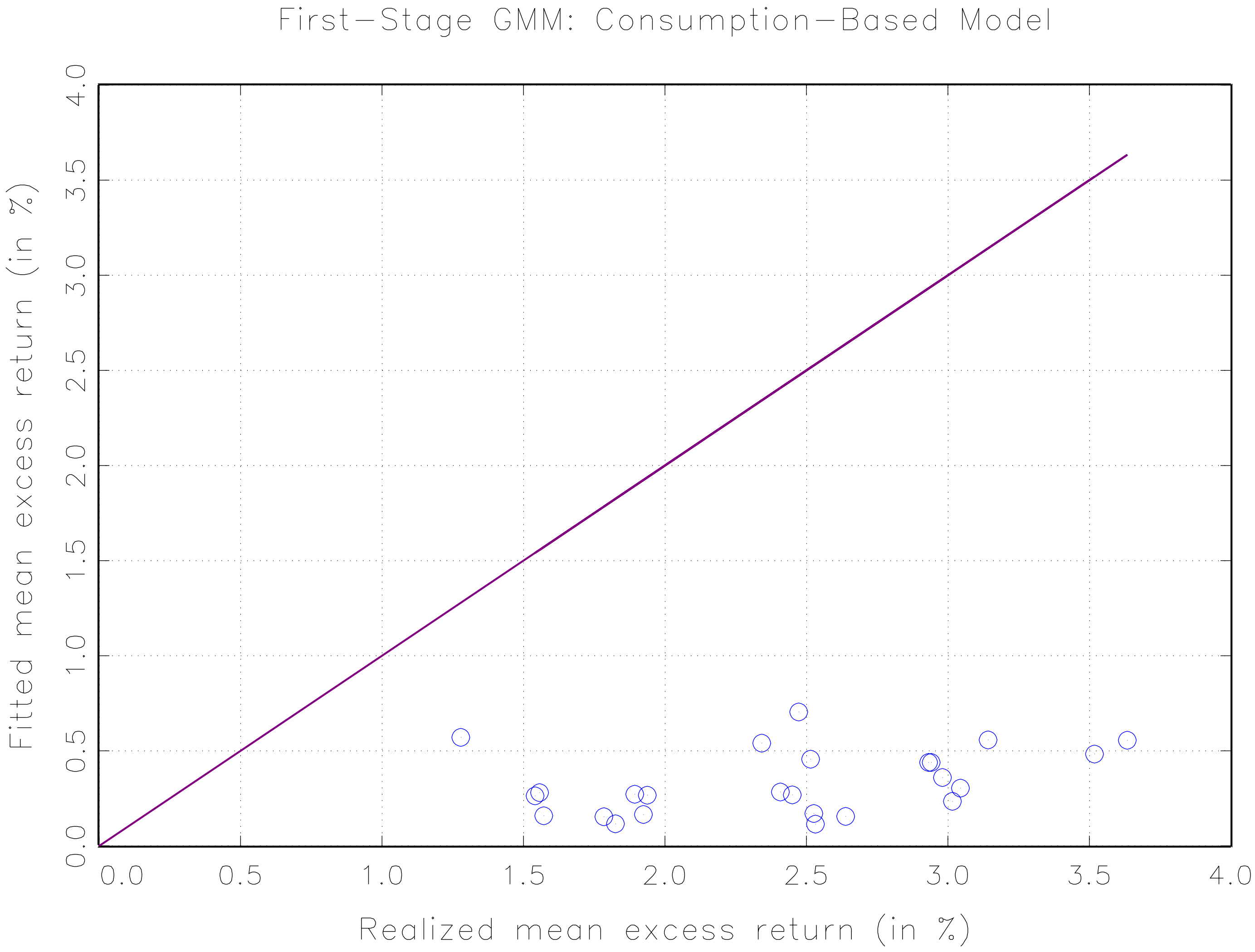
From
$$
\begin{aligned}
0 &= \mathbb{E}(mR^e) \\
0 &= \mathbb{E}(m)\mathbb{E}(R^e) + cov(m, R^e) \\
\mathbb{E}(R^e) &= -\frac{cov(m,R^e)}{\mathbb{E}(m)}
\end{aligned}
$$

Again: replace $\mathbb{E}(\cdot)$ by $\frac{1}{T}\sum(\cdot)$ to obtain $\widehat{\mathbb{E}(R^e)}$

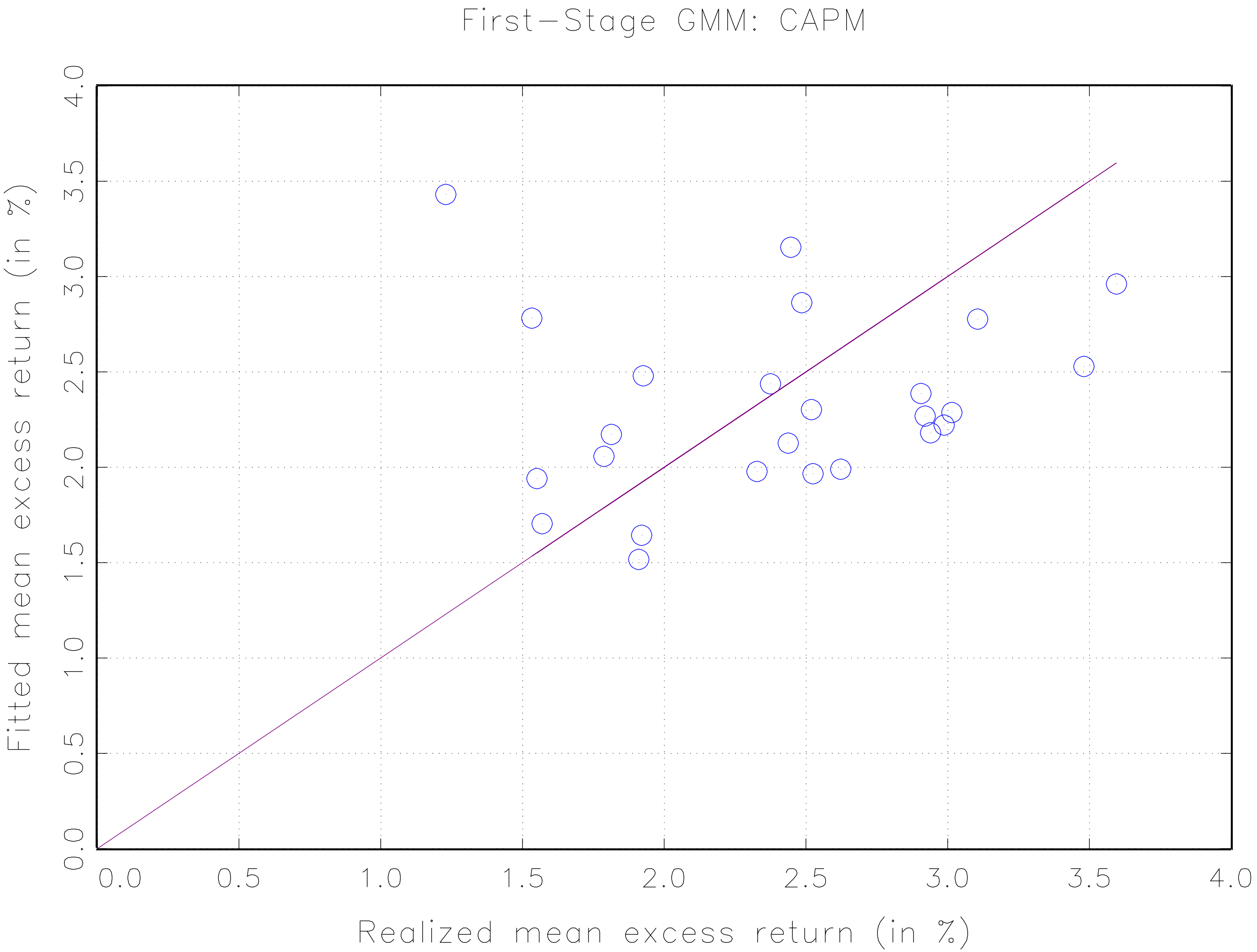
Plot $\widehat{\mathbb{E}(R^e)}$ against $\bar{R}^e$

RMSE $= \sqrt{\sum_{j=1}^{N}\left[\widehat{\mathbb{E}(R^j)} - \bar{R}^j\right]^2}$ or $= \sqrt{\sum_{j=1}^{N}\left[\widehat{\mathbb{E}(R^{ej})} - \bar{R}^{ej}\right]^2}$ used to rank and compare alternative models

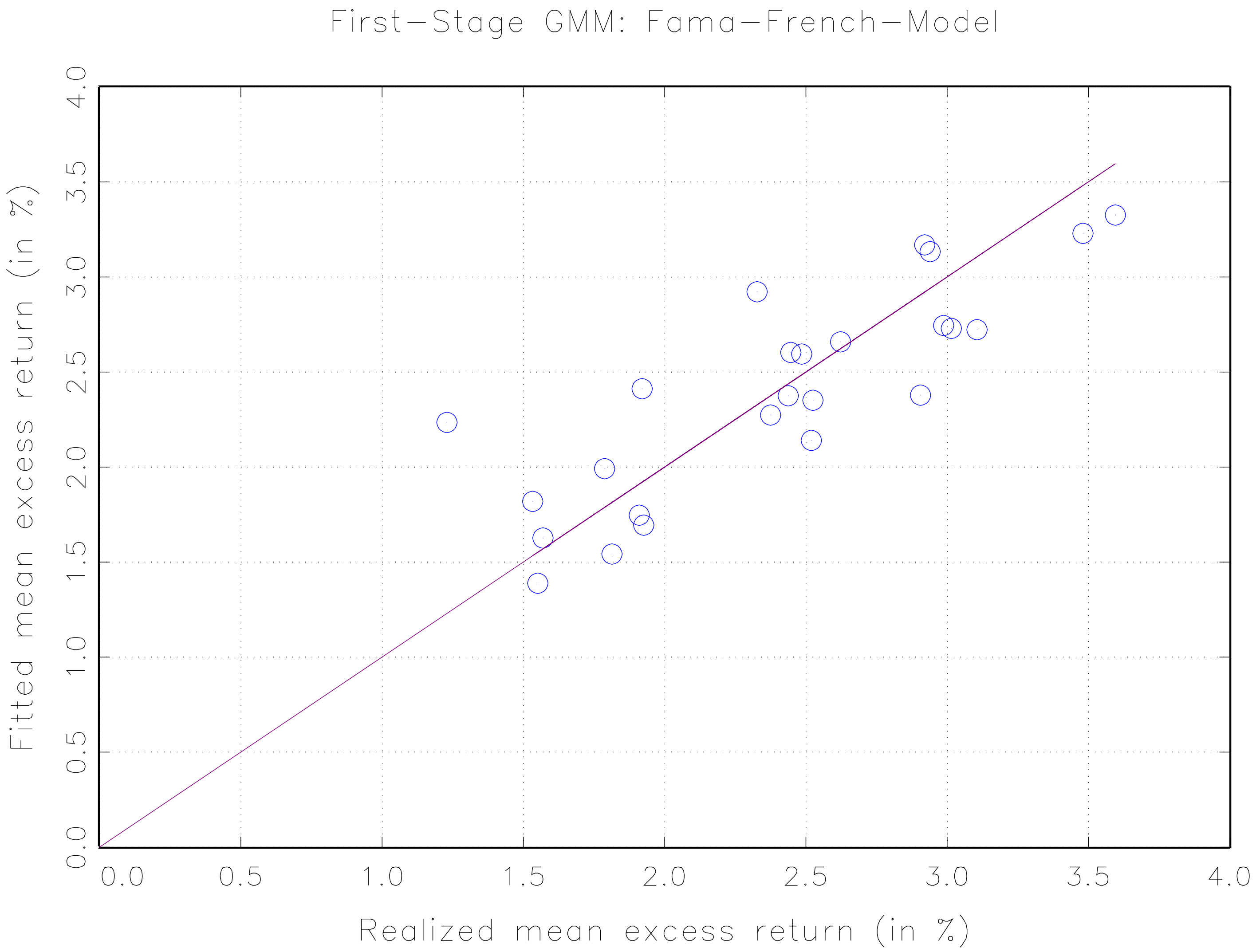
Performance comparison. Example: Consumption-Based Model estimated on 25 Fama-French portfolios



Performance comparison. Example: CAPM estimated on 25 Fama-French portfolios



Performance comparison. Example: Fama-French two factor model estimated on 25 Fama-French portfolios

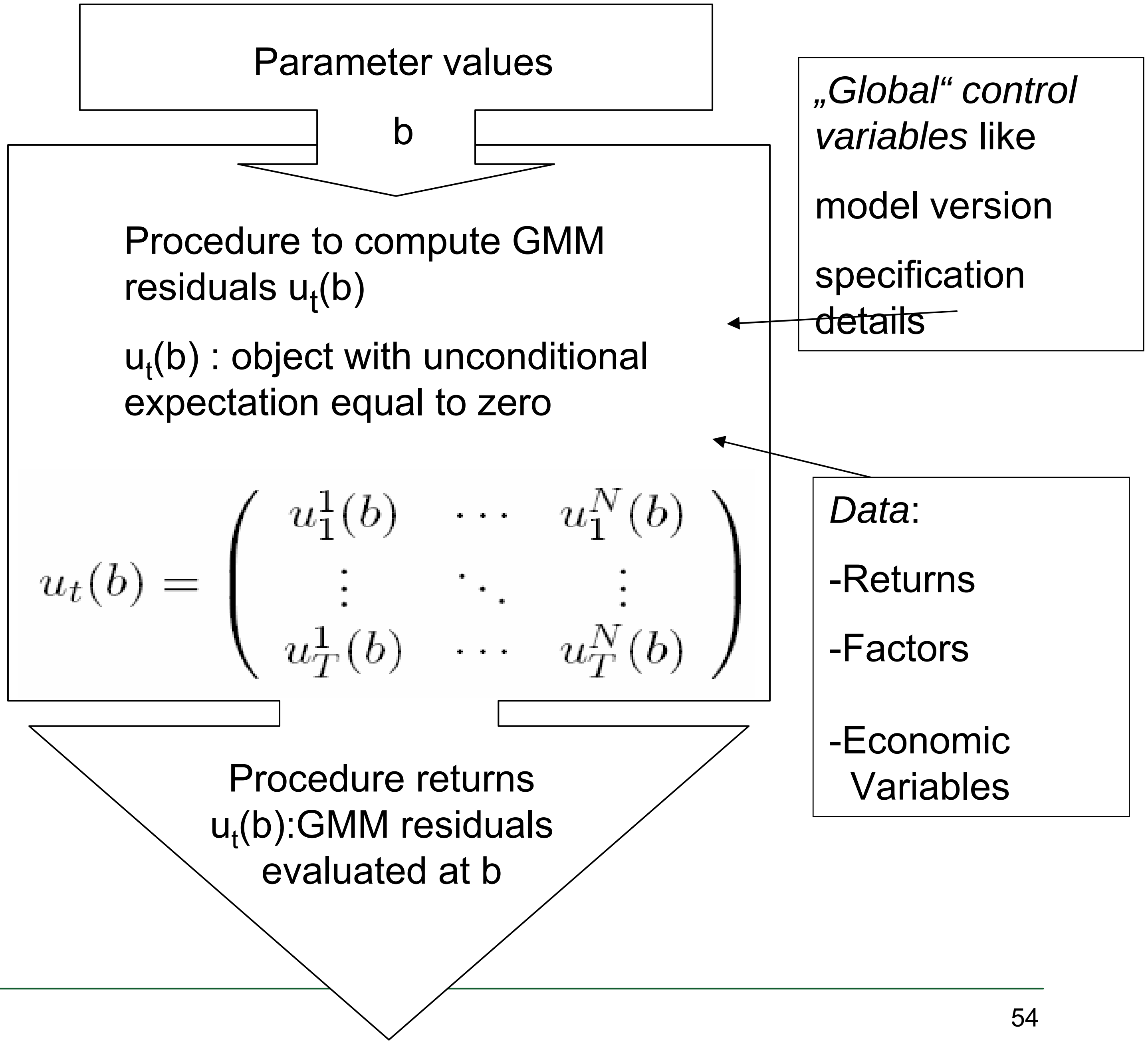


GMM estimation using the Gauss library: Ingredients and recipe

1. Supply data
2. Provide GMM/optimization settings (number of iterations, weighting matrix)
3. Supply initial parameter values
4. Call GMM minimization procedure

   iteratively calls procedure to compute GMM residuals $u_t(b)$

5. Check parameter estimates and test statistics



Parameter values

b

Procedure to compute GMM residuals $u_t(b)$

$u_t(b)$: object with unconditional expectation equal to zero

$$u_t(b) = \begin{pmatrix} u_1^1(b) & \cdots & u_1^N(b) \\ \vdots & \ddots & \vdots \\ u_T^1(b) & \cdots & u_T^N(b) \end{pmatrix}$$

Procedure returns $u_t(b)$:GMM residuals evaluated at b

„Global“ control variables like

model version

specification details

Data:

-Returns

-Factors

-Economic Variables

The canoncical example: Estimate the CBM by GMM

For consumption based model with power utility

$$\mathbb{E}_T(u_t(b)) = \frac{1}{T}\sum_{t=1}^{T} \beta \left(\frac{c_{t+1}}{c_t}\right)^{\gamma} \cdot R_t^i - 1 = 0$$

Exercise: 10 test assets (NYSR decile portfolios)
Perform GMM estimation of γ and β using EXCEL solver.

Input: Time series of returns and consumption growth.

$$\begin{bmatrix} R_1^1 & \cdots & R_1^{10} & R_1^f & dc_1 \\ \vdots & & \vdots & & \vdots \\ R_T^1 & & R_1^{10} & R_1^f & dc_T \end{bmatrix}$$

Newer models consumption based model and habit formation

Garcia et al. (2003)

Period utility function

habit level (external)

$$u(c_t/H_t, H_t) = \frac{\left(\frac{c_t}{H_t}\right)^{1-\gamma} H_t^{1-\psi} - 1}{1-\gamma}$$

Marginal utility

$$u'(c_t) = c_t^{-\gamma} H_t^{\gamma-\psi}$$

Stochastic discount factor

$$m_{t+1} = \delta \left(\frac{c_{t+1}}{c_t}\right)^{-\gamma} \left(\frac{H_{t+1}}{H_t}\right)^{\gamma-\psi}$$

$$\mathbb{E}_t \left[\delta \left(\frac{c_{t+1}}{c_t}\right)^{-\gamma} \left(\frac{H_{t+1}}{H_t}\right)^{\gamma-\psi} R^i_{t+1}\right] = 1$$

Modelling the habit level (1)

$$\begin{aligned}
H_{t+1} &= \mathbb{E}(c_{t+1}|c_t, c_{t-1}, \ldots) \\
\Delta H_{t+1} &= \lambda(c_t - H_t) \qquad 0 \leq \lambda \leq 1 \\
H_{t+1} &= a + \lambda c_t + (1-\lambda) H_t \\
H_{t+1} &= \frac{a}{\lambda} + \lambda \sum_{i=0}^{\infty} (1-\lambda)^i c_{t-i}
\end{aligned}$$

using

$$\begin{aligned}
c_{t+1} &= \frac{a}{\lambda} + \lambda \sum_{i=0}^{\infty} (1-\lambda)^i c_{t-i} + \varepsilon_{t+1} \\
c_{t+1} &= \frac{a}{\lambda} + \lambda c_t + \lambda(1-\lambda) c_{t-1} + \lambda(1-\lambda)^2 c_{t-2} + \ldots + \varepsilon_{t+1} \\
(1-\lambda) c_t &= \frac{a}{\lambda}(1-\lambda) + \lambda(1-\lambda) c_{t-1} + \ldots + (1-\lambda)\varepsilon_t
\end{aligned}$$

Modelling the habit level (2)

Subtracting two previous equations

$$c_{t+1} - (1-\lambda)c_t = a + \lambda c_t + \ldots + \varepsilon_{t+1} - (1-\lambda)\varepsilon_t$$

$$\Delta c_{t+1} = a - (1-\lambda)\varepsilon_t + \varepsilon_{t+1}$$

ARIMA(0,1,1) model - Estimation by Maximum Likelihood

Use parameter estimates of a and λ to iterate on

$$H_{t+1} = a + \lambda c_t + (1-\lambda)H_t.$$

to estimate habit level

Plug in GMM objective function

An alternative model for the habit process (1)

Log habit growth (unobservable)

$$
\begin{aligned}
\Delta h_{t+1} &= \ln(H_{t+1}) - ln(H_t) \\
\Delta h_{t+1} &= a_0 + \sum_{i=1}^{n} a_i \cdot \Delta \ln c_{t+1-i} + b \cdot \overset{\substack{\text{log return market portfolio} \\ \downarrow}}{r^m_{t+1}}
\end{aligned}
$$

with

$$
\begin{aligned}
\Delta h_{t+1} &= \mathbb{E}(\Delta \ln c_{t+1} | \Delta \ln c_t, \Delta \ln c_{t-1}, \ldots) \\
\Delta \ln c_{t+1} &= a_0 + \sum_{i=1}^{n} a_i \cdot \Delta \ln c_{t+1-i} + b \cdot r^m_{t+1} + \overset{\substack{\text{orhogonal forecast error} \\ \downarrow}}{\varepsilon_{t+1}}
\end{aligned}
$$

$a_0, a_1, \ldots, b$ can be estimated by GMM additional moment restrictions

An alternative model for the habit process (2)

Estimation

Add to usual moment conditions additional moment restrictions from habit equation:

$$
\begin{aligned}
\text{use} \qquad & \mathbb{E}(m_{t+1}R^{i}_{t+1} - 1) = 0 \\
& \vdots \\
& \mathbb{E}(m_{t+1}R^{N}_{t+1} - 1) = 0 \\
\text{along with} \qquad & \mathbb{E}(\varepsilon_{t+1}m_{t+1}) = 0 \\
& \mathbb{E}(\varepsilon_{t+1}\Delta \ln c_t) = 0 \\
& \vdots
\end{aligned}
$$

An alternative model for the habit process (3)

Habit growth is then

$$\frac{H_{t+1}}{H_t} = \underset{\substack{\nearrow \\ \exp(a_0)}}{A} \prod_{i=0}^{n} \left[\frac{c_{t+1-i}}{c_{t-i}}\right]^{a_i} \left(R^m_{t+1}\right)^b$$

Stochastic discount factor

$$m_{t+1} = \delta A^{\gamma-\psi} \left[\frac{c_{t+1}}{c_t}\right]^{-\gamma} \prod_{i=0}^{n} \left[\frac{c_{t+1-i}}{c_{t-i}}\right]^{a_i(\gamma-\psi)} \left(R^m_{t+1}\right)^{b(\gamma-\psi)}$$

Used for estimation

$$m_{t+1} = \delta^* \left[\frac{c_{t+1}}{c_t}\right]^{-\gamma} \prod_{i=0}^{n} \left[\frac{c_{t+1-i}}{c_{t-i}}\right]^{\frac{a_i \cdot \kappa}{b}} \left(R^m_{t+1}\right)^{\kappa}$$

We estimate using

$n = 0$ ”Epstein-Zin SDF”

$n = 1$

Some more models (1)

- Linearized consumption based model

  $m_{t+1} = b_0 + b_{\Delta c} \Delta \ln c_{t+1}$

  Taylor approximation of $\frac{u'(c_{t+1})}{u'(c_t)}$

- CAPM

  $m_{t+1} = b_0 + b_m R^m_{t+1}$

- Scaled CAPM by Lettau and Ludvigson (2001)

  $m_{t+1} = b_0 + b_{cay} cay_t + b_m R^m_{t+1} + b_{caym} cay_t R^m_{t+1}$

Is a conditional asset pricing model testable at all?

Most asset pricing models imply **conditional** moment restrictions

$$1 = \mathbb{E}\left(m_{t+1}(b_t) \cdot R_{t+1} | I_t\right)$$

e.g. CAPM $m_{t+1} = a_t - b_t R^W_{t+1}$.

Parameters of factor pricing model vary over time.
⇒unconditioning via l.i.e. no longer possible:

$$1 = \mathbb{E}\left(m_{t+1}(b_t) \cdot R_{t+1} | I_t\right)$$

does NOT imply

$$1 = \mathbb{E}\left(m_{t+1}(b) \cdot R_{t+1}\right)$$

this is not repaired by using scaled returns. GMM estimation no possible.

Hansen and Richard critique: CAPM (or other factor model) is not testable.

Scaled factors are a partial solution to the problem

With linear factor model

$$m_{t+1} = b_t' \underbrace{f_{t+1}}_{K \times 1}$$

use of "scaled factors" a partial solution:

"Blow up" number of factors by scaling factors with $(M \times 1)$ instruments vector z_t observable at t

$$m_{t+1} = b' \underbrace{(f_{t+1} \otimes z_t)}_{KM \times 1}$$

Unconditioning via l.i.e. and GMM procedure as above.

Time varying parameters lead to scaled factors (single factor case)

Motivation

Consider linear one factor model $m_{t+1} = a_t + b_t f_{t+1}$ (f_{t+1} scalar)
Assume Parameters vary with $M \times 1$ instruments vector z_t.

$$m_{t+1} = a(z_t) + b(z_t) f_{t+1}$$

With linear functions

$$a(z_t) = a' z_t \quad \text{and} \quad b(z_t) = b' z_t$$

$$\Rightarrow m_{t+1} = a' z_t + (b' z_t) f_{t+1}$$

Mathematically equivalent to

$$m_{t+1} = \tilde{b}' (\tilde{f}_{t+1} \otimes z_t)$$

where $\tilde{b} = \begin{bmatrix} a \\ b \end{bmatrix}$, $\tilde{f}_{t+1} = \begin{bmatrix} 1 \\ f_{t+1} \end{bmatrix}$
Number of parameters to estimate $2 \cdot M$

Time varying parameters lead to scaled factors (multi factor case)

Multi-factor case:

$$m_{t+1} = b_t' \underbrace{f_{t+1}}_{K\times 1}$$

Again: Time varying parameters linear functions of $M \times 1$ vector of observables z_t.

$$m_{t+1} = b(z_t)' f_{t+1} \quad \text{with} \quad b(z_t) = \underbrace{B}_{K\times M} z_t$$

Equivalent to $m_{t+1} = \tilde{b}' \underbrace{(f_{t+1} \otimes z_t)}_{K\times N}$ where $\tilde{b} = vec(B)$

In practical application some elements of B may be set to zero.

Using scaled factors we can condition down and apply GMM

Conditioning down and GMM estimation possible

$$\mathbb{E}_t\left(\underbrace{\left(\tilde{b}'(f_{t+1}\otimes z_t)\right)}_{m_{t+1}} R_{t+1}\right) = 1 \quad \text{l.i.e.} \Rightarrow \quad \underbrace{\mathbb{E}\left(\left(\tilde{b}'(f_{t+1}\otimes z_t)\right) R_{t+1} - 1\right) = 0}_{\text{unconditional moment restrictions}}$$

Scaled factors and managed portfolios can be combined.
(z_t might be the same).

$$\Rightarrow \mathbb{E}(\tilde{b}'(f_{t+1}\otimes z_t)R_{t+1} - 1\,]\otimes z_t) = 0$$

- Inclusion of conditioning information as managed portfolios (scaled returns, increases number of test assets.

- Scaled factors increase number of unknown parameters

Cochranes (1996) CAPM with scaled factors

$$f = \begin{pmatrix} 1 \\ R^W \end{pmatrix} z_t = \begin{pmatrix} 1 \\ \frac{P}{D} \\ term \end{pmatrix} B = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \end{bmatrix}$$

$$f \otimes z = \begin{pmatrix} 1 \\ R^W \\ \frac{P}{D} \\ R^W \cdot \frac{P}{D} \\ term \\ R^W \cdot term \end{pmatrix} \tilde{b} = \left(b_{11}, b_{21}, b_{12}, b_{22}, b_{13}, b_{23} \right)'$$

$$m = \tilde{b}'(f \otimes z) = b_{11} + b_{12}\frac{P}{D} + b_{13} term + b_{21} R^W + b_{22} R^W \cdot \frac{P}{D} + b_{23} R^W \cdot term$$

In application Cochrane (1996) restricts b_{12} and b_{13} to zero

Some more models (2)

- Scaled CBM by Lettau and Ludvigson (2001)

$$m_{t+1} = b_0 + b_{cay} cay_t + b_{\Delta c} \Delta \ln c_{t+1} + b_{cay \Delta c} cay_t \Delta \ln c_{t+1}$$

- Fama French model

$$m_{t+1} = b_0 + b_m R^{em}_{t+1} + b_{SMB} SMB_{t+1} + b_{HML} HML_{t+1}$$

Model comparison (practical exercise)

- 10 decile portfolios and t-bill rate (Cochrane 1996)
- 25 size/book-to-market portfolios and t-bill rate
- Excess returns or gross returns as test assests
- Estimation using GMM (alternatives ⇒ course 1)
- J-test
- RMSE comparisons (plots)

Models:

$*$ Consumption Based Model (CBM), CAPM, Scaled (LL) CBM, Scaled (LL) CAPM, various habit model variants

II. Econometrics of Financial Market Microstructure (I)

References:

- Boehmer (2004)
- Glosten and Harris (1988)
- Harris (2003)
- Hasbrouck (2004)
- Henker and Wang (2005)
- Huang and Stoll (1997)
- Madhavan, Richardson, Roomans (1997)
- SEC (2001)

II.1 Important Empirical Concepts

Basic concepts of (empirical) financial market microstructure (1)

(best) *ask price* (or *offer price*)	*depth* at best ask price
(best) *bid price*	*depth* at best bid price

→ *best quotes*

Inside spread or *quoted spread*: ask price – bid price

Spread: natural measure of liquidity and market quality and (implicit) transaction costs (cost of “round trip“)

midprice or *midquote* or *midpoint*
(ask price + bid price)

relative (quoted) spread: $\left(\frac{\text{quoted spread}}{\text{midquote}}\right) \cdot (100\%)$

Basic concepts of (empirical) financial market microstructure (2)

Trades occur at the ask or bid

→transaction price = bid price or ask price

or inside the quoted spread

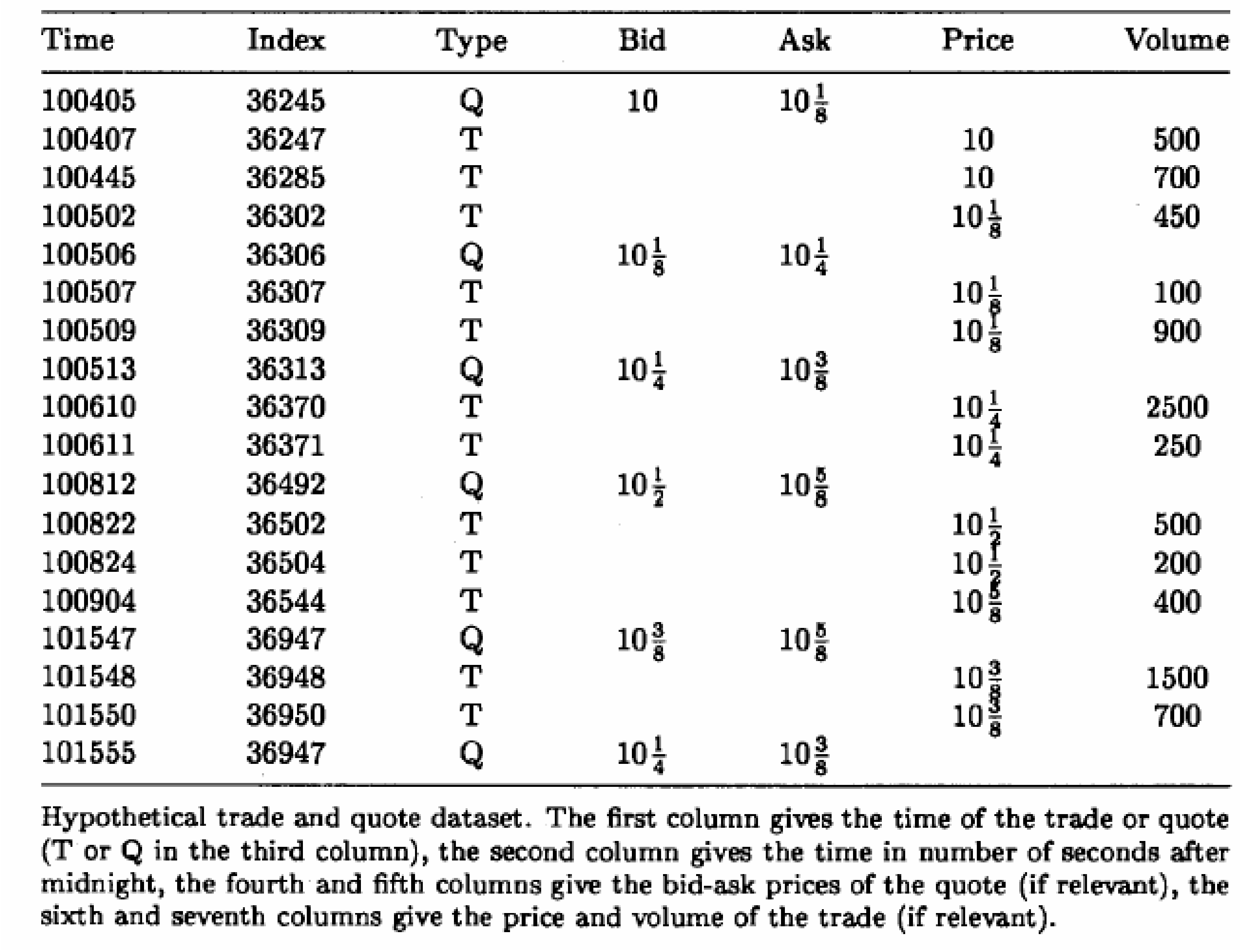
→ transaction price of buyer initiated trade < ask price

respectively

transaction price of seller initiated trade > bid price

or outside the quoted spread (if trading volume exceeds depth)

A sequence of quote changes and trade events (transactions)



Time	Index	Type	Bid	Ask	Price	Volume
100405	36245	Q	10	$10\frac{1}{8}$		
100407	36247	T			10	500
100445	36285	T			10	700
100502	36302	T			$10\frac{1}{8}$	450
100506	36306	Q	$10\frac{1}{8}$	$10\frac{1}{4}$		
100507	36307	T			$10\frac{1}{8}$	100
100509	36309	T			$10\frac{1}{8}$	900
100513	36313	Q	$10\frac{1}{4}$	$10\frac{3}{8}$		
100610	36370	T			$10\frac{1}{4}$	2500
100611	36371	T			$10\frac{1}{4}$	250
100812	36492	Q	$10\frac{1}{2}$	$10\frac{5}{8}$		
100822	36502	T			$10\frac{1}{2}$	500
100824	36504	T			$10\frac{1}{2}$	200
100904	36544	T			$10\frac{5}{8}$	400
101547	36947	Q	$10\frac{3}{8}$	$10\frac{5}{8}$		
101548	36948	T			$10\frac{3}{8}$	1500
101550	36950	T			$10\frac{3}{8}$	700
101555	36947	Q	$10\frac{1}{4}$	$10\frac{3}{8}$		

Hypothetical trade and quote dataset. The first column gives the time of the trade or quote (T or Q in the third column), the second column gives the time in number of seconds after midnight, the fourth and fifth columns give the bid-ask prices of the quote (if relevant), the sixth and seventh columns give the price and volume of the trade (if relevant).

Basic concepts of (empirical) financial market microstructure (3)

Ask/ bid prices and depths provided by liquidity suppliers

- *market makers* (NYSE: specialist, NASDAQ: dealer)
- limit order traders (Xetra, Euronext, virt-x)

limit buy order: buy order with upper price limit and given buy volume

limit sell order: sell order with lower price limit and given sell volume

Non-executable limit orders (LO) constitute the *limit order book*

market order (MO): no price limit (but buy and sell volume)

MO: liquidity demand

non-executed LO: supply liquidity

marketable limit order: like MO

Three components influence the spread

spread

order processing costs

(also opportunity costs of market making)

inventory holding costs

(of liquidity supply market maker holds suboptimal portfolio, risk aversion)

adverse selection costs or asymmetric information costs

(when some traders are better informed about true asset price than market maker)

costs incurred by market maker (liquidity supplier)

spread has to compensate for these costs

Competition among liquidity suppliers reduces gains of liquidity supply in excess of these costs

Measures of market quality (execution quality)

quoted spread		ask price – bid price
effective spread	measured at time of execution of order	2 · (execution price – midquote) for buy order 2 · (midquote – execution price) for sell order
realized spread		2 · (execution price (-t) – midquote (t + x)) for buy order 2 · (midquote (t + x) – midquote (t)) for sell order
price impact		= (effective spread – realized spread)/2 = midquote (t+x) – midquote (t)

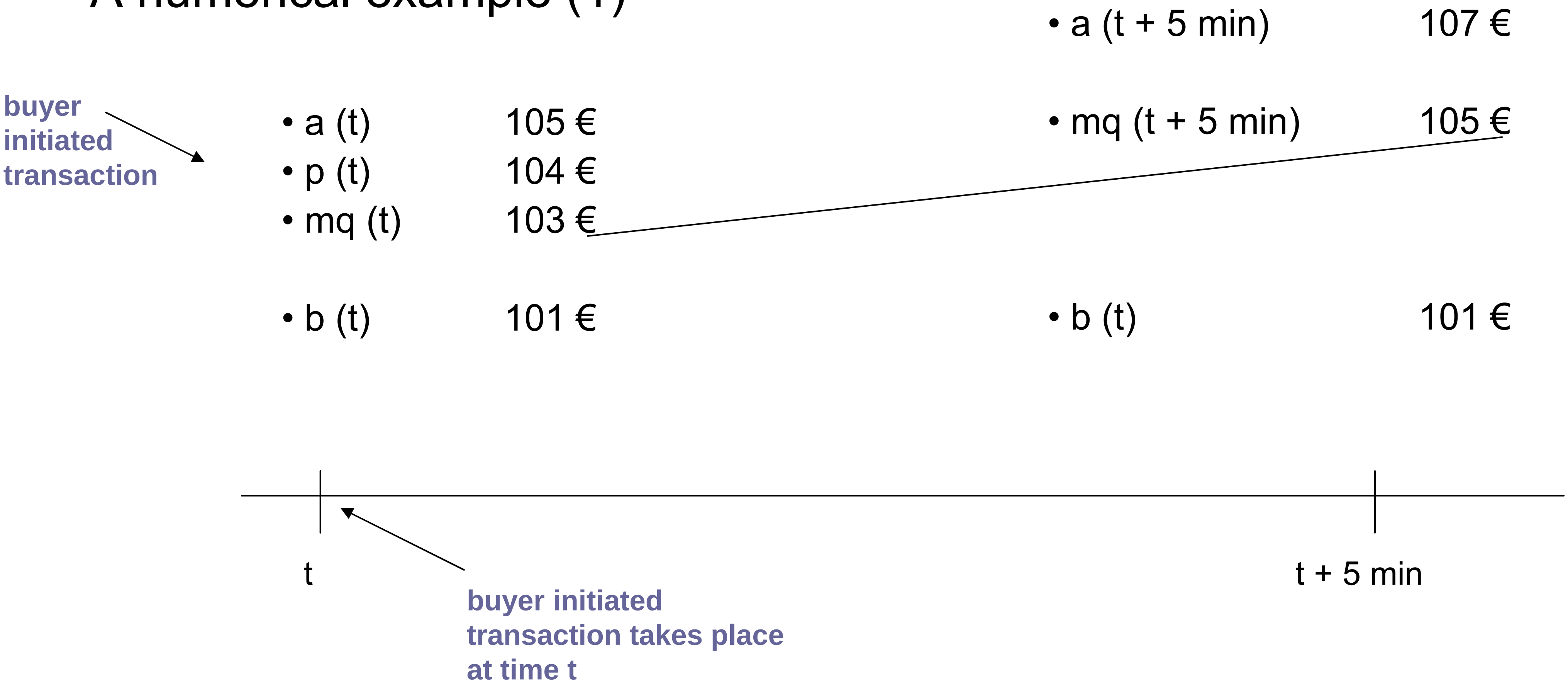
SEC Rule 11AC1-5 (Nash-5): Nov. 2000: US market centers (NYSE, Nasdaq, AMEX et cetera) have to report effective, realized and quoted spreads

SEC Rule 11AC-5 x = 5 min

Relative quoted, effective, realized spread and price impact: Relative to time t midquote

Average (relative) quoted, effective, realized spread: sample means over all transactions

A numerical example (1)



buyer initiated transaction

- a (t) 105 €
- p (t) 104 €
- mq (t) 103 €
- b (t) 101 €

- a (t + 5 min) 107 €
- mq (t + 5 min) 105 €
- b (t) 101 €

buyer initiated transaction takes place at time t

- a (t): ask price
- b (t): bid price
- mq (t): midquote time t
- p (t): execution price t

A numerical example (2)

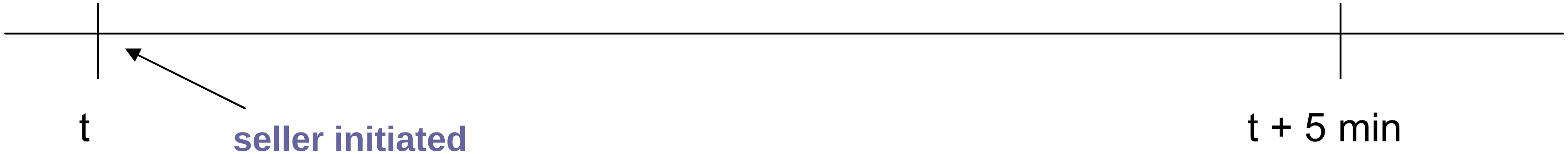
quoted spread	$105\,€ - 101€ = 4\,€$	relative quoted spread	$\frac{4\,€}{103\,€} \cdot 100 = 3.88\ \%$
effective spread	$2\,(104\,€ - 103\,€) = 2\,€$	relative effective spread	$\frac{2\,€}{103\,€} \cdot 100 = 0.97\%$
realized spread	$2\,(104\,€ - 105\,€) = -2\,€$	relative realized spread	$\frac{-2\,€}{103\,€} \cdot 100 = -1.94\%$
price impact	$\frac{(2\,€ - (-\,2\,€))}{2} = 105\,€ - 104\,€ = 1€$	relative price impact	$\frac{1€}{103\,€} \cdot 100 = 0.97\%$

Another empirical example (1)

- a (t) 106 €
- mq (t) 103 €
- p (t) = b (t) 101 €

seller initiated transaction

- a (t + 5 min) 105 €
- mq (t + 5 min) 103 €
- b (t) 101 €



seller initiated transaction takes place at time t

Another empirical example

quoted spread $106\,€ - 101€ = 5\,€$

relative quoted spread $\frac{5\,€}{103\,€} \cdot 100 = 4.85\%$

effective spread $2\,(103€ - 101€) = 4\,€$

relative effective spread $\frac{4\,€}{103\,€} \cdot 100 = 3.88\%$

realized spread $2\,(103€ - 101€) = 4\,€$

relative realized spread $\frac{4\,€}{103\,€} \cdot 100 = 3.88\%$

price impact $\frac{(4\,€ - 4\,€)}{2} = 103\,€ - 103\,€ = 0\,€$

relative price impact $\frac{0\,€}{103\,€} \cdot 100 = 0\,\%$

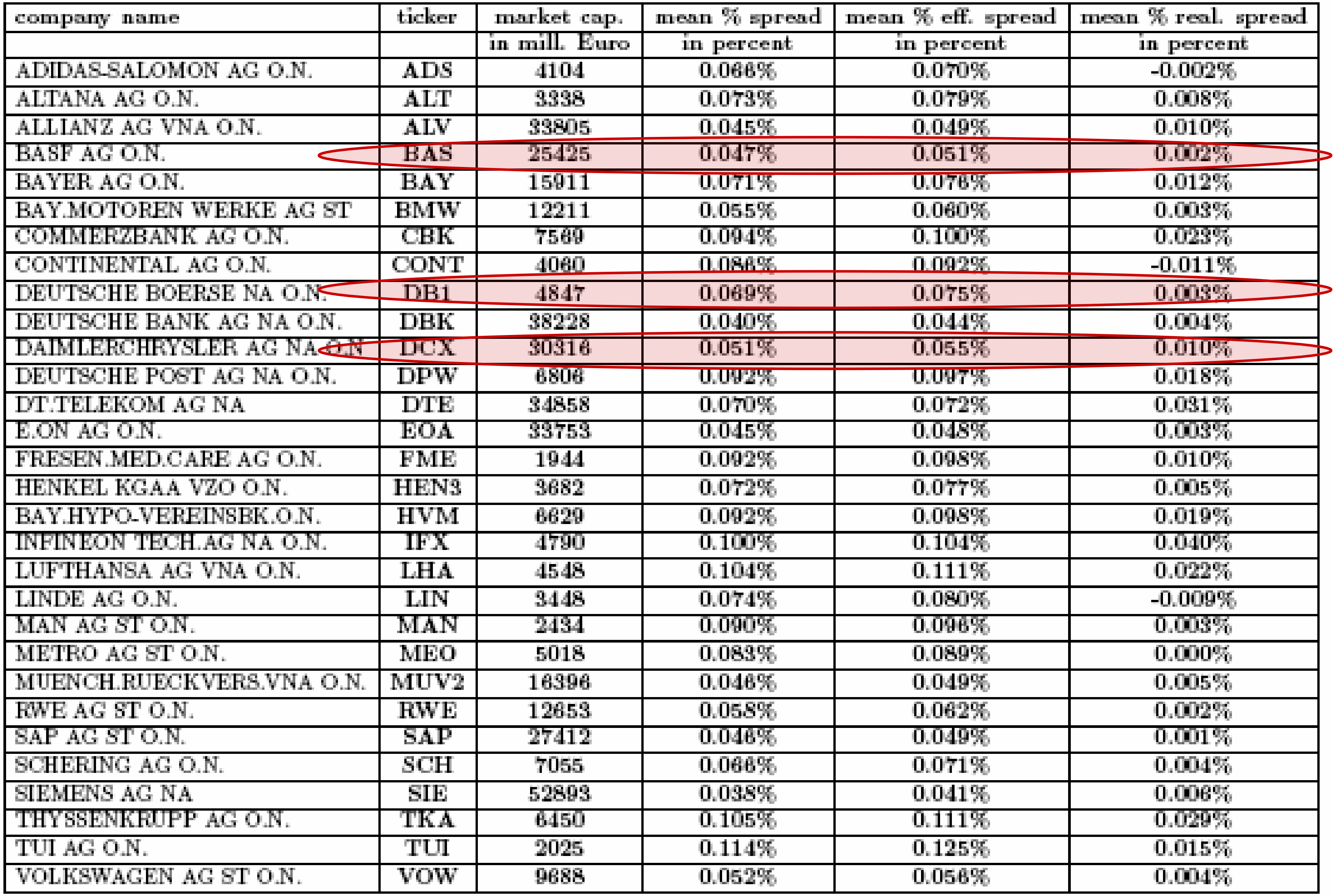
Interpretation of market quality measures

Effective spread: incorporates costs of liquidity supply and adverse selection costs

Realized spread: Transaction cost or liquidity measure “purged“ of adverse selection costs

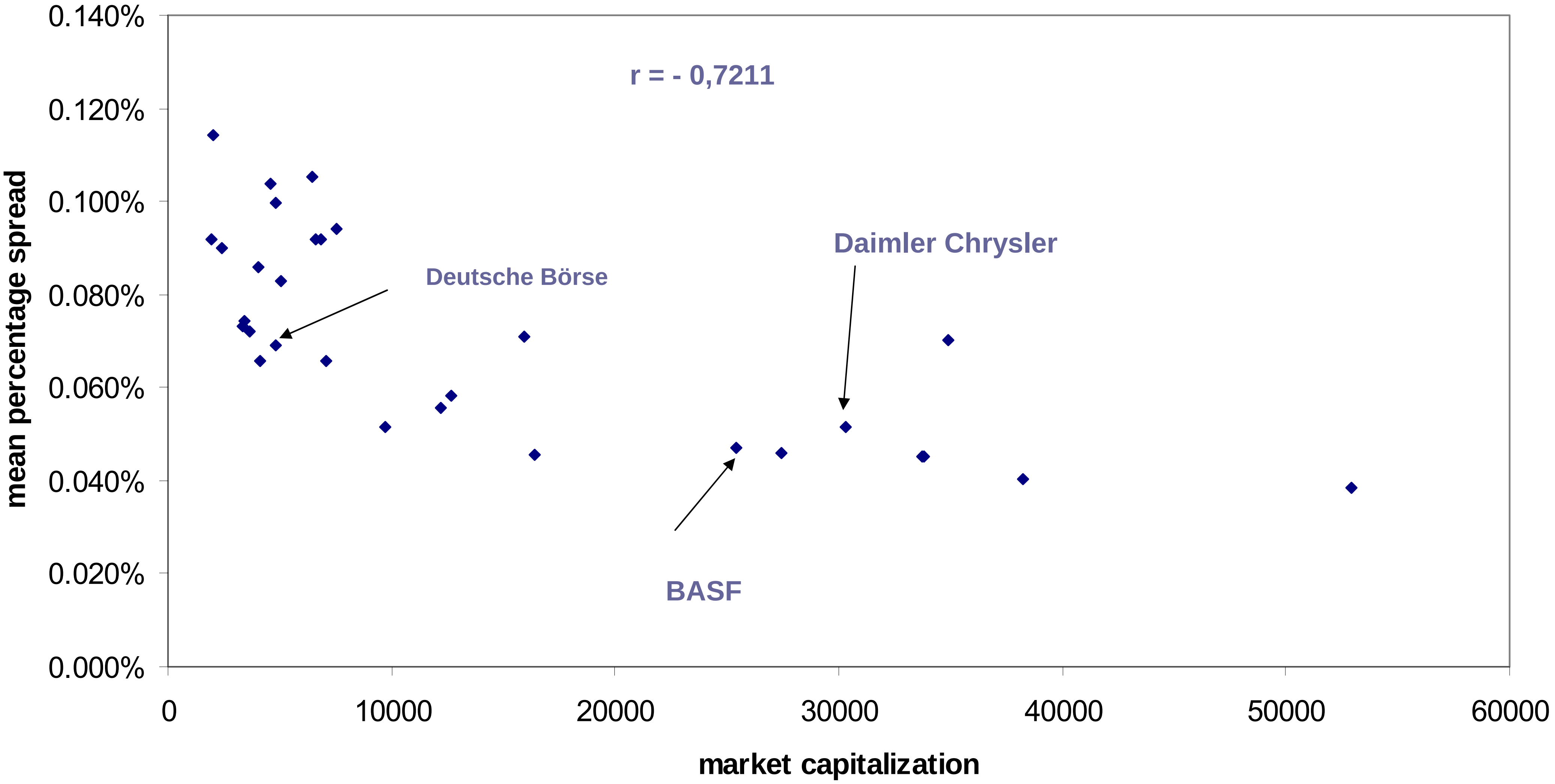
Price impact: Adverse selection cost part of the effective spread

Empirical example quoted effective, realized spreads

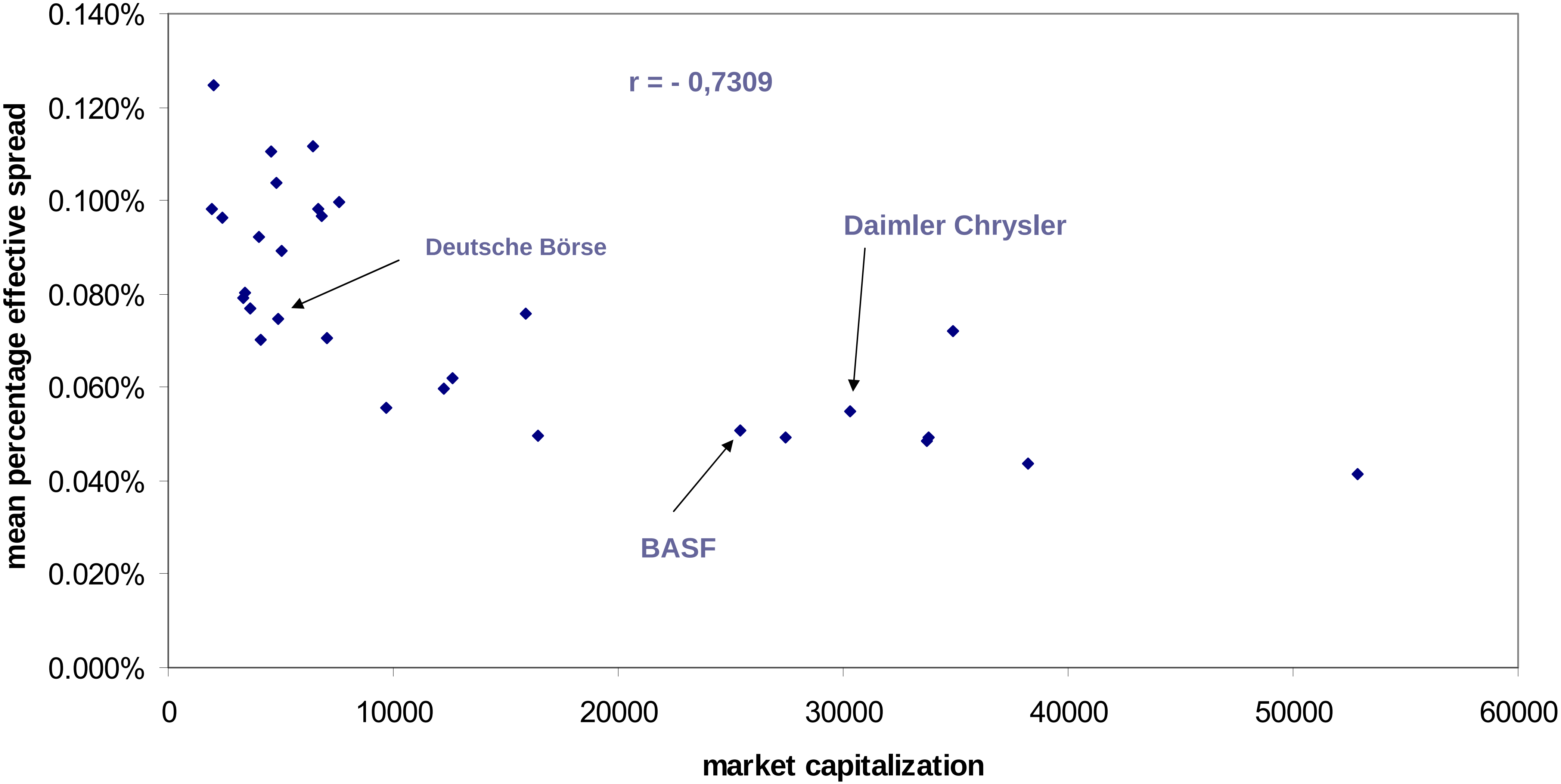


company name	ticker	market cap. in mill. Euro	mean % spread in percent	mean % eff. spread in percent	mean % real. spread in percent
ADIDAS-SALOMON AG O.N.	ADS	4104	0.066%	0.070%	-0.002%
ALTANA AG O.N.	ALT	3338	0.073%	0.079%	0.008%
ALLIANZ AG VNA O.N.	ALV	33805	0.045%	0.049%	0.010%
BASF AG O.N.	BAS	25425	0.047%	0.051%	0.002%
BAYER AG O.N.	BAY	15911	0.071%	0.076%	0.012%
BAY.MOTOREN WERKE AG ST	BMW	12211	0.055%	0.060%	0.003%
COMMERZBANK AG O.N.	CBK	7569	0.094%	0.100%	0.023%
CONTINENTAL AG O.N.	CONT	4060	0.086%	0.092%	-0.011%
DEUTSCHE BOERSE NA O.N.	DB1	4847	0.069%	0.075%	0.003%
DEUTSCHE BANK AG NA O.N.	DBK	38228	0.040%	0.044%	0.004%
DAIMLERCHRYSLER AG NA O.N	DCX	30316	0.051%	0.055%	0.010%
DEUTSCHE POST AG NA O.N.	DPW	6806	0.092%	0.097%	0.018%
DT.TELEKOM AG NA	DTE	34858	0.070%	0.072%	0.031%
E.ON AG O.N.	EOA	33753	0.045%	0.048%	0.003%
FRESEN.MED.CARE AG O.N.	FME	1944	0.092%	0.098%	0.010%
HENKEL KGAA VZO O.N.	HEN3	3682	0.072%	0.077%	0.005%
BAY.HYPO-VEREINSBK.O.N.	HVM	6629	0.092%	0.098%	0.019%
INFINEON TECH.AG NA O.N.	IFX	4790	0.100%	0.104%	0.040%
LUFTHANSA AG VNA O.N.	LHA	4548	0.104%	0.111%	0.022%
LINDE AG O.N.	LIN	3448	0.074%	0.080%	-0.009%
MAN AG ST O.N.	MAN	2434	0.090%	0.096%	0.003%
METRO AG ST O.N.	MEO	5018	0.083%	0.089%	0.000%
MUENCH.RUECKVERS.VNA O.N.	MUV2	16396	0.046%	0.049%	0.005%
RWE AG ST O.N.	RWE	12653	0.058%	0.062%	0.002%
SAP AG ST O.N.	SAP	27412	0.046%	0.049%	0.001%
SCHERING AG O.N.	SCH	7055	0.066%	0.071%	0.004%
SIEMENS AG NA	SIE	52893	0.038%	0.041%	0.006%
THYSSENKRUPP AG O.N.	TKA	6450	0.105%	0.111%	0.029%
TUI AG O.N.	TUI	2025	0.114%	0.125%	0.015%
VOLKSWAGEN AG ST O.N.	VOW	9688	0.052%	0.056%	0.004%

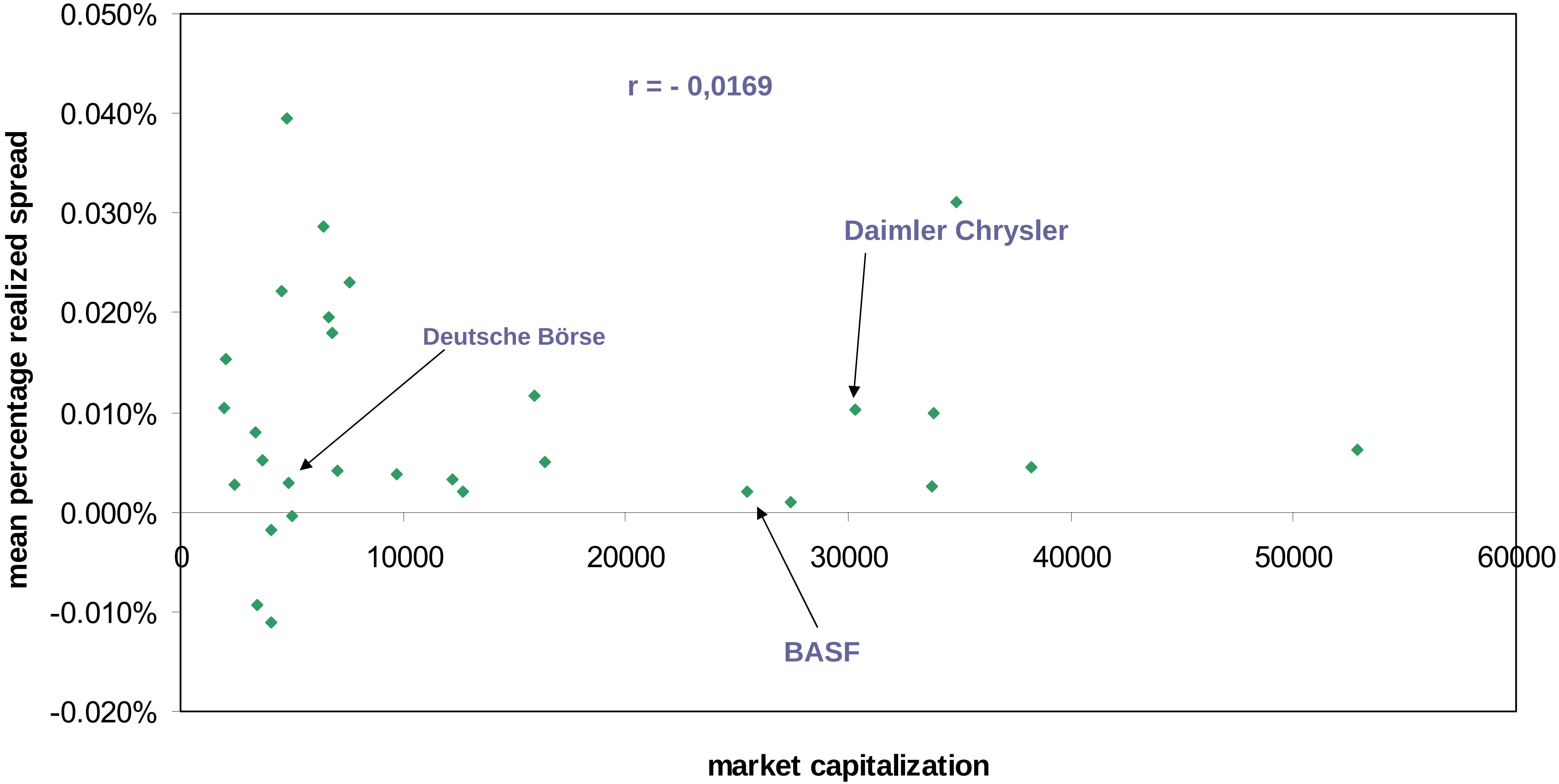
Mean percentage spread versus market capitalization



Mean percentage effective spread versus market capitalization

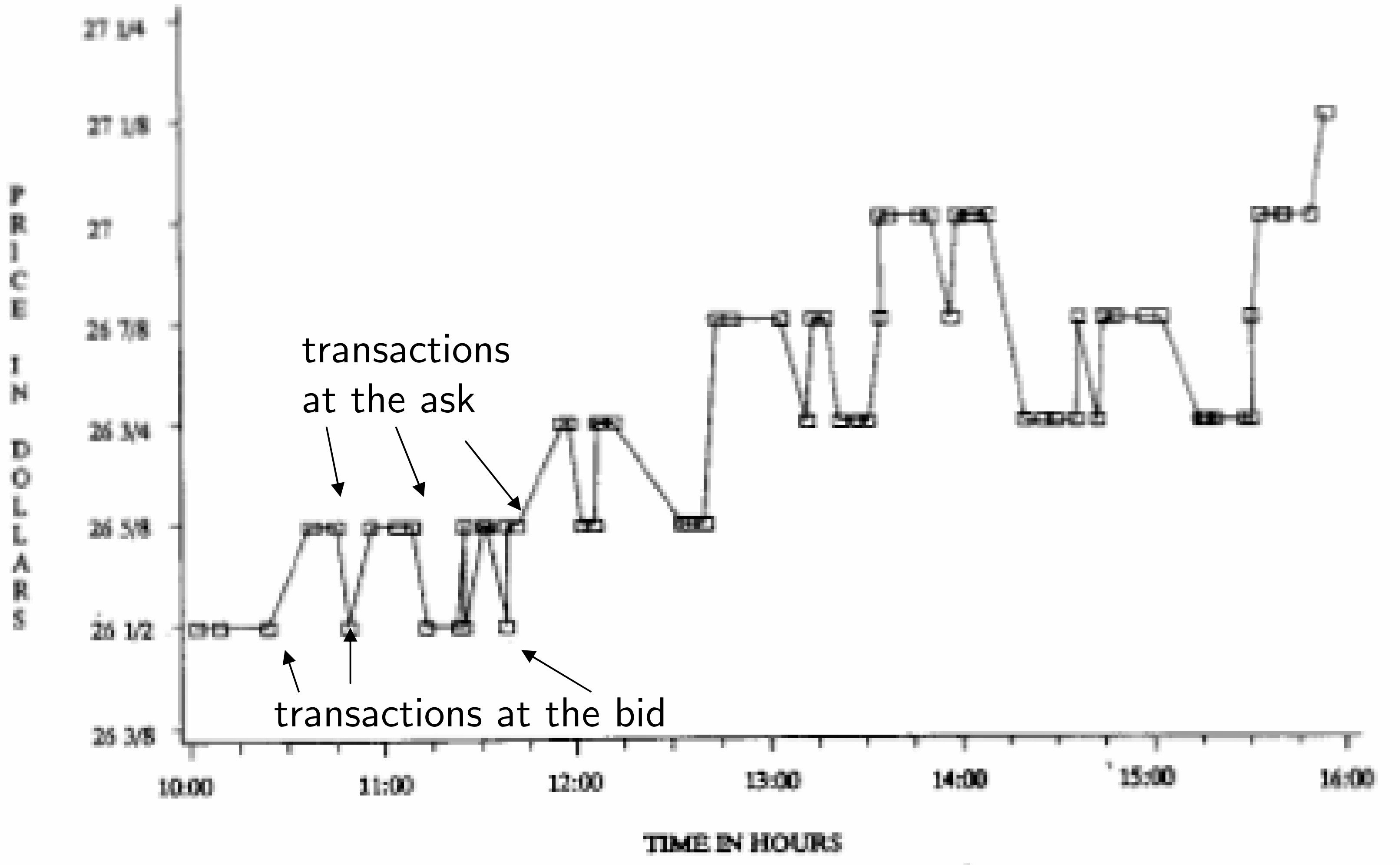


Mean percentage realized spread versus market capitalization



II.2 Trade indicator models and estimation

A transaction price series (from Glosten and Harris (1988))



transactions at the ask

transactions at the bid

Fig. 1. Complete NYSE transaction price time series for Alcoa Aluminum on December 1, 1981; 78 trades.

Trade indicator models decompose components of the spread

Structural models

- Assumptions about how trades move fundamental asset values, midquotes and transaction prices

- liquidity suppliers account for order processing costs, inventory holding costs and adverse selection costs when posting bid- ask quotes

Important contributions

- Glosten and Harris (1988): seminal model
- Huang and Stoll (1997): disentangles inventory and adverse selection component
- Madhavan, Richards and Roomans (1997): correlation in order flow, explain time of day effects of spreads and volatility

Estimation by GMM or OLS

Glosten and Harris (1988) model (basic version)

Data required:

Sequence of transaction prices with associated volumes and trade side indicator

P_t : transaction (or execution) price of trade at time t

V_t : volume (number of shares) traded

Q_t : trade side indicator $Q_t = +1$ if trade buyer initiated

$Q_t = -1$ if trade seller initiated

Glosten and Harris (1988) model: the model structure

Evolution of efficient asset price

$$m_t = m_{t-1} + e_t + Q_t Z_t$$

efficient price

new public information (i.i.d. normal) accumulated since last trade

impact of time t trade event on efficient price (permanent)

$$Z_t = z_0 + z_1 V_t$$

parameter

Glosten and Harris (1988) model: the model structure (2)

$$C_t = c_0 + c_1 V_t$$

transitory component, order processing cost

$$P_t = m_t + Q_t C_t$$

adverse selection component of spread

$$= \underbrace{m_{t-1} + e_t}_{} + \underbrace{Q_t Z_t + Q_t C_t}_{}$$

efficient price
without time t trade
information

half of the spread

Note: liquidity supplier anticipates transistory and permanent component

transaction price of a buy Q $_{t+1}$

$$Z_t + C_t \begin{cases} P_t \\ \\ m_{t-1} + e_t \end{cases}$$

liquidity supplier take into account public information when selling bid and ask prices

a buy transaction at time t

same goes for sell transaction…

Glosten and Harris (1988) model: the model structure (3)

Combining

$$\begin{aligned} P_t - P_{t-1} =& Q_t C_t - Q_{t-1} C_{t-1} + Q_t Z_t + e_t \\ =& c_0(Q_t - Q_{t-1}) + c_1(Q_t V_t - Q_{t-1} V_{t-1}) \\ & + z_0 Q_t + z_1 Q_t V_t + e_t. \end{aligned}$$

- Estimation by OLS possible
- Account for conditional heteroskedasticity and serial correlation in e_t by using robust standard errors.

Eviews Application Glosten/Harris (1988) model

Glosten and Harris (1988) model: the model structure (4)

“Implied“ spread

$$2(C_t + Z_t) = 2(c_0 + c_1 V_t) + 2(z_o + z_1 V_t)$$

Share of implied spread attributable to adverse selection costs

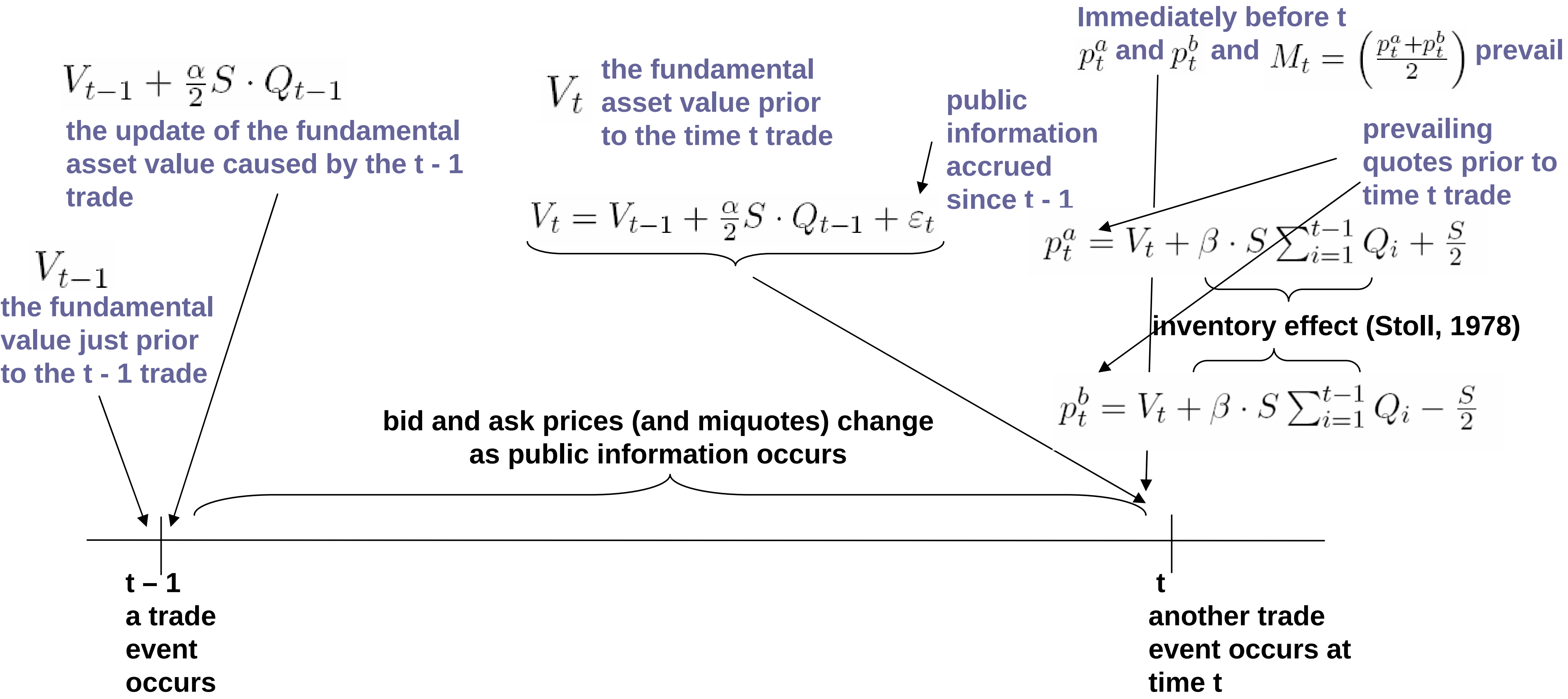
$$\alpha = \frac{z_0 + z_1 V_t}{z_0 + z_1 V_t + c_0 + c_1 V_t}$$

Share of implied spread attributable to order processing costs

$$\gamma = \frac{c_0 + c_1 V_t}{z_0 + z_1 V_t + c_0 + c_1 V_t}$$

- Plot evolution of α_t and γ_t and implied spread (time of day patterns)
- sample averages of α_t and γ_t for comparison across stocks and trading venues

Timing: Quote setting and price formation in Huang & Stoll (1997)



$V_{t-1} + \frac{\alpha}{2} S \cdot Q_{t-1}$

the update of the fundamental asset value caused by the t - 1 trade

V_t the fundamental asset value prior to the time t trade

public information accrued since t - 1

Immediately before t p_t^a and p_t^b and $M_t = \left(\frac{p_t^a + p_t^b}{2}\right)$ prevail

$$\underbrace{V_t = V_{t-1} + \frac{\alpha}{2} S \cdot Q_{t-1} + \varepsilon_t}$$

prevailing quotes prior to time t trade

$$p_t^a = V_t + \underbrace{\beta \cdot S \sum_{i=1}^{t-1} Q_i} + \frac{S}{2}$$

inventory effect (Stoll, 1978)

$$p_t^b = V_t + \overbrace{\beta \cdot S \sum_{i=1}^{t-1} Q_i} - \frac{S}{2}$$

V_{t-1}

the fundamental value just prior to the t - 1 trade

bid and ask prices (and miquotes) change as public information occurs

t – 1
a trade
event
occurs

t
another trade
event occurs at
time t

trades occur at bid or ask ←

$$\underbrace{M_t = V_t + \beta \cdot \frac{S}{2} \sum_{i=1}^{t-1} Q_i}$$

prevailing midquote prior to time t trade

$$P_t = M_t + \frac{S}{2} Q_t + \underbrace{\eta_t}_{\text{Price discreteness}}$$

Note: $\alpha + \beta \neq 1$

The Huang and Stoll (1997) base model

Accounts for inventory cost component

Dynamics of fundamental asset value (Huang/Stoll (1997) notation)

“traded“ spread (Huang and Stoll (1997))

public information

$$V_t = V_{t-1} + \underbrace{\underbrace{\left(\alpha \cdot \frac{S}{2}\right)}_{\text{Impact coefficient}} Q_{t-1}}_{\text{asymmetric information}} + \varepsilon_t$$

only trade indicator, not volume as in Glosten/Harris

equiv. to m_t in Glosten/Harris

Quote midpoint contains inventory control mechanism

midquote, do not confuse with m_t in Glosten/Harris (1988)

When number of buy order exceed sell orders market maker increases ask and bid price to discourage further buys and encourage sells.

$$M_t = V_t + \beta \cdot \frac{S}{2} \sum_{i=1}^{t-1} Q_i$$

The Huang and Stoll (1997) base model

Midquote dynamics

$$\Delta M_t = (\alpha + \beta)\tfrac{S}{2}Q_{t-1} + \varepsilon_t$$

$M_t - M_{t-1}$

Midquote change affected by adverse selection and inventory effects, fundamental value only affected by adverse selection component

Combining we get

$$\Delta P_t = \frac{S}{2}(Q_t - Q_{t-1}) + \underbrace{v}_{(\alpha+\beta)}\frac{S}{2}Q_{t-1} + \underbrace{e_t}_{(\varepsilon_t+\Delta\eta_t)}$$

α and β not separately identified

Inventory and adverse selection component lumped together unknown parameters: S (traded spread); $\gamma = (\alpha + \beta)$

GMM estimation base model Huang and Stoll (1997)

Moment conditions

$$E(e_t Q_t) = 0$$

$$E(e_t Q_{t-1}) = 0$$

2 moment conditions, 2 parameters

→ exact identification

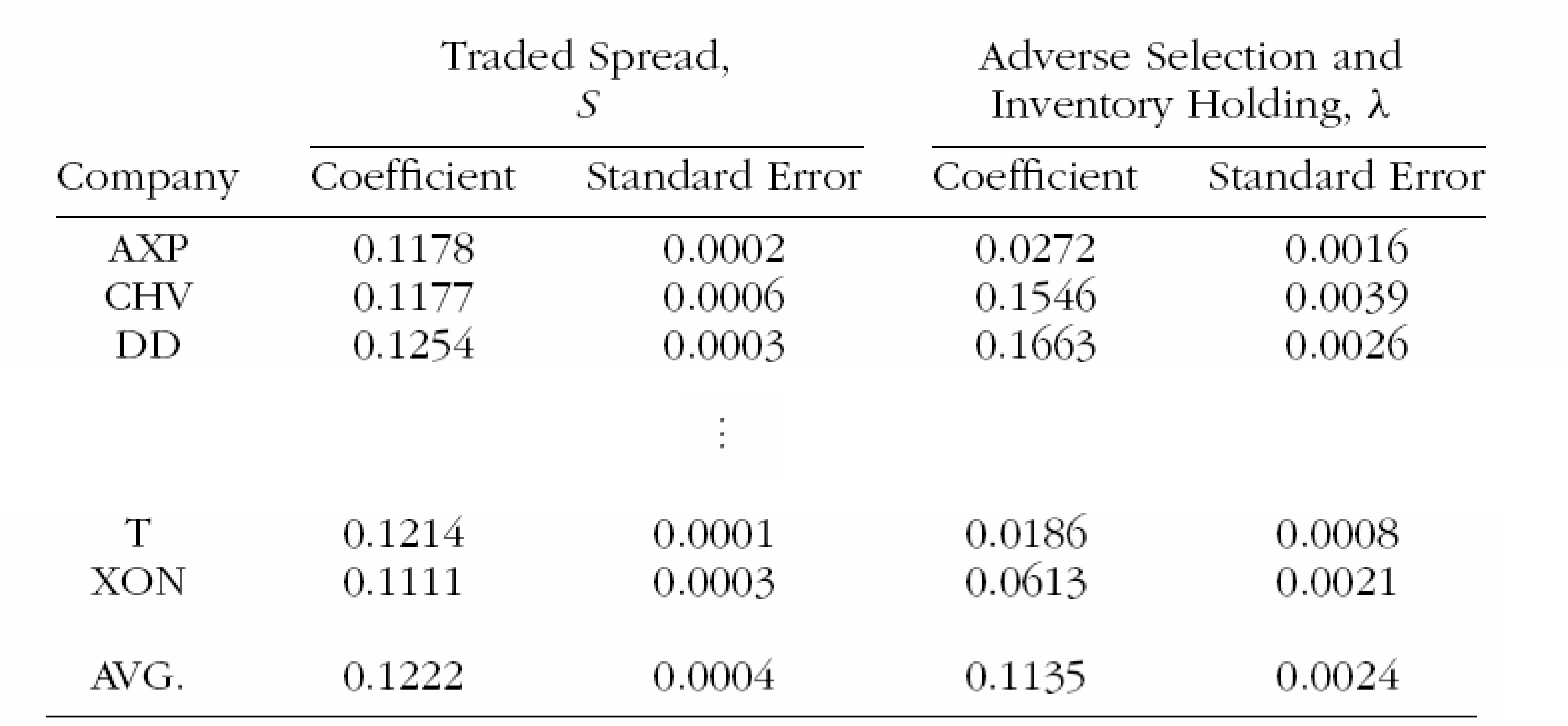
- could be estimated by OLS $\Delta P_t = \beta_0 \Delta Q_t + \beta_1 Q_{t-1} + e_t$
- advantage of GMM: standard errors robust against conditional heteroskedasticity and serial correlation in e_t

EVIEWS: use Newey-West standard errors

use Delta Method to obtain standard errors for S and v from $\beta_0 = \frac{S}{2}$ $\beta_1 = v\frac{S}{2}$

$$S = 2\beta_0 \quad v = \frac{\beta_1}{\beta_0}$$

Estimation results Huang and Stoll base model



Company	Traded Spread, S		Adverse Selection and Inventory Holding, λ	
	Coefficient	Standard Error	Coefficient	Standard Error
AXP	0.1178	0.0002	0.0272	0.0016
CHV	0.1177	0.0006	0.1546	0.0039
DD	0.1254	0.0003	0.1663	0.0026
		⋮		
T	0.1214	0.0001	0.0186	0.0008
XON	0.1111	0.0003	0.0613	0.0021
AVG.	0.1222	0.0004	0.1135	0.0024

Huang and Stoll‘s (1997) model with trade size categories (1)

Define

$$\begin{aligned} D_t^s &= Q_t && \text{if share volume at } t \leq 1000 \text{ shares} \\ &= 0 && \text{otherwise} \end{aligned}$$

$$\begin{aligned} D_t^m &= Q_t && 1000 \text{ shares} < \text{if share volume at } t < 10{,}000 \text{ shares} \\ &= 0 && \text{otherwise} \end{aligned}$$

$$\begin{aligned} D_t^l &= Q_t && \text{if share volume at } t \geq 10{,}000 \text{ shares} \\ &= 0 && \text{otherwise} \end{aligned}$$

Evalution of fundamental value subject to trade information

$$V_t = V_{t-1} + \alpha^s \frac{S^s}{2} D_{t-1}^s + \alpha^m \frac{S^m}{2} D_{t-1}^m + \alpha^l \frac{S^l}{2} D_{t-1}^l + \epsilon_t$$

Huang and Stoll‘s (1997) model with trade size categories (2)

Midquote (as above) affected by inventory effects

$$M_t = V_t + \sum_j \left[\beta^j \frac{S^j}{2} \sum_{i=1}^{t-1} D_i^j \right]$$

for $j \in \{s, m, l\}$

Combining we get

$$\triangle M_t = \triangle V_t + \beta^s \frac{S^s}{2} D_{t-1}^s + \beta^m \frac{S^m}{2} D_{t-1}^m + \beta^l \frac{S^l}{2} D_{t-1}^l$$

$$\triangle M_t = (\alpha^s + \beta^s) \frac{S^s}{2} D_{t-1}^s + (\alpha^m + \beta^m) \frac{S^m}{2} D_{t-1}^m + (\alpha^l + \beta^l) \frac{S^l}{2} D_{t-1}^l + \epsilon_t$$

Huang and Stoll‘s (1997) model with trade size categories (3)

As above: transaction price incorporates the half spread

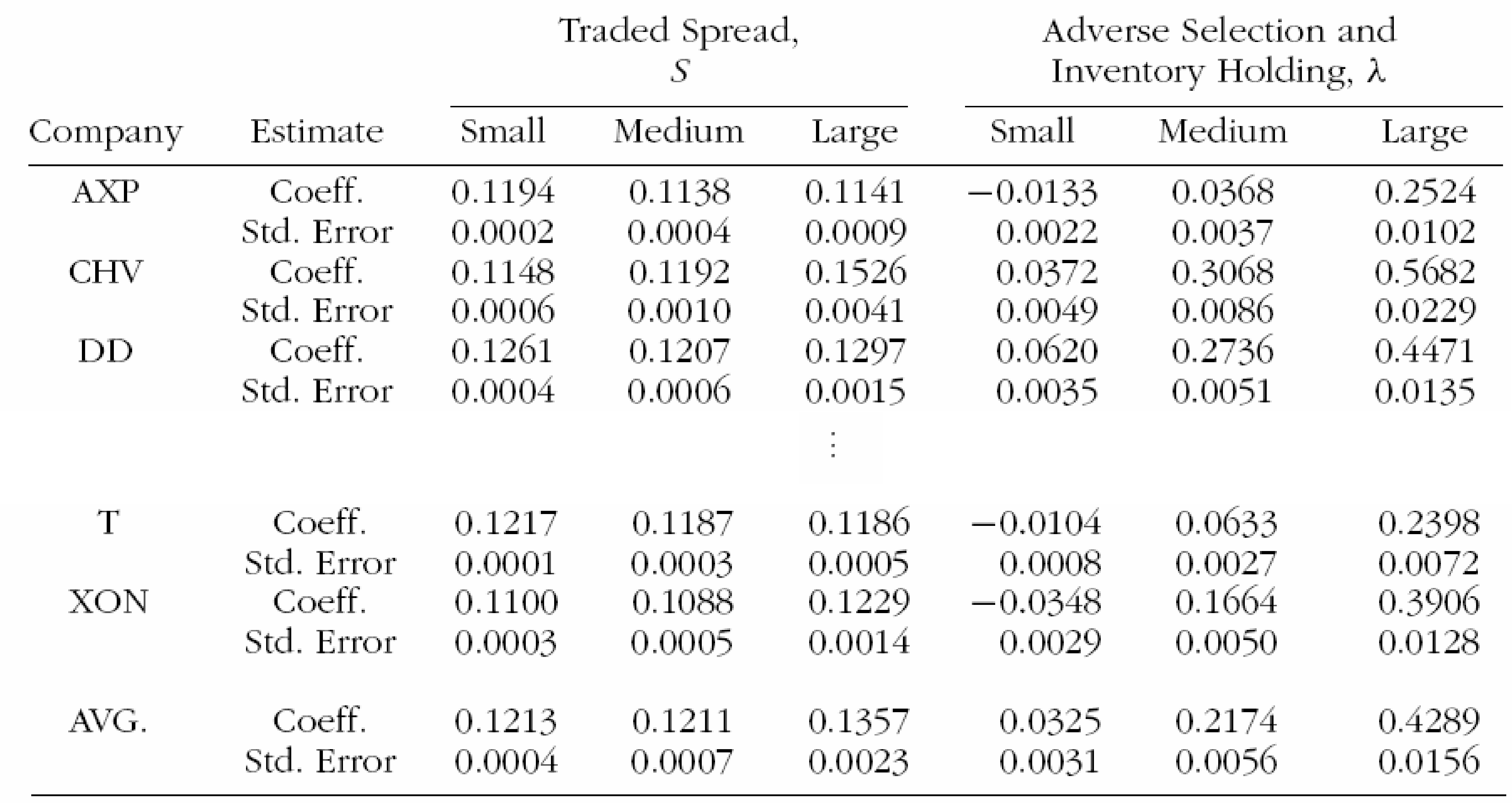
$$P_t = M_t + \frac{S^s}{2} D_t^s + \frac{S^m}{2} D_t^m + \frac{S^l}{2} D_t^l + \eta_t$$

Equation estimated by GMM

$$\begin{aligned} \triangle P_t &= \frac{S^s}{2} D_t^s + (\lambda^s - 1) \frac{S^s}{2} D_{t-1}^s + \frac{S^m}{2} D_t^m + (\lambda^m - 1) \frac{S^m}{2} D_{t-1}^m \\ &+ \frac{S^l}{2} D_t^l + (\lambda^l - 1) \frac{S^l}{2} D_{t-1}^l + e_t \end{aligned}$$

where $j = s, m, l$ for $\lambda_j = \alpha_j + \beta_j$

Estimation results Huang and Stoll model with trade size categories



		Traded Spread, S			Adverse Selection and Inventory Holding, λ		
Company	Estimate	Small	Medium	Large	Small	Medium	Large
AXP	Coeff.	0.1194	0.1138	0.1141	−0.0133	0.0368	0.2524
	Std. Error	0.0002	0.0004	0.0009	0.0022	0.0037	0.0102
CHV	Coeff.	0.1148	0.1192	0.1526	0.0372	0.3068	0.5682
	Std. Error	0.0006	0.0010	0.0041	0.0049	0.0086	0.0229
DD	Coeff.	0.1261	0.1207	0.1297	0.0620	0.2736	0.4471
	Std. Error	0.0004	0.0006	0.0015	0.0035	0.0051	0.0135
				⋮			
T	Coeff.	0.1217	0.1187	0.1186	−0.0104	0.0633	0.2398
	Std. Error	0.0001	0.0003	0.0005	0.0008	0.0027	0.0072
XON	Coeff.	0.1100	0.1088	0.1229	−0.0348	0.1664	0.3906
	Std. Error	0.0003	0.0005	0.0014	0.0029	0.0050	0.0128
AVG.	Coeff.	0.1213	0.1211	0.1357	0.0325	0.2174	0.4289
	Std. Error	0.0004	0.0007	0.0023	0.0031	0.0056	0.0156

Huang and Stoll (1997): Three way decomposition (extended model)

Allows to distinguish adverse selection (α) and inventory component (β) of traded spread

Identified if Q_t (positively) serially correlated $P(Q_t = Q_{t-1}) \neq 0.5$

Why?

Splitting of large orders to cushion price impact

Identified through predictability of midquote changes

Why?

Liquidity suppliers raise *midquote* after buyer initiated transaction

lower *midquote* after seller initiated transaction

Inventory effect on expected midquote change

serial correlation of Q_t → predictability of midquote change $\Delta M_t = f(Q_{t-1})$

Huang and Stoll (1997) extended model (1)

Define

$\pi = P(Q_t \neq Q_{t-1})$ allegedly: $\pi > 0.5$ (bid-ask-bounce due to inventory effect)

no trades inside the quote considered!

$$1 - \pi = P(Q_t = Q_{t-1})$$

→ $\mathbb{E}(Q_t | Q_{t-1}) = (1 - 2\pi) Q_{t-1}$ order flow predictable!

Evolution of fundamental value

Surprises in order flow matter

$$V_t = V_{t-1} + \alpha \frac{S}{2} (\underbrace{Q_t - \mathbb{E}(Q_t | Q_{t-1})}_{\text{surprise}}) + \varepsilon_t$$

$$= V_t + \alpha \frac{S}{2} (1 - 2\pi) Q_{t-1} + \varepsilon_t$$

Huang and Stoll (1997) extended model (2)

change in fundamental value unpredictable

$\mathbb{E}(\Delta V_t | Q_{t-2}, V_{t-1}) = 0$ → $\left\{ \Delta V_t \right\}$ m.d.s.

Still: V_t timed prior to time t trade

using Q_t instead of $Q_t - \mathbb{E}(Q_t | Q_{t-1})$

with $\pi \neq 0.5$ would imply predictability of ΔV_t

Midquote evolves as above

$$M_t = V_t + \underbrace{\beta \frac{S}{2} \sum_{i=1}^{t-1} Q_i}_{\textbf{inventory effect}}$$

$$\Delta M_t = \Delta V_t + \beta \frac{S}{2} Q_{t-1}$$

Huang and Stoll (1997) extended model (3)

Writing extensively

$$\underbrace{M_t = M_{t-1} + (\alpha + \beta)\frac{S}{2}Q_{t-1} - \alpha\frac{S}{2}(1 - 2\pi)Q_{t-2} + \varepsilon_t}_{\text{could be used for estimation when midquotes available}}$$

As $P_t = M_t + \frac{S}{2}Q_t + \eta_t$ (as above)

we have

$$\Delta P_t = \frac{S}{2}Q_t + (\alpha + \beta - 1)\frac{S}{2}Q_{t-1} - \alpha\frac{S}{2}(1 - 2\pi)Q_{t-2} + \underbrace{\varepsilon_t + \eta_t + \eta_{t-1}}_{e_t}$$

GMM estimation possible when series of transaction prices and trade indicators available

Huang and Stoll (1997) GMM estimation extended model (1)

GMM residuals

$$\underbrace{\Delta P_t - \frac{S}{2}Q_t - (\alpha + \beta - 1)\frac{S}{2}Q_{t-1} + \alpha\frac{S}{2}(1 - 2\pi)Q_{t-2}}_{e_t}$$

$$\begin{array}{l} e_tQ_t \\ e_tQ_{t-1} \\ e_tQ_{t-2} \end{array}$$

→

$$\left[\mathbb{E}(e_t) = 0\right]$$

$$\left.\begin{array}{l} \mathbb{E}(e_tQ_{t-1}) = 0 \\ \mathbb{E}(e_tQ_{t-2}) = 0 \\ \mathbb{E}(e_tQ_t) = 0 \end{array}\right\}$$

3 moment conditions

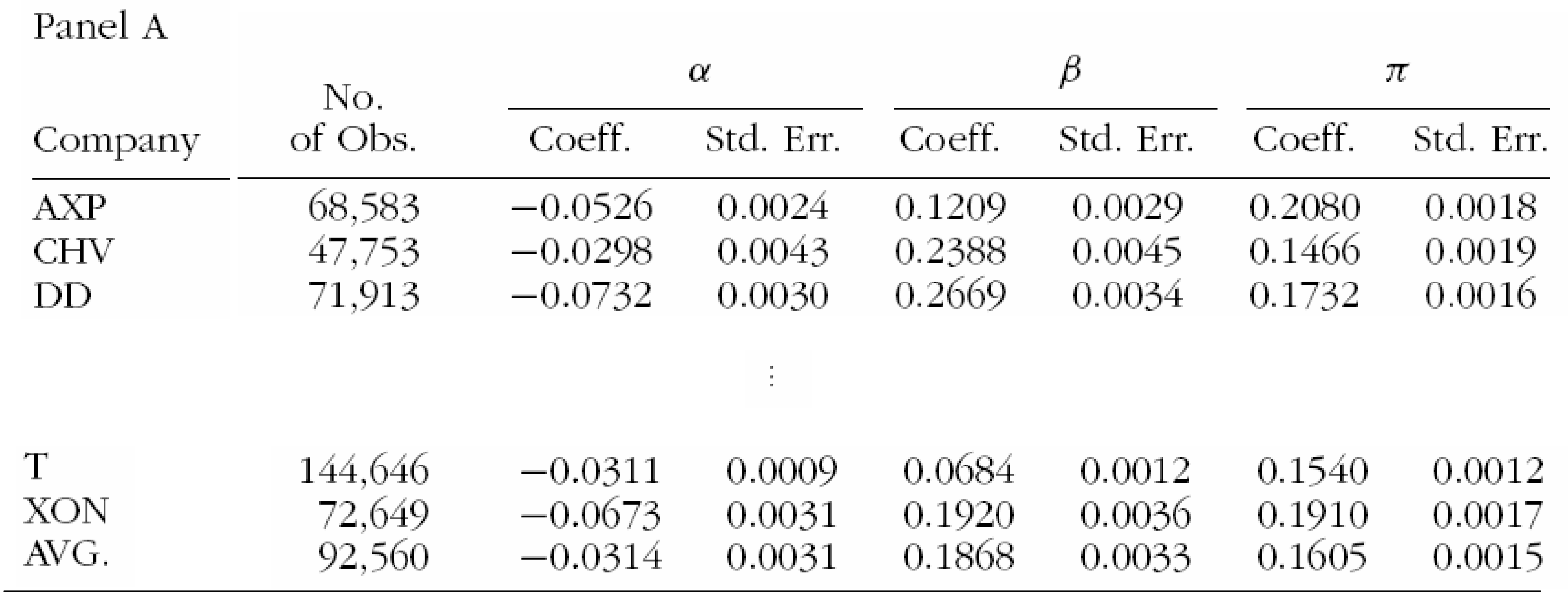
4 parameters

additional moment restriction required

$$\mathbb{E}(Q_t - \mathbb{E}(Q_t|Q_{t-1})) = 0$$

$$\mathbb{E}(\underbrace{Q_t - (1 - 2\pi)Q_{t-1}}_{\text{another GMM residual}}) = 0$$

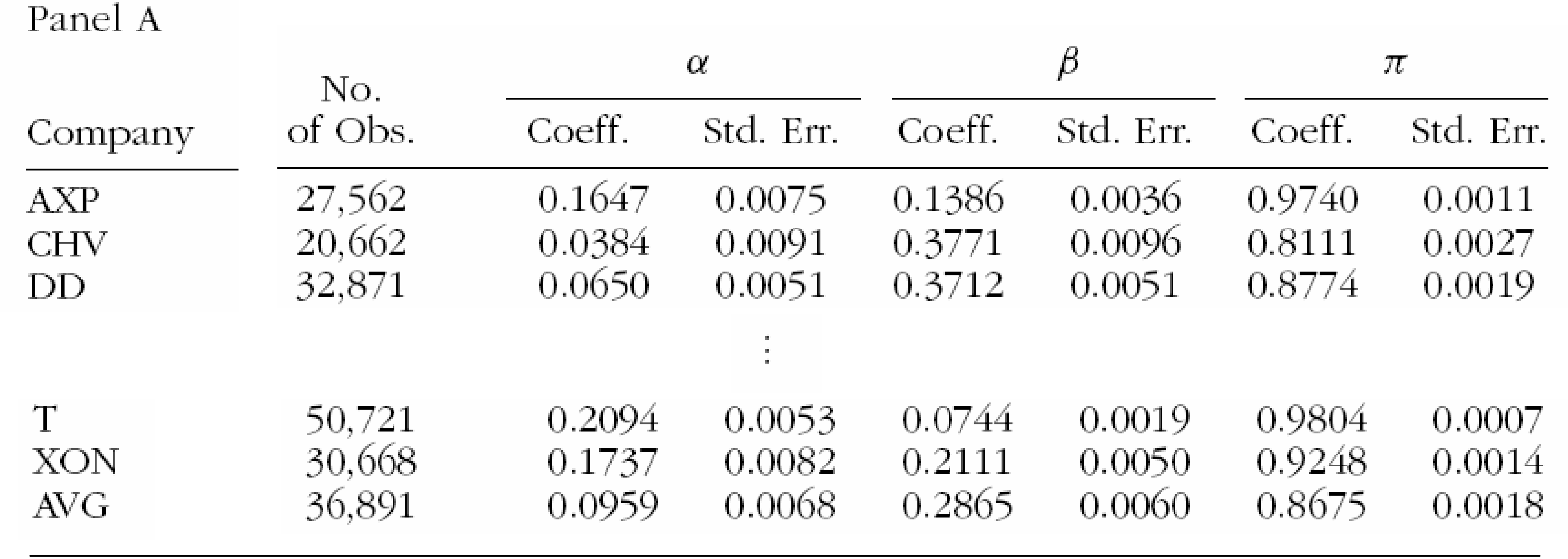
Huang and Stoll (1997) extended model estimation results



Panel A

		α		β		π	
Company	No. of Obs.	Coeff.	Std. Err.	Coeff.	Std. Err.	Coeff.	Std. Err.
AXP	68,583	−0.0526	0.0024	0.1209	0.0029	0.2080	0.0018
CHV	47,753	−0.0298	0.0043	0.2388	0.0045	0.1466	0.0019
DD	71,913	−0.0732	0.0030	0.2669	0.0034	0.1732	0.0016
			⋮				
T	144,646	−0.0311	0.0009	0.0684	0.0012	0.1540	0.0012
XON	72,649	−0.0673	0.0031	0.1920	0.0036	0.1910	0.0017
AVG.	92,560	−0.0314	0.0031	0.1868	0.0033	0.1605	0.0015

Huang and Stoll (1997) extended model estimation results with bunching



Panel A

		α		β		π	
Company	No. of Obs.	Coeff.	Std. Err.	Coeff.	Std. Err.	Coeff.	Std. Err.
AXP	27,562	0.1647	0.0075	0.1386	0.0036	0.9740	0.0011
CHV	20,662	0.0384	0.0091	0.3771	0.0096	0.8111	0.0027
DD	32,871	0.0650	0.0051	0.3712	0.0051	0.8774	0.0019
			⋮				
T	50,721	0.2094	0.0053	0.0744	0.0019	0.9804	0.0007
XON	30,668	0.1737	0.0082	0.2111	0.0050	0.9248	0.0014
AVG	36,891	0.0959	0.0068	0.2865	0.0060	0.8675	0.0018

Huang and Stoll´s (1997) alternative using spread information

Huang and Stoll suggest to use

$$\Delta M_t = (\alpha + \beta)\frac{S_{t-1}}{2}Q_{t-1} - \alpha(1 - 2\pi)\boxed{\frac{S_{t-2}}{2}}Q_{t-2} + e_t$$

$$= \beta_1 X_{t1} + \beta_2 X_{t2} + e_t$$

→ form orthogonality conditions and use GMM

Use $\mathbb{E}(Q_t - (1 - 2\pi)Q_{t-1}) = 0$

to identify structural parameters

Recent research: Henker and Wang (2005) doubt validity and consistency of procedure using St sequence: Timing incorrect.

Correct midquote dynamics with time varying spreads

$$\Delta M_t = (\alpha + \beta)\frac{S_{t-1}}{2}Q_{t-1} - \alpha(1 - 2\pi)\boxed{\frac{S_{t-1}}{2}}Q_{t-2} + e_t$$

The model by Madhavan, Richardson and Roomans (1997) (1)

- Accounts for serial correlation in order flow (in Q_t)

  → predictability of order flow

reasons: splitting of large orders to cushion price impact

- fundamental asset value affected by surprise in order flow
- three states of trade indicator

purchases at ask	$Q_t = +1$
sales at bid	$Q_t = -1$
„crosses" inside the spread	$Q_t = 0$

Why not model trade volume?

- splitting of orders → volume not informative
- parsimonity
- block trades (at NYSE upstairs market) have peculiar effect (non anonymous market, uninformed trades)

The model by Madhavan, Richardson and Roomans (1997) (2)

Ingredients

$P(Q_t = 0) = \lambda$ probability of crossing (may be zero like in Xetra system)

$P(Q_t = -1) = P(Q_t = +1) = \frac{1-\lambda}{2}$ unconditional probability

→ $\mathbb{E}(Q_t) = 0$ $var(Q_t) = \mathbb{E}(Q_t^2) = 1 - \lambda$

Evolution of fundamental value (MRR: expected value of stock given public information)

θ > 0 measures degree of information asymmetry

public information (accrued since t-1)

$$\mu_t = \mu_{t-1} + \theta\underbrace{[Q_t - \mathbb{E}(Q_t|Q_{t-1})]}_{\text{surprise in order flow}} + \varepsilon_t$$

post (!) trade value

surprise in order flow

The model by Madhavan, Richardson and Roomans (1997) (3)

market makers anticipate price impact of trade and costs for supplying liquidity

$$p_t^a = \mu_{t-1} + \theta[1 - \mathbb{E}(Q_t|Q_{t-1})] + \phi + \varepsilon_t$$ (assuming Q_t = +1)

$$p_t^b = \mu_{t-1} + \underbrace{\theta[-1 - \mathbb{E}(Q_t|Q_{t-1})]}_{\text{market maker lowers bid price depending on surprise in order flow}} - \phi + \varepsilon_t$$ (assuming Q_t = -1)

cost for supplying liquidity

- **-transaction costs (order processing)**
- **-inventory costs**
- **-risk bearing**
- **-monopolist gain**

MRR account for trades at midquote $\frac{p_t^a + p_t^b}{2}$

(pre-negotiated trades in upstairs market)

Expression for transaction price (valid for Q_t = +1, Q_t = -1, Q_t = 0)

$$P_t = \mu_t + \phi Q_t + \xi_t$$

account for effects of rounding (1/16 ticks at NYSE 1 cent ticks in Xetra)

The model by Madhavan, Richardson and Roomans (1997) (4)

Collecting

$$P_t = \mu_{t-1} + \theta[Q_t - \underbrace{\mathbb{E}(Q_t|Q_{t-1})}_{\text{to derive an estimable version use}}] + \phi Q_t + \varepsilon_t + \xi_t$$

$$P(Q_t = Q_{t-1}|Q_{t-1} \neq 0) = \gamma$$

Probability of two ask successions or two bid successions identical

First order autocorrelation $\rho = \frac{cov(Q_t, Q_{t-1})}{\sqrt{var(Q_t)}\sqrt{var(Q_{t-1})}} = \frac{E(Q_t Q_{t-1})}{var(Q_t)}$

we have $\rho = 2\gamma - (1 - \lambda)$

$$\mathbb{E}(Q_t|Q_{t-1}) = \rho Q_{t-1}$$

Note: $\mathbb{E}(Q_t) = 0$

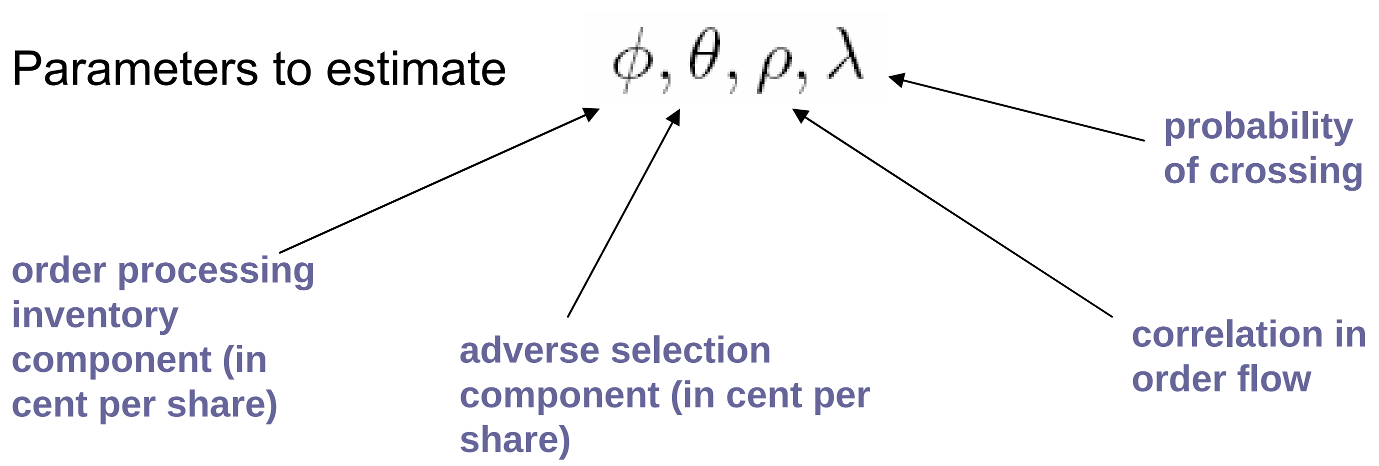
$$Var(Q_t) = 1 - \lambda$$

The model by Madhavan, Richardson and Roomans (1997) (5)

Combining we have

$$\underbrace{P_t - P_{t-1}}_{\text{observed}} = (\phi + \theta)Q_t - (\phi + \rho\theta)Q_{t-1} + \underbrace{\varepsilon_t + \xi_t - \xi_{t-1}}_{\text{residual}}$$

observed



Parameters to estimate $\phi, \theta, \rho, \lambda$

probability of crossing

order processing inventory component (in cent per share)

adverse selection component (in cent per share)

correlation in order flow

The model by Madhavan, Richardson and Roomans (1997) (6)

Estimation by GMM

Moment conditions

Define $u_t = \varepsilon_t + \xi_t - \xi_{t-1} = \Delta P_t - (\phi + \theta)Q_t + (\phi + \rho\theta)Q_{t-1}$

$$\mathbb{E}(u_t) = 0$$

$$\mathbb{E}(u_t Q_t) = 0$$

$$\mathbb{E}(u_t Q_{t-1}) = 0$$

$\mathbb{E}(Q_t^2 - (1 - \lambda)) = 0$ from $var(Q_t) = 1 - \lambda$

$\mathbb{E}(Q_t Q_{t-1} - \rho Q_t^2) = 0$ from $\frac{\mathbb{E}(Q_t Q_{t-1})}{\mathbb{E}(Q_t^2)} = \rho$

The model by Madhavan, Richardson and Roomans (1997) (7)

GMM residuals are

$$u_t = \Delta P_t - (\phi + \theta)Q_t + (\phi + \rho\theta)Q_{t-1}$$

$$u_t Q_t$$

$$u_t Q_{t-1}$$

$$Q_t^2 - (1 - \lambda)$$

$$Q_t Q_{t-1} - \rho Q_t^2$$

Built series from data and plug in GMM tool

Note: MRR introduce a drift parameter in

$$\mu_t = \mu_{t-1} + \alpha + \theta[Q_t - \mathbb{E}(Q_t|Q_{t-1})] + \varepsilon_t$$

but α small

The model by Madhavan, Richardson and Roomans (1997) (8)

- MRR estimate model for 4 day time intervals
- study time of day evolution of asymmetric information component θ and cost of liquidity supply ϕ

Explain stylized facts

”U-Shape” of bid-ask spread and return volatility

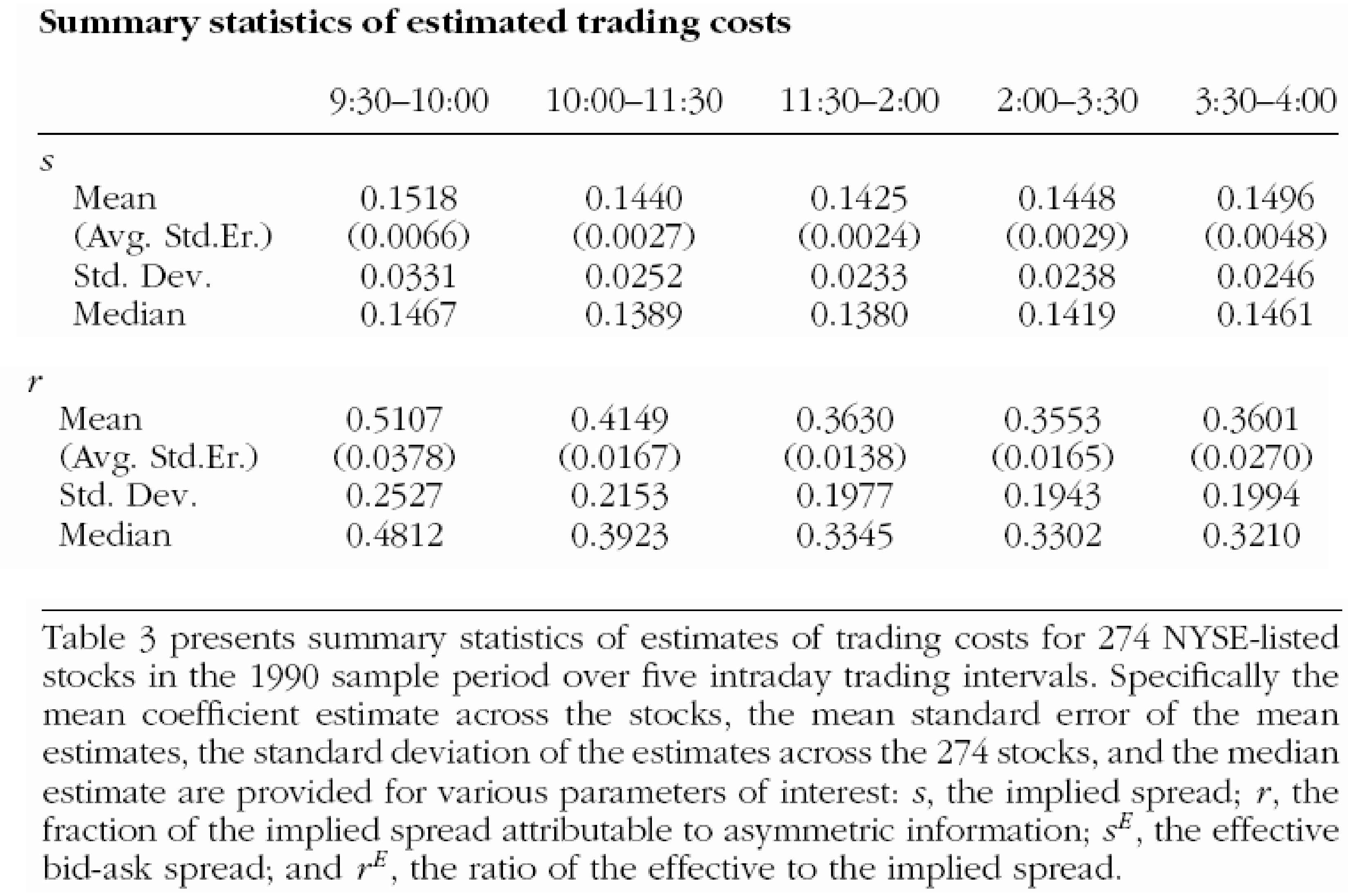
Results

θ ”L-shaped” (asymmetric information declines over the day)

ϕ increases over the day inventory costs (overnight risk)

explain stylized facts well.

The model by Madhavan, Richardson and Roomans (1997) (9)



Summary statistics of estimated trading costs

	9:30–10:00	10:00–11:30	11:30–2:00	2:00–3:30	3:30–4:00
s					
Mean	0.1518	0.1440	0.1425	0.1448	0.1496
(Avg. Std.Er.)	(0.0066)	(0.0027)	(0.0024)	(0.0029)	(0.0048)
Std. Dev.	0.0331	0.0252	0.0233	0.0238	0.0246
Median	0.1467	0.1389	0.1380	0.1419	0.1461
r					
Mean	0.5107	0.4149	0.3630	0.3553	0.3601
(Avg. Std.Er.)	(0.0378)	(0.0167)	(0.0138)	(0.0165)	(0.0270)
Std. Dev.	0.2527	0.2153	0.1977	0.1943	0.1994
Median	0.4812	0.3923	0.3345	0.3302	0.3210

Table 3 presents summary statistics of estimates of trading costs for 274 NYSE-listed stocks in the 1990 sample period over five intraday trading intervals. Specifically the mean coefficient estimate across the stocks, the mean standard error of the mean estimates, the standard deviation of the estimates across the 274 stocks, and the median estimate are provided for various parameters of interest: *s*, the implied spread; *r*, the fraction of the implied spread attributable to asymmetric information; s^E, the effective bid-ask spread; and r^E, the ratio of the effective to the implied spread.

$$s = 2(\hat{\phi} + \hat{\theta})$$

$$r = \frac{\hat{\theta}}{(\hat{\phi}+\hat{\theta})}$$

III. Event study methodology

References:

- Boehmer et al. (2002), Ch. 5 + 6
- Campbell et al. (1997), Ch. 4

Notational conventions

In asset pricing module we have used notation R_t^i to denote **gross** return of asset i in period t.

$$R_t^i = \frac{p_t^i + d_t^i}{p_{t-1}^i}$$

dividend

price

Following Campbell et al. (1997)

$$R_{it} = \frac{p_t^i + d_t^i}{p_{t-1}^i} - 1$$

denotes net returns of asset i in period t.

Event study philosophy (1)

Studies of stock market responses (valuation of firms) to public announcements of new value-relevant information:

- stock splits Fama (1969) pioneering work
- earnings announcements
- announcement of merger or acquisition
- macroeconomic announcements (interest rates, unemployment)
- liquidity stocks (Gomber et al. 2004)

Event study philosophy (2)

Studies of stock market responses (valuation of firms) to public announcements of new value-relevant information:

- regulatory environment
- issues of new debt or equity
- IPOs (a special case: no estimation period available)

One of the most widely used techniques in empirical finance.

Event study philosophy (3)

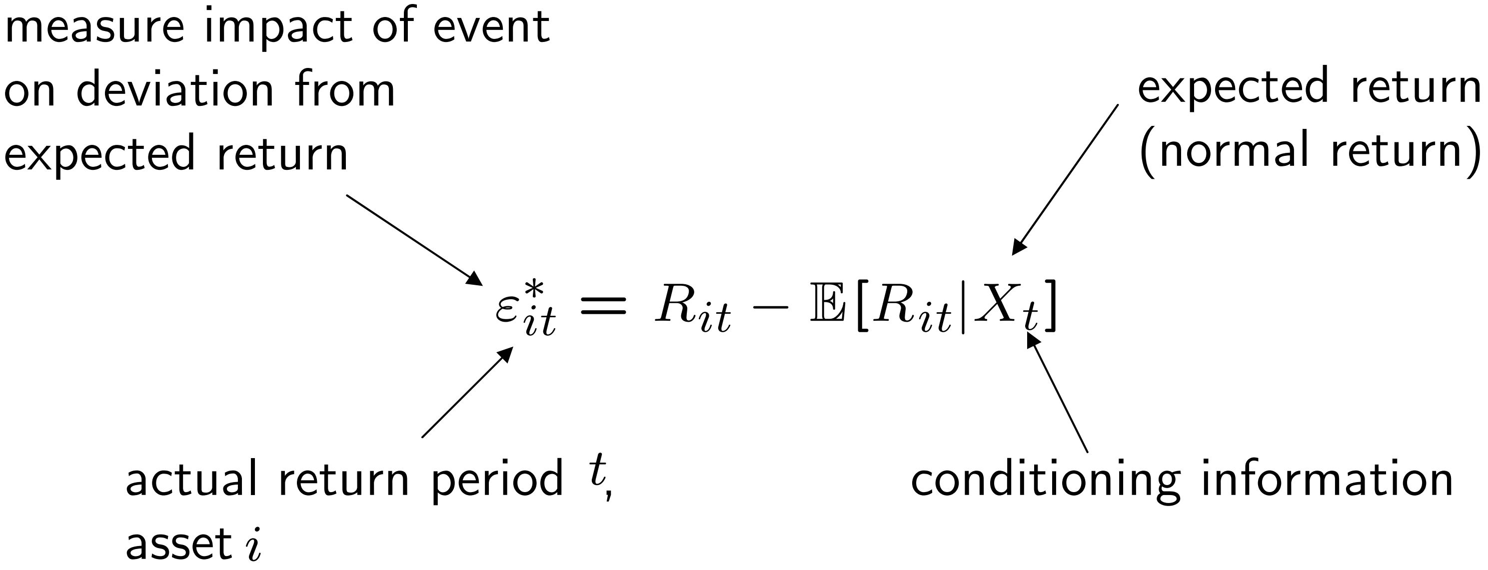
Idea:
Markets are efficient w.r.t public information. Asset prices should reflect relevant information i.e. react quickly to value relevant effect.

Measure price changes around events and compare "normal" or "expected" price changes.

Expected price change calculated based on conditioning information (e.g. market-wide changes)

Event study methodology focusses on abnormal returns

Can deviations from normal (or expected) returns be attributed to event or is it just a random fluctuation due to public information?



measure impact of event on deviation from expected return

expected return (normal return)

$$\varepsilon^*_{it} = R_{it} - \mathbb{E}[R_{it}|X_t]$$

actual return period t, asset i

conditioning information

Needed:
Distribution of abnormal return under null hypothesis of no effect of event

Classical event study methodology based on the multivariate normal assumption of cross-sectional returns (1)

R_t $(N \times 1)$ vector of asset returns for calendar time period t.
R_t independently multivariate normally distributed with mean μ and covariance matrix Ω for all t.

$$cov(R_t) = \begin{bmatrix} var(R_{1t}) & \cdots & \\ cov(R_{1t}, R_{2t}) & & \\ \vdots & \ddots & \\ cov(R_{1t}, R_{Nt}) & & var(R_{Nt}) \end{bmatrix} = \begin{bmatrix} \sigma_1^2 & \cdots & \sigma_{1N} \\ \sigma_{12} & & \\ \vdots & \ddots & \\ \sigma_{1N} & & \sigma_N^2 \end{bmatrix}$$

Classical event study methodology based on the multivariate normal assumption of cross-sectional returns (2)

R_t $(N \times 1)$ vector of asset returns for calendar time period t.
R_t independently multivariate normally distributed with mean μ and covariance matrix Ω for all t.

$$\mu = \begin{bmatrix} \mathbb{E}(R_{1t}) \\ \mathbb{E}(R_{2t}) \\ \vdots \\ \mathbb{E}(R_{Nt}) \end{bmatrix} = \begin{bmatrix} \mu_1 \\ \mu_2 \\ \vdots \\ \mu_N \end{bmatrix}$$

The constant return model does not use conditioning information

$$R_{it} = \mu_i + \xi_{it}$$

$$\mathbb{E}(\xi_{it}) = 0 \quad var(\xi_{it}) = \sigma^2_{\xi_i} = \sigma^2_i$$

$$cov \begin{pmatrix} \xi_{it} \\ \vdots \\ \xi_{Nt} \end{pmatrix} = cov(R_t) = \Omega$$

The market model is widely used

$$R_{it} = \alpha_i + \beta_i R_{mt} + \varepsilon_{it}$$

$$\mathbb{E}(\varepsilon_{it}) = 0 \quad var(\varepsilon_{it}) = \sigma^2_{\varepsilon_i}$$

Note:
This is not the CAPM!
R_{mt} is just one (yet powerful) conditioning variable
Other statistical models: factor models

Effect of conditioning: Variance of abnormal returns reduced
More precise detection of effect of event otherwise drowned in noise

Statistical background and motivation for the market model (1)

We assumed: $R_t = \mathcal{N}(\mu, \Omega)$

Market portfolio return is a linear combination of jointly normally distributed variables (the stock returns)

$$\begin{pmatrix} R_t \\ R_t^m \end{pmatrix} \sim \mathcal{N}(\mu^*, \Omega^*) \quad \mu^* = \begin{pmatrix} \mu \\ \mu_m \end{pmatrix} \quad \Omega^* = \begin{pmatrix} \Omega & \Sigma' \\ \Sigma & \sigma_m^2 \end{pmatrix}$$

bivariate normal distribution

$$\Sigma = [cov(R_{1t}, R_{mt}), \ldots, cov(R_{Nt}, R_{mt})] = [\sigma_{1m}, \ldots, \sigma_{Nm}]$$

Recall:

linear combinations of (multivariate) normally distributed random variables yield (multivariate) normal random vectors!

Statistical background and motivation for the market model (2)

Thus

$$\begin{pmatrix} R_{it} \\ R_{mt} \end{pmatrix} \sim \mathcal{N}\left[\begin{pmatrix} \mu_i \\ \mu_m \end{pmatrix}, \begin{pmatrix} \sigma^2_{\xi_i} & \sigma_{im} \\ \sigma_{im} & \sigma^2_m \end{pmatrix}\right]$$

A familiar result $R_{it}|R_{mt} \sim \mathcal{N}[\mathbb{E}(R_{it}|R_{mt}), var(R_{it}|R_{mt})]$ where

$$\mathbb{E}(R_{it}|R_{mt}) = \mu_i + \frac{cov(R_{mt}, R_{it})}{var(R_{mt})} \ \ [R_{mt} - \mu_m] = \alpha_i + \beta_i \ \ R_{mt}$$

$$\alpha_i = \mu_i - \beta_i \mu_m \qquad \beta_i = \frac{cov(R_{mt}, R_{it})}{var(R_{mt})}$$

$$var(R_{it}|R_{mt}) = \sigma^2_{\xi_i}(1 - \rho^2_{im}) \qquad \rho_{im} = corr(R_{it}, R_{mt})$$

Statistical background and motivation for the market model (3)

Define $\varepsilon_{it} = R_{it} - \mathbb{E}(R_{it}|R_{mt})$
We have

$$\mathbb{E}(\varepsilon_{it}|R_{mt}) = 0$$

$$var(\varepsilon_{it}|R_{mt}) = var(R_{it}|R_{mt}) = \sigma^2_{\xi_i}(1 - \rho^2_{im}) = \sigma^2_{\varepsilon_i}$$

and $\varepsilon_{it}|R_{mt} \sim \mathcal{N}(0, \sigma^2_{\varepsilon_i})$ homoskedastic innovations
Hence

$$R_{it} = \mathbb{E}(R_{it}|R_{mt}) + \varepsilon_{it} \;\; = \;\; \alpha_i + \beta_i R_{mt} + \varepsilon_{it}$$

where $\alpha_i = \mu_i - \beta_i \mu_m \qquad \beta_i = \frac{cov(R_{mt}, R_{it})}{var(R_{mt})}$

$\Rightarrow$ Market model follows from assumption of joint normality (and not asset pricing theory)

Why not using asset pricing model (like CAPM) to form normal or expected returns? (1)

- You could (actually frequently done in earlier studies), but currently out of fashion
- empirical failure of models
- other successful models in sight?
- time varying parameters

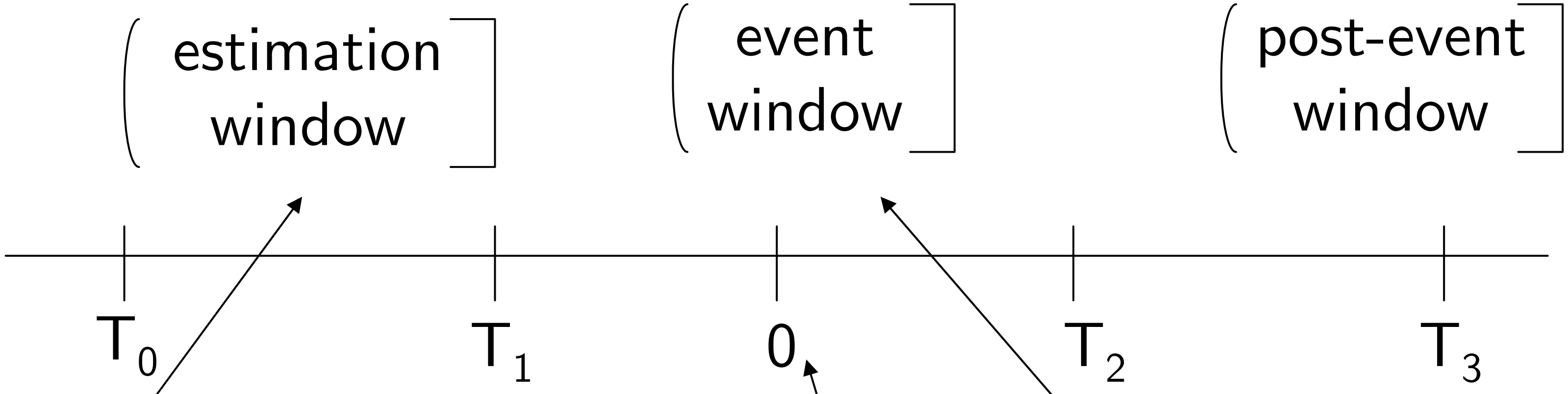
Why not using asset pricing model (like CAPM) to form normal or expected returns? (2)

- more complicated estimation, constant mean-return model does basically same job!
- rather rely on statistical assumptions than on false economic model
- using conditioning information allows more precise conclusions

Campbell et al. (1997): ”There seems to be no good reason to use an economic model rather than a statistical model in an event study”

Time line for an event study

Time line:



we use returns of this period to estimate the parameters of the market model

the event date τ

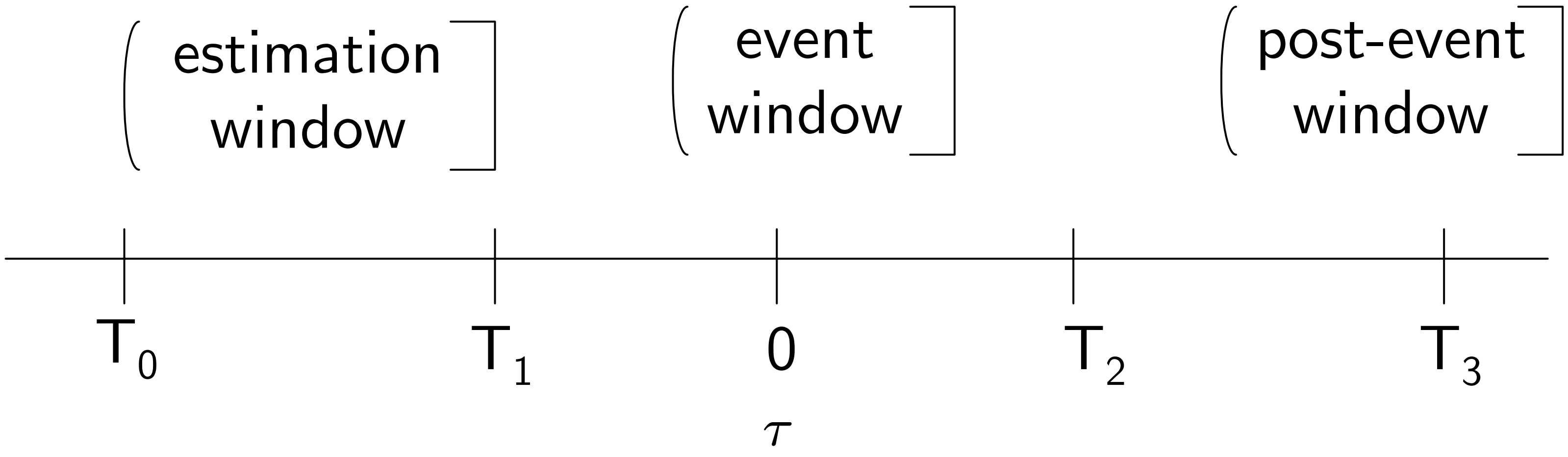
So-called event time (time relative to event)

We measure abnormal returns in this period
Can we attribute them to
The occurrence of the event
Event window possibly before event date (information leakage)

sometimes no data for estimation available, e.g. IPO‘s then use market adjusted return model

$$R_{it} = \beta R_m$$

Time line and timing convention (1)



$\tau = 0$ event date

$\tau = T_1 + 1$ to $\tau = T_2$ event window

$L_1 = T_1 - T_0$ length of estimation window

$L_2 = T_2 - T_1$ length of event window

$L_3 = T_3 - T_2$ length of post event window

Time line and timing convention (2)

In order to get a ”clean” estimation period: Stop some time before event window.

Usually: daily frequency

Analyze abnormal returns in sub-interval of event window $[\tau_1, \tau_2]$
$\tau_1 \geq T_1$ $\tau_2 \leq T_2$

Example: $T_0 = -250$
$T_1 = -20$
event period $\tau \in [-5, 0]$
days $-19, -18, ... - 6$ not considered for estimation and not used for analysis of abnormal returns.

The recipe for an event study (1)

1. Define event of interest: Make sure you can allocate the event date (or at least the event window)

2. Define criteria to select firms/events
   Examples:

   - membership in an industry

   - small/large firms

   - tech-stocks

3. Decide model for measuring normal performance (e.g. market model)

The recipe for an event study (2)

4. Estimate parameters (of market model or mean return) using data in estimation window

5. Calculate abnormal returns in event period
   Test significance of abnormal returns

6. Empirical results, interpretation and conclusions

The recipe for an event study (3)

Basic question:
Is the unconditional distribution of the abnormal returns different from the distribution of abnormal returns conditional on the occurrence of the event
(i.e. the distribution of abnormal returns in the event period)

Null-hypothesis: conditional distribution of abnormal returns is not affected by event!

Estimation of market model (1)

Note:
Event assumed to be **exogenous** w.r.t. market value of security

Undisputed for macroeconomic announcements, regulatory environment changes

Problematic for IPO, stock splits

Revision of value of firm caused by event and not the other way round

Estimation of market model (2)

Estimation of market model parameters using estimation period data by OLS (or give GMM interpretation)

$$R_{i\tau} = \alpha_i + \beta_i R_{m\tau} + \varepsilon_{i\tau}$$

OLS yields

Estimation window
Sample covariance

$$\widehat{\alpha}_i = \frac{1}{L_1} \sum_{\tau=T_0+1}^{T_1} R_{i\tau} \qquad \widehat{\beta}_i = \frac{1}{L_1} \sum_{\tau=T_0+1}^{T_1} R_{m\tau}$$

$$\widehat{\beta}_i = \frac{L_1 \sum R_{i\tau} \cdot R_{m\tau} - \sum R_{i\tau} \sum R_{m\tau}}{L_1 \sum R_{m\tau}^2 - [\sum R_{m\tau}]^2}$$

Estimation window
Sample variance

Properties of abnormal returns (1)

In matrix notation

$$R_i = X_i\theta_i + \varepsilon_i$$

$$R_i = [R_{iT_0+1}, \ldots, R_{iT_1}]'$$

$$X_i = [\iota, R_m] \quad R_m = [R_{mT_0+1}, \ldots, R_{mT_1}]' \quad \iota = [1, 1, \ldots, 1]'$$

$$\theta_i = (\alpha_i, \beta_i)'$$

$$min \sum (R_{i\tau} - \alpha_i - \beta_i R_{m\tau})^2$$

sum over estimation period data

yields

$$\widehat{\theta}_i = (X_i' X_i)^{-1} X_i' R_i$$

standard OLS formula

Properties of abnormal returns

an unbiased estimate of $var(\varepsilon_{i\tau}) = \sigma^2_{\varepsilon i}$:

$$\widehat{\sigma}^2_{\varepsilon_i} = \frac{1}{L_1 - 2}\widehat{\varepsilon}'_i\widehat{\varepsilon}_i$$

$$\widehat{\varepsilon}_i = R_i - X_i\widehat{\theta}_i$$

$$var[\widehat{\theta}_i] = (X'_iX_i)^{-1}\sigma^2_{\varepsilon_i}$$

Standard OLS results under conditional homoskedasticity and absence of serial correlation of residuals and predetermined regressors or strict exogeneity

Properties of estimators are as usual returns in event period

Assuming:

$$\mathbb{E}(\varepsilon_{i\tau}|R_m) = 0$$

(strict exogeneity) OLS unbiased

With

$$var(\varepsilon_{i\tau}|R_m) = \sigma^2_{\varepsilon_i} \quad cov(\varepsilon_{i\tau}, \varepsilon_{i\tau'}) = 0 \quad \forall\tau' \neq \tau$$

assuming only $\mathbb{E}(\varepsilon_{i\tau}, R_{m\tau}) = 0$: OLS consistent

Properties of abnormal returns in event period (under null-hypothesis of no effect of event) (1)

With only one day event window

Estimated abnormal return at event day

$$\widehat{\varepsilon}^{*}_{i0} = R_{i0} - \widehat{\alpha}_i - \widehat{\beta}_i R_{m0} = R_{i0} - X_{i0}\widehat{\theta}_i$$

$$X_{i0} = (1, R_{m0}) \quad \widehat{\theta}_i = \left[\begin{array}{c} \widehat{\alpha}_i \\ \widehat{\beta}_i \end{array} \right]$$

Properties of abnormal returns in event period (under null-hypothesis of no effect of event) (2)

$$
\begin{aligned}
\mathbb{E}(\widehat{\varepsilon}^{*}_{i0}|R_{m0}) &= \mathbb{E}(R_{i0} - x_{i0}\widehat{\theta}|R_{m0}) \\
&= \mathbb{E}(R_{i0} - x_{i0}\theta_i - x_{i0}(\widehat{\theta}_i - \theta_i)|R_{m0}) \\
&= \mathbb{E}(R_{i0} - x_{i0}\theta_i|R_{m0}) - x_{i0}\mathbb{E}(\widehat{\theta}_i - \theta_i|R_{m0}) \\
&= 0 - 0 = 0
\end{aligned}
$$

Properties of abnormal returns in event period (under null-hypothesis of no effect of event) (3)

Same goes for multiple day event window

$$\widehat{\varepsilon}_i^* = R_i^* - \widehat{\alpha}_i \iota - \widehat{\beta}_i R^* = R_i^* - X_i^* \widehat{\theta}_i$$

$$\widehat{\varepsilon}_i^* = (\widehat{\varepsilon}_{iT_1+1}, \ldots, \widehat{\varepsilon}_{iT_2})' \longleftarrow$$

vector of abnormal return

$$\underbrace{R^* = [R_{iT_1+1}, \ldots, R_{iT_2}]' \quad X_i^* = [\iota, R_m^*] \quad R_m^* = [R_{mT_{1+1}}, \ldots, R_{mT_2}]'}_{\text{event period data}}$$

$$\begin{aligned} \mathbb{E}[\widehat{\varepsilon}_i^* | X_i^*] &= \mathbb{E}[R_i^* - X_i^* \widehat{\theta}_i | X_i^*] \\ &= \mathbb{E}[(R_i^* - X_i^* \theta_i) - X_i^* (\widehat{\theta}_i - \theta_i) | X_i^*] = 0 \end{aligned}$$

(under H_0)

Note:

We implicitly assume strict exogeneity, i.e.

$\mathbb{E}(\varepsilon_{i\tau} | R_m^*) = 0 \quad \forall \tau$ where $\varepsilon_{i\tau} = R_{i\tau} - \alpha_i - \beta_i R_{m\tau}$

Properties of abnormal returns in event period (1)

$$\begin{aligned}
V_i &= \mathbb{E}\Big[\widehat{\varepsilon}_i^*\widehat{\varepsilon}_i^{*\prime}|X_i^*\Big] \\
&= \mathbb{E}\Big[[\varepsilon_i^* - X_i^*(\widehat{\theta}_i - \theta_i)][\varepsilon_i^* - X_i^*(\widehat{\theta}_i - \theta_i)]'|X_i^*\Big] \\
&= \mathbb{E}\Big[\varepsilon_i^*\varepsilon_i^{*\prime} - \varepsilon_i^*(\widehat{\theta}_i - \theta_i)'X_i^{*\prime} - X_i^*(\widehat{\theta}_i - \theta_i)\varepsilon_i^{*\prime} + X_i^*(\widehat{\theta}_i - \theta_i)(\widehat{\theta}_i - \theta_i)'X_i^{*\prime}|X_i^*\Big] \\
&= I\sigma_{\varepsilon_i}^2 + X_i^*(X_i'X_i)^{-1}X_i^{*\prime}\sigma_{\varepsilon_i}^2
\end{aligned}$$

Properties of abnormal returns in event period (2)

More precisely we would write $\mathbb{E}(\widehat{\varepsilon}_i^* \widehat{\varepsilon}_i^{*\prime} | X_i^*, X_i, R_i)'$.

For a single day event window this becomes

$$var(\widehat{\varepsilon}_{i0}|R_{m0}) = \mathbb{E}(\widehat{\varepsilon}_{i0}|R_{m0}) = \sigma^2_{\varepsilon_i}\Big(1 + \underbrace{(1, R_{m0})(X_i'X_i)^{-1}\begin{bmatrix} 1 \\ R_{m0} \end{bmatrix}\Big)}_{\text{“parameter uncertainty“}}$$

$\widehat{\varepsilon}_i^*|X_i^* \sim \mathcal{N}(0, V_i)$ or more precisely $\widehat{\varepsilon}_i^*|X_i^*, X_i, R_i \sim \mathcal{N}(0, V_i)$

Properties of abnormal returns in event period (some details)

Note that for

$$\widehat{\varepsilon}_{i0} = R_{i0} - \left[\begin{array}{c} 1 \\ R_{m0} \end{array} \right] \widehat{\theta} = R_{i0} - \left[\begin{array}{c} 1 \\ R_{m0} \end{array} \right] [X_i' X_i]^{-1} X' R_i$$

conditioning on R_{m0} and X_i and R_i:
$\widehat{\varepsilon}_{i0}$ a normally distributed random variable.

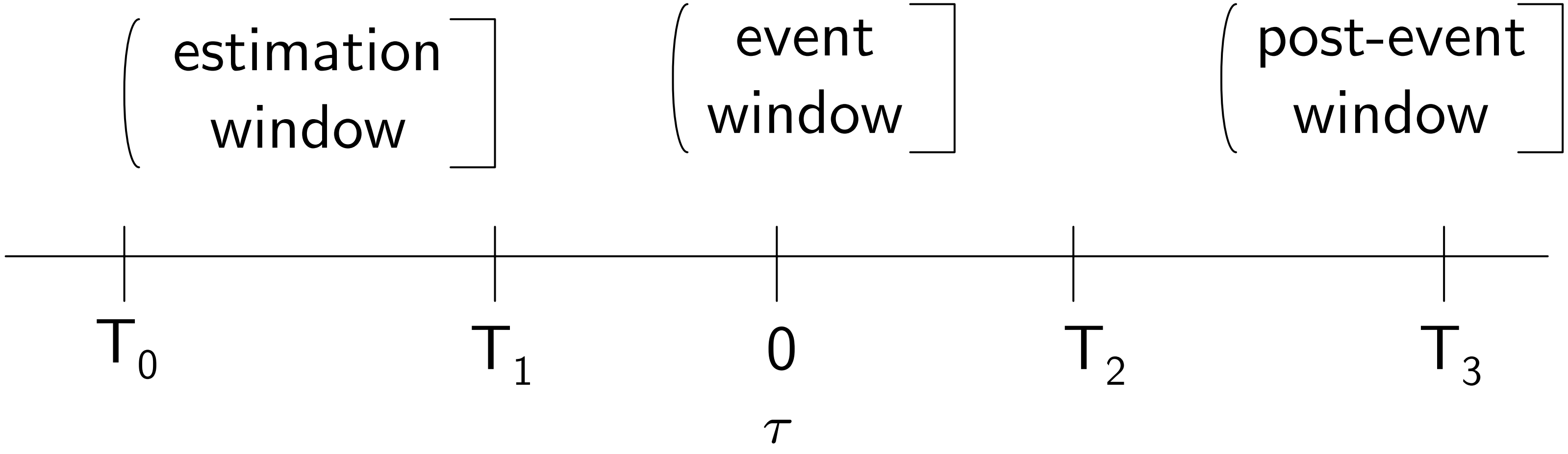
Same goes for

$$\widehat{\varepsilon}_i^* = [\widehat{\varepsilon}_{iT_1+1}, \ldots, \widehat{\varepsilon}_{iT_2}]'$$

conditioning on R_m^*, R_i and X_i
$\widehat{\varepsilon}_i^*$ results from linear combinations of normally distributed random variables $\underbrace{(R_{iT_1+1}, \ldots, R_{iT_2})'}_{\text{event period returns}}$

Abnormal returns of subinterval of event period are summed up to generate accumulative abnormal returns (1)

Campbell at al. (1997) convention

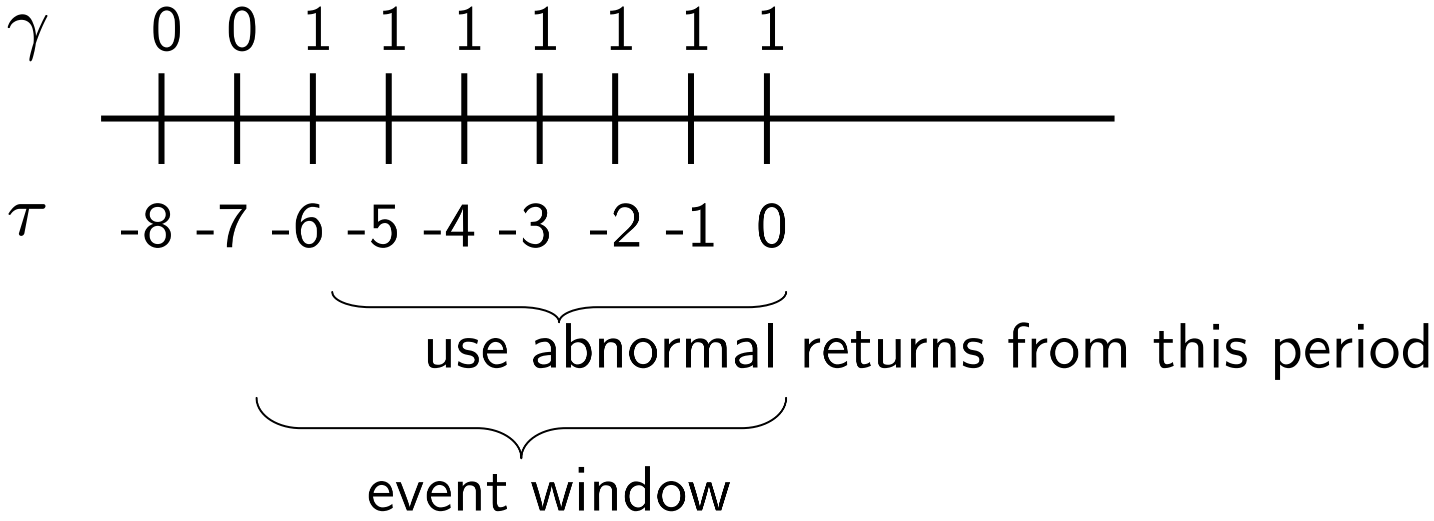


$\tau = 0$	event date
$\tau = T_1 + 1$ to $\tau = T_2$	event window
$L_1 = T_1 - T_0$	length of estimation window
$L_2 = T_2 - T_1$	length of event window
$L_3 = T_3 - T_2$	length of post event window

Abnormal returns of subinterval of event period are summed up to generate accumulative abnormal returns (2)

Pick subinterval from $(T_1, T_2]$, $[\tau_1, \tau_2]$ $\tau_1 > T_1$ $\tau_2 \leq T_2$ and cumulate abnormal returns $\Rightarrow$ $\text{CAR}(\tau_1, \tau_2)$

Define γ a $(L_2 \times 1)$ vector



γ 0 0 1 1 1 1 1 1 1

τ -8 -7 -6 -5 -4 -3 -2 -1 0

use abnormal returns from this period

event window

Abnormal returns of subinterval of event period are summed up to generate accumulative abnormal returns (3)

$$\widehat{CAR}(\tau_1, \tau_2) \equiv \gamma' \widehat{\varepsilon}_i^*$$

$$\Rightarrow \quad \mathbb{E}(\widehat{CAR}(\tau_1, \tau_2)) = 0 \quad \text{as } \mathbb{E}(\widehat{\varepsilon}_i^*) = 0$$

$$var[\widehat{CAR}(\tau_1, \tau_2)] = \sigma_i^2(\tau_1, \tau_2) = \gamma' V_i \gamma$$

Since linear combinations of normally distributed random variables are normally distributed

$$\widehat{CAR}(\tau_1, \tau_2) \sim \underbrace{\mathcal{N}\Big(0, \sigma_i^2(\tau_1, \tau_2)\Big)}_{\text{univariate normal}}$$

Abnormal returns of subinterval of event period are summed up to generate accumulative abnormal returns (4)

In $V_i = \sigma^2_{\varepsilon_i}(I - X_i^*(X_i'X_i)^{-1}X_i^{*\prime})$ $\sigma^2_{\varepsilon_i}$ not known,

consistently estimated by

$$\widehat{\sigma}^2_{\varepsilon_i} = \frac{1}{L_1 - 2}\sum \widehat{\varepsilon}_i^2 = \frac{1}{L_1 - 2}\widehat{\varepsilon}_i'\widehat{\varepsilon}_i$$

$$\widehat{V}_i = \widehat{\sigma}^2_{\varepsilon_i}(I - X_i^*(X_i'X_i)^{-1}X_i^{*\prime})$$

Abnormal returns of subinterval of event period are summed up to generate accumulative abnormal returns (5)

$$\widehat{V}_i = \widehat{\sigma}^2_{\varepsilon_i}(I - X^*_i(X'_iX_i)^{-1}X^{*\prime}_i)$$

Hence $\widehat{var}\Big(CAR(\tau_1, \tau_2)\Big) = \widehat{\sigma}^2_i(\tau_1, \tau_2) = \gamma'\widehat{V}_i\gamma$

or $\widehat{\sigma}^2_{\varepsilon_i}\gamma'\Big(I - X^*_i(X'_iX_i)^{-1}X^{*\prime}_i\Big)\gamma$

Distribution of test statistic under H_0

$$\widehat{SCAR}(\tau_1, \tau_2) = \frac{\widehat{CAR}(\tau_1, \tau_2)}{\widehat{\sigma}_i(\tau_1, \tau_2)} \sim t(L_1 - 2)$$

Proof analogous to proof that OLS t-statistic is t-distributed with degrees of freedom equal to a number of observations - number of regressors.

Abnormal returns of subinterval of event period are summed up to generate accumulative abnormal returns (6)

Properties of t-distribution

$$\mathbb{E}(t(k)) = 0 \qquad var(t(k)) = \frac{k}{k-2}$$

Hence

$$\mathbb{E}(\widehat{SCAR}(\tau_1, \tau_2)) = 0 \qquad var(\widehat{SCAR}(\tau_1, \tau_2)) = \frac{L_1 - 2}{L_1 - 4}$$

For L_1 (estimation window large) use standard normal approximation of $t(L_1 - 2)$
Single event: test significance of $\widehat{SCAR}(\tau_1, \tau_2)$ under H_0.
Fix α (significance level).
Reject for large or small values of test statistic.

We analyse typically many events. Then we average abnormal returns (1)

For a sample of N events

Compute market model estimates and abnormal returns per event

Compute sample averages of abnormal returns

$$\overline{\varepsilon}^* = \frac{1}{N}\sum_{i=1}^{N}\overline{\varepsilon}_i^* = \underbrace{\begin{bmatrix} \frac{1}{N}\sum_{i=1}^{N}\overline{\varepsilon}_{iT_1+1}^* \\ \vdots \\ \frac{1}{N}\sum_{i=1}^{N}\overline{\varepsilon}_{iT_2}^* \end{bmatrix}}_{\text{averages per event day averaged over events}}$$

We analyse typically many events. Then we average abnormal returns (2)

Assume independence of abnormal returns across events (no overlap of event windows) and also ignore dependence induced by estimating $\widehat{\alpha}_i$ and $\widehat{\beta}_i$ that may exist.

$$\mathbb{E}(\overline{\varepsilon}^*) = 0$$

$cov(\overline{\varepsilon}^*) = V = \frac{1}{N^2}\sum_{i=1}^{N} V_i$ for one event day $var(\overline{\varepsilon}_0^*) = \frac{1}{N^2}\sum var(\widehat{\varepsilon}_{i0})$

variance of average
event day abnormal returns

Using the averaged abnormal returns we proceed as in the single event case (1)

$$\overline{CAR}(\tau_1, \tau_2) \equiv \gamma' \overline{\varepsilon}^*$$

$$var[\overline{CAR}(\tau_1, \tau_2)] = \overline{\sigma}^2(\tau_1, \tau_2) = \gamma' V \gamma$$

Equivalently average the cumulative abnormal returns across securities

$$\left. \overline{CAR}(\tau_1, \tau_2) = \frac{1}{N} \sum_{i=1}^{N} \widehat{CAR}_i(\tau_1, \tau_2) \quad \right\} \begin{array}{l} \text{approximately} \\ \text{normal for} \\ N \text{ large} \end{array}$$

$$var[\overline{CAR}(\tau_1, \tau_2)] = \overline{\sigma}^2(\tau_1, \tau_2) = \frac{1}{N^2} \sum_{i=1}^{N} \sigma_i^2(\tau_1, \tau_2)$$

Using the averaged abnormal returns we proceed as in the single event case (2)

$$\overline{CAR}(\tau_1, \tau_2) \sim \mathcal{N}\Big(0, \bar{\sigma}^2(\tau_1, \tau_2)\Big)$$

we can use

$$\frac{1}{N^2}\sum_{i=1}^{N} \hat{\sigma}_i^2(\tau_1, \tau_2) \equiv \hat{\bar{\sigma}}^2(\tau_1, \tau_2)$$

$\gamma' \widehat{V}_i \gamma$

variance of abnormal return

to consistently estimate $\sigma^2(\tau_1, \tau_2)$

Using a central limit theorem

$$\left. J_1 = \frac{\overline{CAR}(\tau_1, \tau_2)}{[\hat{\bar{\sigma}}^2(\tau_1, \tau_2)]^{\frac{1}{2}}} \overset{a}{\sim} \mathcal{N}(0, 1) \quad \right\}$$

approximation works well for large number of events

under H_0 that event does not influence distribution of abnormal returns

An alternative method averages standardized cumulative returns (1)

Equal weighting of events using

$$\overline{SCAR}(\tau_1, \tau_2) = \frac{1}{N} \sum_{i=1}^{N} \underbrace{\widehat{SCAR}_i(\tau_1, \tau_2)}$$

$$\downarrow$$

distributed (see above)

$$t(L_1 - 2) \quad \mathbb{E}(\widehat{SCAR}(\tau_1, \tau_2)) = 0$$

$$var(\widehat{SCAR}(\tau_1, \tau_2)) = \tfrac{L_1-2}{L_1-4}$$

$$var(\overline{SCAR}(\tau_1, \tau_2)) = \tfrac{1}{N^2} N \tfrac{L_1-2}{L_1-4} = \tfrac{1}{N}\left[\tfrac{L_1-2}{L_1-4}\right]$$

An alternative method averages standardized cumulative returns (2)

coverges in distribution to a normal distribution (for large sample of events)

By a central limit theorem

$$J_2 = \frac{\overline{SCAR}(\tau_1, \tau_2)}{\sqrt{\frac{L_1-2}{NL_1-4}}} \overset{a}{\sim} \mathcal{N}(0, 1)$$

Advantage:

Reduces effect of stocks with large return standard deviations on test statistic

Patell´s test statistic accounts for different estimation period lenghts

CAR standardized by estimation period standard deviation

$$t_{Patell} = \frac{\sum\limits_{i=1}^{N} \widehat{SCAR}_i(\tau_1, \tau_2)}{\sqrt{\sum\limits_{i=1}^{N} \frac{L_{1i}-2}{L_{1i}-4}}} \overset{a}{\sim} N(0,1)$$

~N for longer estimation periods

length of estimation period varies across events

$$t_{Patell} = \frac{\sum\limits_{i=1}^{N} \widehat{SCAR}_i(\tau_1, \tau_2)}{\sqrt{N}} \overset{a}{\sim} N(0,1)$$

Campbell et al.‘s (1997) application (1)

600 earnings announcements

Dow Jones firms Jan. 1989 - Dec. 1993

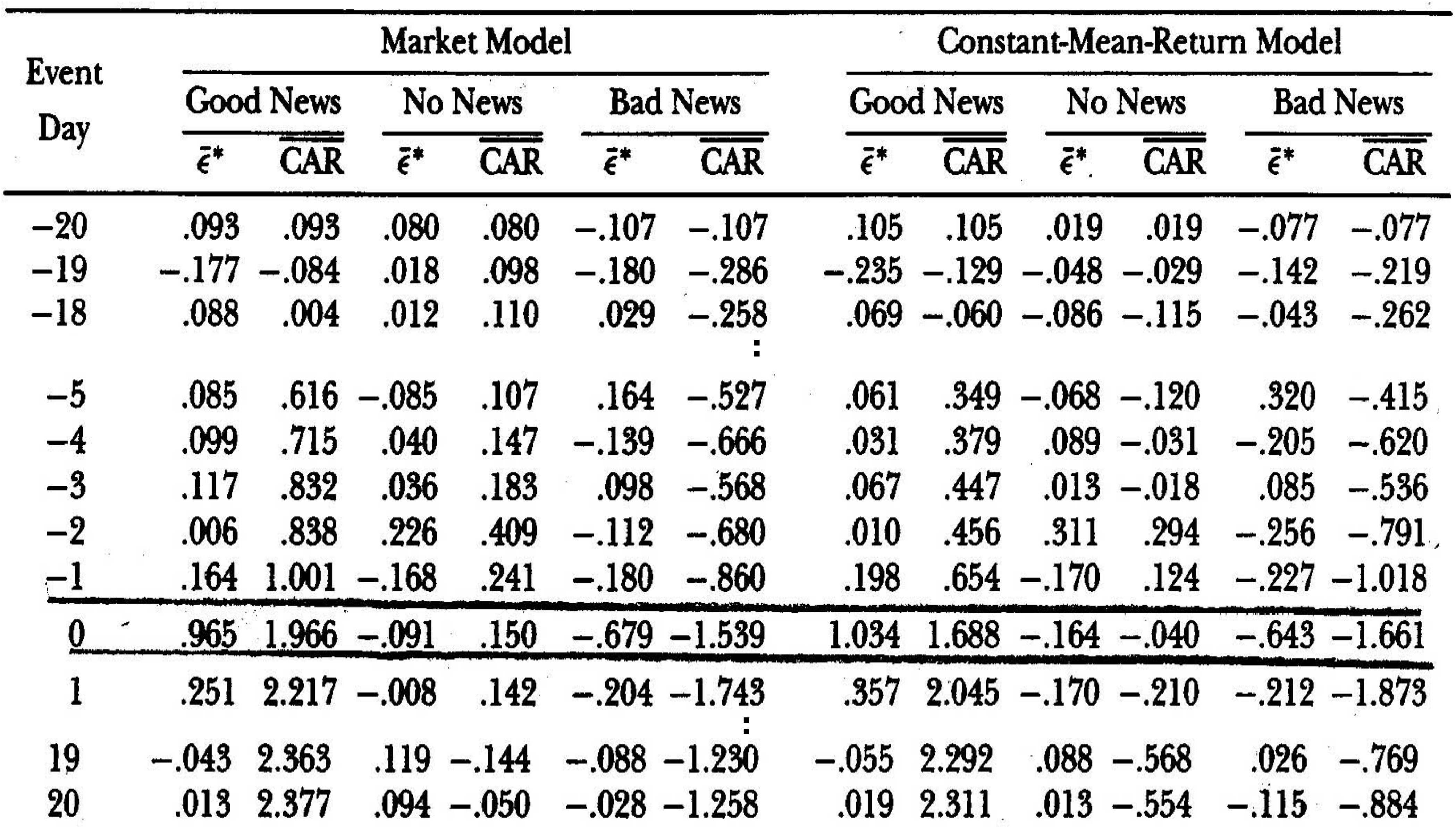
Data stream: data of announcement

Compustat: actual earnings (per quarter)

Institutional Brokers Estimate System (I|B|E|S): mean (over analysts) quarterly earnings forecast

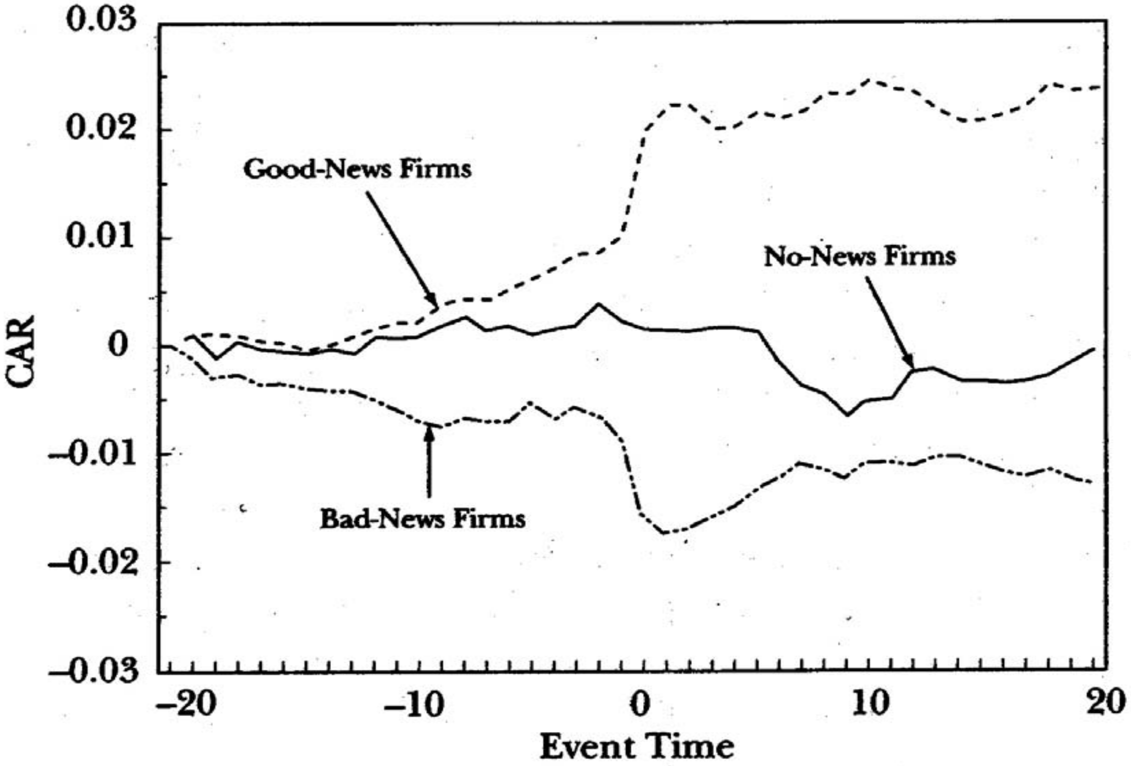
⇒ Deviation of actual earnings from forecast: good news (189), no news (173), bad news (238)

Campbell et al.‘s (1997) application (2)



Event Day	Market Model						Constant-Mean-Return Model					
	Good News		No News		Bad News		Good News		No News		Bad News	
	$\bar{\epsilon}^*$	$\overline{CAR}$	$\bar{\epsilon}^*$	$\overline{CAR}$	$\bar{\epsilon}^*$	$\overline{CAR}$	$\bar{\epsilon}^*$	$\overline{CAR}$	$\bar{\epsilon}^*$	$\overline{CAR}$	$\bar{\epsilon}^*$	$\overline{CAR}$
−20	.093	.093	.080	.080	−.107	−.107	.105	.105	.019	.019	−.077	−.077
−19	−.177	−.084	.018	.098	−.180	−.286	−.235	−.129	−.048	−.029	−.142	−.219
−18	.088	.004	.012	.110	.029	−.258	.069	−.060	−.086	−.115	−.043	−.262
						⋮						
−5	.085	.616	−.085	.107	.164	−.527	.061	.349	−.068	−.120	.320	−.415
−4	.099	.715	.040	.147	−.139	−.666	.031	.379	.089	−.031	−.205	−.620
−3	.117	.832	.036	.183	.098	−.568	.067	.447	.013	−.018	.085	−.536
−2	.006	.838	.226	.409	−.112	−.680	.010	.456	.311	.294	−.256	−.791
−1	.164	1.001	−.168	.241	−.180	−.860	.198	.654	−.170	.124	−.227	−1.018
0	.965	1.966	−.091	.150	−.679	−1.539	1.034	1.688	−.164	−.040	−.643	−1.661
1	.251	2.217	−.008	.142	−.204	−1.743	.357	2.045	−.170	−.210	−.212	−1.873
						⋮						
19	−.043	2.363	.119	−.144	−.088	−1.230	−.055	2.292	.088	−.568	.026	−.769
20	.013	2.377	.094	−.050	−.028	−1.258	.019	2.311	.013	−.554	−.115	−.884

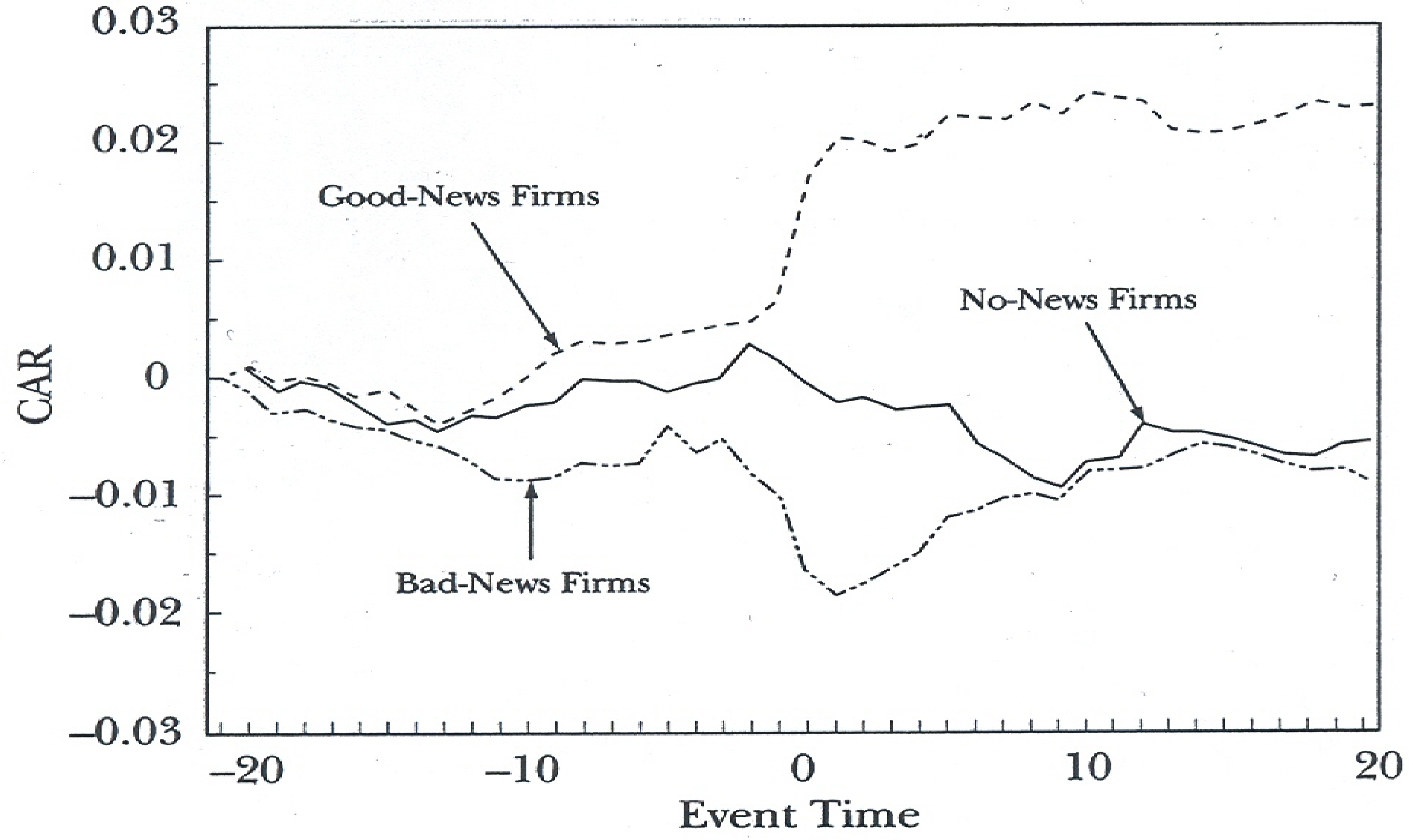
Campbell et al.‘s (1997) application (3)



Focusing only on event day $\widehat{\text{CAR}} = \overline{\varepsilon^*} = 0.965\%$
$\sqrt{\frac{1}{\theta}^2(0,0)} = 0.104\%$ $J_1 = 9.28$

under H_0: $J_1 \overset{a}{\sim} N(0,1) \Rightarrow H_0$ rejected on conventional significance levels

Campbell et al.‘s (1997) application (4)



Why using the market model instead of the constant-mean-return model?

Variance of abnormal return for market model

$$
\begin{aligned}
\sigma^2_{\varepsilon_i} &= var[R_{it} - \alpha_i - \beta_i R_{mt}] \\
&= var[R_{it}] - \beta_i^2 var[R_{mt}] \\
&= (1 - \beta_i^2) var[R_{mt}] \\
&= (1 - R_i^2) var[R_{it}]
\end{aligned}
$$

$R_i^2 = R^2$ of market-model regression for security i

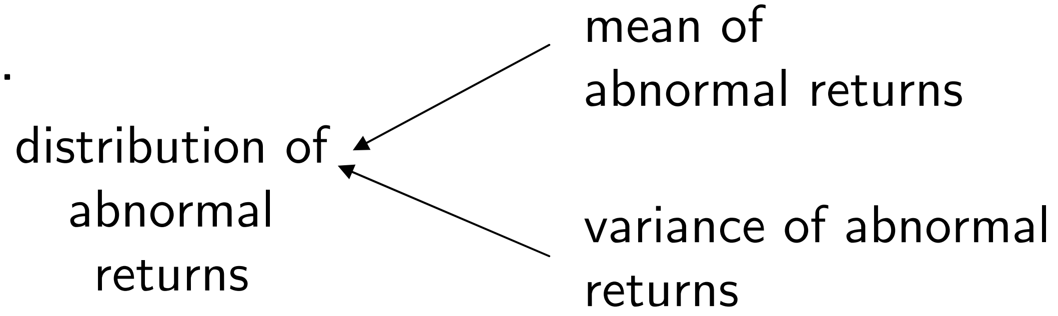
$$
\begin{aligned}
\sigma^2_{\xi_i} &= var[R_{it} - \mu_i] = var[R_{it}] \\
\sigma^2_{\varepsilon_i} &= (1 - R_i^2)\sigma^2_{\xi_i}
\end{aligned}
$$

Since R_i^2 lies between 0 and 1 abnormal return variance of market model less than or equal to abnormal return using constant-mean-return model

Boehmer et. al (1991) propose a modified Null Hypothesis

H_0: given event has no impact on behavior (more precisely: distribution) of security returns

we assume normality i.e.



distribution of abnormal returns

mean of abnormal returns

variance of abnormal returns

Boehmer er al. (1991): Isolation of and testing for mean effect cross sectional approach to estimate abnormal return variances so far: estimation of variance of abnormal returns using estimation period data!

$$V_i = I\hat{\sigma}^2_{\varepsilon_i} + X_i^*(X_i'X_i)^{-1}X_i^{*\prime}\hat{\sigma}^2_{\varepsilon_i}$$

Two ways to account for event induced abnormal return variances

„Standard cross sectional test“:
Standardize cross section average CAR by event period cross sectional standard deviation**:**

$$\widehat{Var}\left[\overline{CAR}(\tau_1,\tau_2)\right] = \frac{1}{N^2}\sum_{i=1}^{N}\left(CAR_i(\tau_1,\tau_2)-\overline{CAR}(\tau_1,\tau_2)\right)^2$$

$$\widehat{Var}\left[\overline{SCAR}(\tau_1,\tau_2)\right] = \frac{1}{N^2}\sum_{i=1}^{N}\left(SCAR_i(\tau_1,\tau_2)-\overline{SCAR}(\tau_1,\tau_2)\right)^2$$

Standardize cross sectional average SCAR by cross sectional variance of the SCAR

Consistency of estimators requires abnormal returns to be cross sectionally uncorrelated (covariances have to vanish).

Avoid overlapping event periods (clustering)

Boehmer (2002) considers two test statistics that account for event induced abnormal variance changes

Standard cross sectional test

$$t_{cs} = \frac{\overline{CAR}(\tau_1, \tau_2)}{\sqrt{\widehat{VAR}(\overline{CAR}(\tau_1, \tau_2))}} \overset{a}{\sim} N(0,1)$$

Problem: not a consistent estimate if event induced variance different across stocks/events

Standardized cross sectional test (Boehmer et al. 1991)

$$t_{BMP} = \frac{\overline{SCAR}(\tau_1, \tau_2)}{\sqrt{\widehat{VAR}(\overline{SCAR}(\tau_1, \tau_2))}} \overset{a}{\sim} N(0,1)$$