Testing conditional predictions of asset pricing models:
Scaled returns (managed portfolios) and scaled factors
Readings: Cochrane (2002), Ch. 8, 10

We use instruments to test the conditional predictions of asset pricing models

$p_t = \mathbb{E}\left(m_{t+1}(b) \cdot x_{t+1} | I_t\right)$ or $1 = \mathbb{E}\left(m_{t+1}(b) \cdot R_{t+1} | I_t\right)$
or $0 = \mathbb{E}\left(m_{t+1}(b) \cdot R^e_{t+1} | I_t\right)$

I.i.e "integrates out" conditional implications, lets us focus on unconditional implications of asset pricing model (model for S.D.F.):
$\mathbb{E}\left(m_{t+1}(b) \cdot R_{t+1} - 1\right) = 0$

To test conditional implications write
$\mathbb{E}\left(Y_{t+1} | I_t\right) = 0$ where $Y_{t+1} = (m_{t+1}(b) \cdot R_{t+1} - 1)$ or ...
$\{Y_{t+1}\}$ a martingale difference sequence.

Properties of m.d.s include:
$cov\left(y_{t+1}, z_t\right) = 0 \quad \forall \quad z_t \in I_t$
$\mathbb{E}\left(y_{t+1} z_t\right) = 0$ since $1 \in I_t$
Testable restrictions therefore: $\mathbb{E}\left[(m_{t+1}(b) \cdot R_{t+1} - 1) z_t\right] = 0 \ \forall \quad z_t \in I_t$

The use of instruments has an economic interpretation: Can the model price "managed portfolios"?

$\tilde{x}_{t+1} = x^i_{t+1} z_t$ conceived as (payoff of) **managed portfolios**,
i.e. artificial assets.
Example: $z_t = \frac{d_t}{p_t}$ invest if $z_t \uparrow$
$\tilde{x}_{t+1}$ conceived as another payoff wtih price $z_t p_t$
If model correct, it prices any asset, also mgt. portfolios.

$$\underbrace{z_t p_t}_{p(\tilde{x}_{t+1})} = \mathbb{E}_t(m_{t+1}(b) \cdot \underbrace{x_{t+1} z_t}_{\tilde{x}_{t+1}}) \text{ or } z_t = \mathbb{E}_t\left(m_{t+1}(b) \cdot R_{t+1} z_t\right)$$

i.e.

$$\mathbb{E}(z_t) = \mathbb{E}(m_{t+1} R_{t+1} z_t) \quad \text{or} \quad \mathbb{E}[(m_{t+1} R_{t+1} - 1) z_t] = 0$$

To test the conditional implications you simply “blow up“ the number of assets by including meaningful managed portfolios and proceed as before.

Practice: N assets, M instruments
M moment restrictions

$$\mathbb{E}\left(\left[m_{t+1}(b)\,R_{t+1} - 1\right] \otimes z_t\right) = 0$$

With two assets and two instruments $z_t = (1, z_t^1)'$

$$\mathbb{E}\begin{bmatrix} m_{t+1}(b)\,R^a_{t+1} - 1 \\ m_{t+1}(b)\,R^b_{t+1} - 1 \\ (m_{t+1}(b)\,R^a_{t+1} - 1)z_t^1 \\ (m_{t+1}(b)\,R^b_{t+1} - 1)z_t^1 \end{bmatrix} = 0$$

or, emphasizing the managed portfolio interpretation

$$\mathbb{E}(m_{t+1}(b)\underbrace{R_{t+1} \otimes z_t}_{\text{payoff}} - \underbrace{1 \otimes z_t}_{\text{price}}) = 0$$

$$\mathbb{E}(m_{t+1}(b)\underbrace{x_{t+1} \otimes z_t}_{\text{payoff}} - \underbrace{p_t \otimes z_t}_{\text{price}}) = 0$$

You should include economically meaningful instruments (managed portfolios)

- $p = \mathbb{E}(mx)$ should price any asset, also managed portfolios
- if model prices all managed portfolios, conditional asset pricing model true.
- select few selected instruments (we also select few assets from millions available). New managed funds example
- Select meaningful instruments: Those affecting conditional distribution of returns
- Any $z_t \in I_t$ qualifies as an instruments, but if $corr((m_{t+1}R_{t+1}), z_t) = 0$ but $corr(R_{t+1}, z_t)$ small: weak instrument
- danger of using weak instruments (Hamilton, 1994, p. 426 for references)

Some more details and intuition on the choice of instruments

$$p_t z_t = \mathbb{E}_t(m_{t+1} x_{t+1} z_t) \quad \text{resp.} \quad z_t = \mathbb{E}_t(m_{t+1} R_{t+1} z_t)$$

holds true trivially if $corr((m_{t+1}R_{t+1} - 1), z_t) = 0$
but an interesting instrument implies $corr(R_{t+1}, z_t) \neq 0$ and/or $corr(m_{t+1}, z_t) \neq 0$

$$\text{if} \quad \mathbb{E}_t(R_{t+1}) \uparrow \ \text{when} \quad z_t \uparrow$$

then in

$$1 z_t = z_t \underbrace{\mathbb{E}_t(R_{t+1})}_{\uparrow} \underbrace{\mathbb{E}_t(m_{t+1})}_{\downarrow \ \text{or}} + z_t \underbrace{cov_t(m_{t+1} R_{t+1})}_{\downarrow}$$

Is a conditional asset pricing model testable at all?

Most asset pricing models imply **conditional** moment restrictions

$$1 = \mathbb{E}\left(m_{t+1}(b_t) \cdot R_{t+1} | I_t\right)$$

e.g. CAPM $m_{t+1} = a_t - b_t R^W_{t+1}$.

Parameters of factor pricing model vary over time.
$\Rightarrow$unconditioning via l.i.e. no longer possible:

$$1 = \mathbb{E}\left(m_{t+1}(b_t) \cdot R_{t+1} | I_t\right)$$

does NOT imply

$$1 = \mathbb{E}\left(m_{t+1}(b) \cdot R_{t+1}\right)$$

this is not repaired by using scaled returns. GMM estimation no possible.

Hansen and Richard critique: CAPM (or other factor model) is not testable.

Scaled factors are a partial solution to the problem

With linear factor model

$$m_{t+1} = b_t' \underbrace{f_{t+1}}_{K \times 1}$$

use of "scaled factors" a partial solution:

"Blow up" number of factors by scaling factors with $(M \times 1)$ instruments vector z_t observable at t

$$m_{t+1} = b' \underbrace{(f_{t+1} \otimes z_t)}_{KM \times 1}$$

Unconditioning via l.i.e. and GMM procedure as above.

Time varying parameters lead to scaled factors (single factor case)

Motivation

Consider linear one factor model $m_{t+1} = a_t + b_t f_{t+1}$ (f_{t+1} scalar)
Assume Parameters vary with $M \times 1$ instruments vector z_t.

$$m_{t+1} = a(z_t) + b(z_t) f_{t+1}$$

With linear functions

$$a(z_t) = a' z_t \quad \text{and} \quad b(z_t) = b' z_t$$

$$\Rightarrow m_{t+1} = a' z_t + (b' z_t) f_{t+1}$$

Mathematically equivalent to

$$m_{t+1} = \tilde{b}' (\tilde{f}_{t+1} \otimes z_t)$$

where $\tilde{b} = \begin{bmatrix} a \\ b \end{bmatrix}$, $\tilde{f}_{t+1} = \begin{bmatrix} 1 \\ f_{t+1} \end{bmatrix}$

Number of parameters to estimate $2 \cdot M$

Time varying parameters lead to scaled factors (multi factor case)

Multi-factor case:

$$m_{t+1} = b_t' \underbrace{f_{t+1}}_{K\times 1}$$

Again: Time varying parameters linear functions of $M \times 1$ vector of observables z_t.

$$m_{t+1} = b(z_t)' f_{t+1} \quad \text{with} \quad b(z_t) = \underbrace{B}_{K\times M} z_t$$

Equivalent to $m_{t+1} = \tilde{b}' \underbrace{(f_{t+1} \otimes z_t)}_{K\times N}$ where $\tilde{b} = vec(B)$

In practical application some elements of B may be set to zero.

Using scaled factors we can condition down and apply GMM

Conditioning down and GMM estimation possible

$$\mathbb{E}_t\left(\underbrace{\left(\tilde{b}'(f_{t+1}\otimes z_t)\right)}_{m_{t+1}}R_{t+1}\right)=1 \quad \text{l.i.e.} \Rightarrow \quad \underbrace{\mathbb{E}\left(\left(\tilde{b}'(f_{t+1}\otimes z_t)\right)R_{t+1}-1\right)=0}_{\text{unconditional moment restrictions}}$$

Scaled factors and managed portfolios can be combined.
(z_t might be the same).

$$\Rightarrow \mathbb{E}(\tilde{b}'(f_{t+1}\otimes z_t)R_{t+1}-1\,]\otimes z_t)=0$$

- Inclusion of conditioning information as managed portfolios (scaled returns, increases number of test assets.

- Scaled factors increase number of unknown parameters

Cochranes (1996) CAPM with scaled factors

$$f = \begin{pmatrix} 1 \\ R^W \end{pmatrix} z_t = \begin{pmatrix} 1 \\ \frac{P}{D} \\ term \end{pmatrix} B = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \end{bmatrix}$$

$$f \otimes z = \begin{pmatrix} 1 \\ R^W \\ \frac{P}{D} \\ R^W \cdot \frac{P}{D} \\ term \\ R^W \cdot term \end{pmatrix} \tilde{b} = \left(b_{11}, b_{21}, b_{12}, b_{22}, b_{13}, b_{23} \right)'$$

$$m = \tilde{b}'(f \otimes z) = b_{11} + b_{12}\frac{P}{D} + b_{13} term + b_{21} R^W + b_{22} R^W \cdot \frac{P}{D} + b_{23} R^W \cdot term$$

In application Cochrane (1996) restricts b_{12} and b_{13} to zero