Applied Microeconometrics

Chapter 2

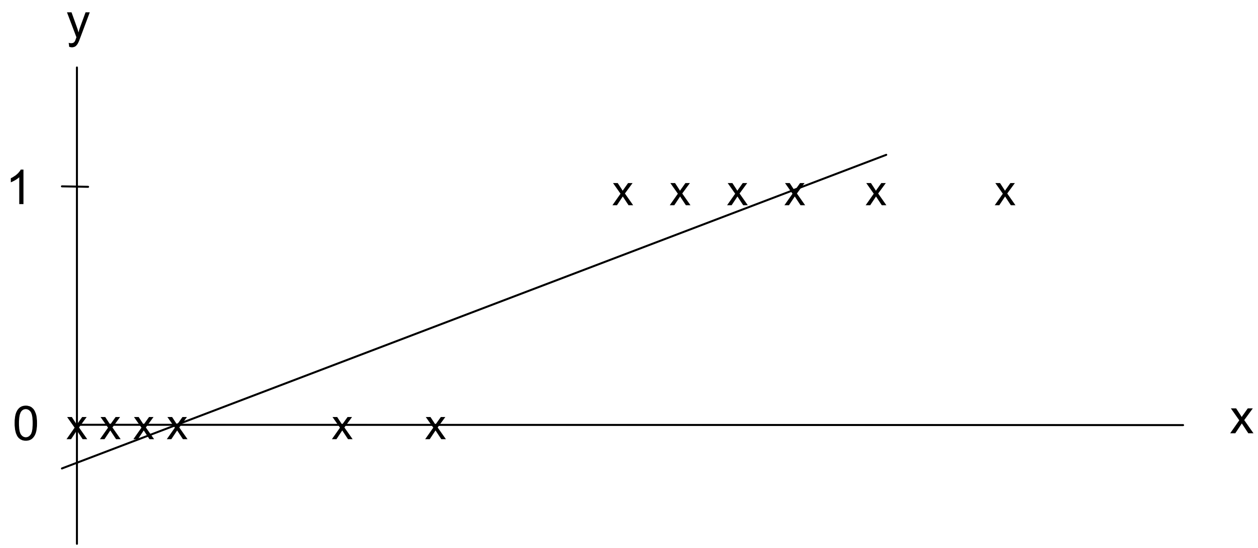
Models with binary dependent variables

1. Introduction to the Probit Model
2. Estimation
3. A Practical Application
4. Coefficients and Marginal Effects
5. Goodness-of-Fit Measures
6. Hypothesis Tests
7. Probit vs. Logit

1. Introduction to the Probit model

Recall our example from the introduction:

- **Binary** choice variable: voting yes-no $y \in \{0,1\}$
- Explanatory variable: household income $x \in \mathbb{R}^{+}$



Introduction to the Probit model – latent variables

- We aim to model the probability that the observed binary variable takes one of its values conditional on x, such as

$$p = P(y_i = 1 | x)$$

where $0 \leq p \leq 1$

- We need to derive this probability to estimate the model by maximum likelihood

Introduction to the Probit model – latent variables

- We think of the process generating observations on discrete outcome y as driven by an unobserved **(latent)** variable y* which can take all values in (-∞, +∞).
- Example: y* = net utility from labour income, y = observed labour market participation
- the underlying model is in terms of the latent variable and is linear

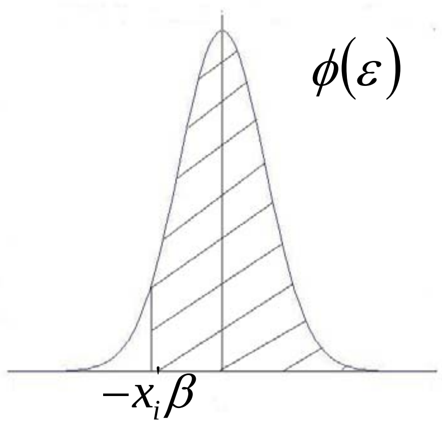
$$y_i = \begin{cases} 1, y_i^* > 0 \\ 0, y_i^* \leq 0 \end{cases}$$

$$y_i^* = x_i^{'}\beta + \varepsilon_i$$

Introduction to the Probit model – latent variables

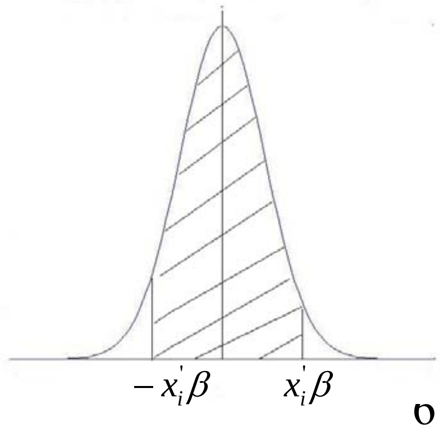
Probit is based on the latent model:

$$
\begin{aligned}
&P(y_i = 1 \mid x) = P(y_i^{*} > 0 \mid x) \\
&= P(x_i^{'}\beta + \varepsilon_i > 0 \mid x) \\
&= P(\varepsilon_i > -x_i^{'}\beta \mid x) \\
&= 1 - F(-x_i^{'}\beta)
\end{aligned}
$$



Assumption: Error terms are independent and normally distributed:

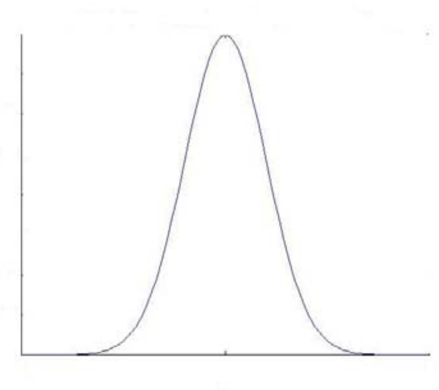
$$
\begin{aligned}
&P(y_i = 1 \mid x) = 1 - \Phi\left(-\frac{x_i^{'}\beta}{\sigma}\right), \sigma \equiv 1 \\
&= \Phi(x_i^{'}\beta) \quad \text{because of symmetry}
\end{aligned}
$$



Background on probability distribution functions (PDF)

- PDF: probability distribution function f(x)
- Example: Normal distribution:

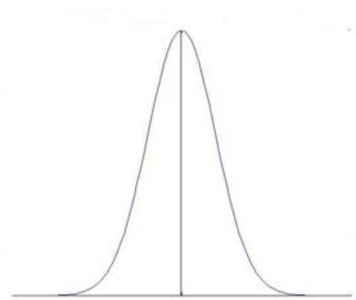
$$\phi(x)=\frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{1}{2}\left[\frac{(x-\mu)^2}{\sigma^2}\right]}$$



- Example: Standard normal distribution:

N(0,1), $\mu = 0$, $\sigma = 1$

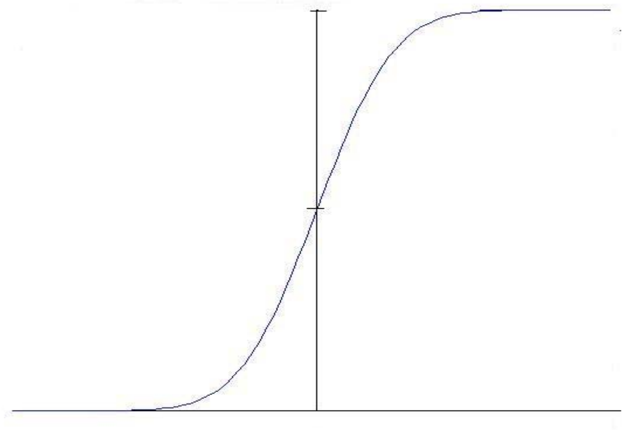
$$\phi(x)=\frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}$$



Notation and statistical foundations – CDF

- CDF: cumulative distribution function F(x)
- Example: Standard normal distribution:

$$\Phi\left(z\right)=\int_{-\infty}^{z}\frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}dx$$



- The cdf is the integral of the pdf. It is bounded between 0 and 1, as required

2. Estimation

- The probability of choosing $y_i = 1$ is
  $\Phi(x_i^{'}\beta)$
- Similarly, the probability of choosing $y_i = 0$ is
  $1 - \Phi(x_i^{'}\beta)$
- Combining these, the likelihood of observing unit i in the state actually chosen is

$$L_i(x_i, \beta) = \Phi(x_i^{'}\beta)^{y_i}\left(1 - \Phi(x_i^{'}\beta)\right)^{1-y_i}$$

Derivation of the log likelihood function

- Taking the product over all units in the sample i = 1,…,n gives the likelihood function

$$L(y \mid x, \beta) = \prod_i \Phi(x_i'\beta)^{y_i} \left[1 - \Phi(x_i'\beta)\right]^{(1-y_i)}$$

$$= \prod_i \Phi_i^{y_i} (1 - \Phi_i)^{1-y_i}$$

- It is more convenient to use the log likelihood function:

$$\ln L = \sum_i y_i \ln \Phi_i + (1 - y_i) \ln(1 - \Phi_i)$$

The ML principle

- The principle of ML: Which value of β maximizes the probability of observing the given sample?

$$\frac{\partial \ln L}{\partial \beta} = \sum_i \left[\frac{y_i \varphi_i}{\Phi_i} + \frac{(1-y_i)(-\varphi_i)}{1-\Phi_i} \right] x_i$$

$$= \sum_i \left[\frac{y_i - \Phi_i}{\Phi_i (1-\Phi_i)} \varphi_i \right] x_i \overset{!}{=} 0$$

- Usually, use k explanatory variables rather than one
- The gradient vector $\partial \ln L(\theta) / \partial \theta$ is also called the score vector

Distribution of the ML estimator

- Under certain regularity conditions (see Cameron / Trivedi, p. 142) the MLE defined by $\partial \ln L(\theta) / \partial\theta = 0$ is consistent for θ_0 and

$$\sqrt{n}(\hat{\theta}_{ML} - \theta_0) \xrightarrow{d} N\left[0, -A_0^{-1}\right]$$

where $A_0 = \operatorname{p\,lim} n^{-1} \left.\dfrac{\partial^2 \ln L(\theta)}{\partial\theta\partial\theta'}\right|_{\theta_0}$

- Then, the asymptotic distribution of the MLE can be written as

$$\hat{\theta}_{ML} \overset{a}{\sim} N\left[\theta, -E\left[\left(\frac{\partial^2 \ln L(\theta)}{\partial\theta\partial\theta'}\right)\right]^{-1}\right]$$

Derivation of the MLE

- It can be shown that the likelihood function for the Probit model is globally concave → there exists only one maximum of the likelihood function
- However, the first-order conditions $\partial \ln L(\theta) / \partial\theta = 0$ cannot be solved analytically
- Hence, need to find numerical solutions
- Mostly used: Newton-Raphson Algorithm

Newton-Raphson Algorithm

- Iterative procedure: from an estimate in the s-th step, apply a rule that finds the next-step estimate
- The rule must be chosen such that it ensures a move towards the maximum
- Process stops if the distance between steps s and s+1 becomes very small

Newton-Raphson Algorithm

- In the Newton-Raphson case, the rule is
  $\hat{\theta}_{s+1} = \hat{\theta}_s - H_s^{-1} g_s$

  where g_s is the gradient $g_s = \partial \ln L(\theta) / \partial\theta \big|_{\theta_s}$ derived from step s and

  $$H_s = \left. \frac{\partial^2 \ln L(\theta)}{\partial\theta\partial\theta'} \right|_{\theta_s}$$

- Intuition: if the score is positive, need to increase θ in order to get closer to maximum (note that H_s is always negative, as claimed previously).

Newton-Raphson Algorithm

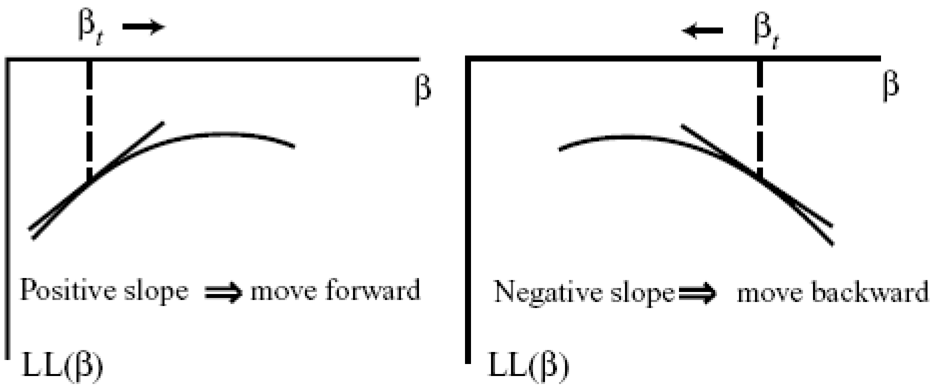


Figure 8.2. Direction of step follows the slope.

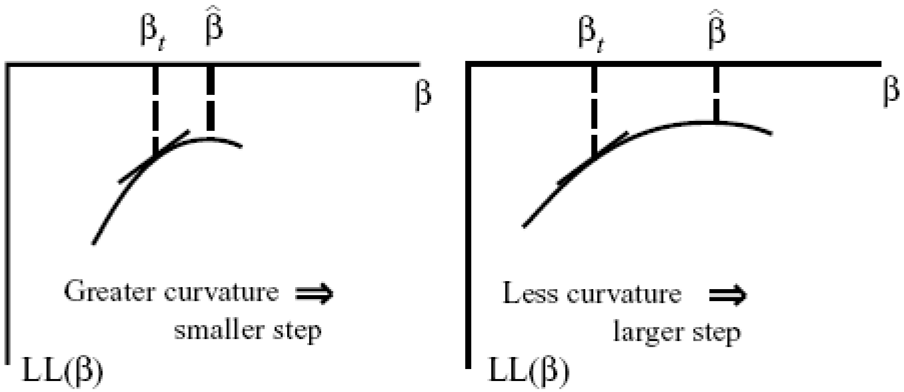


Figure 8.3. Step size is inversely related to curvature.

Taken from:

K. Train (2003), Discrete Choice Methods with Simulation, Cambridge University Press

http://elsa.berkeley.edu/books/choice2.html

(Chapter on numerical maximisation highly recommended!)

Newton-Raphson Algorithm

What happens if the likelihood function is not globally concave?

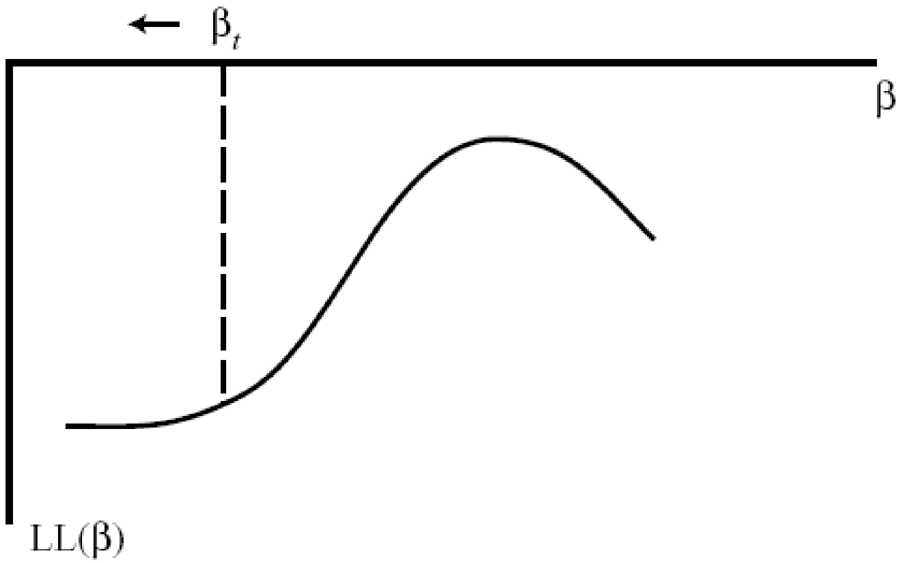


Figure 8.6. NR in the convex portion of LL.

Taken from:

K. Train (2003), Discrete Choice Methods with Simulation, Cambridge University Press

http://elsa.berkeley.edu/books/choice2.html

(Chapter on numerical maximisation highly recommended!)

A Practical Application

- Analysis of the effect of a new teaching method in economic sciences
- Data:

Obs.No.	GPA	TUCE	PSI	Grade	Obs.No.	GPA	TUCE	PSI	Grade
1	2.66	20	0	0	17	2.75	25	0	0
2	2.89	22	0	0	18	2.83	19	0	0
3	3.28	24	0	0	19	3.12	23	1	0
4	2.92	12	0	0	20	3.16	25	1	1
5	4	21	0	1	21	2.06	22	1	0
6	2.86	17	0	0	22	3.62	28	1	1
7	2.76	17	0	0	23	2.89	14	1	0
8	2.87	21	0	0	24	3.51	26	1	0
9	3.03	25	0	0	25	3.54	24	1	1
10	3.92	29	0	1	26	2.83	27	1	1
11	2.63	20	0	0	27	3.39	17	1	1
12	3.32	23	0	0	28	2.67	24	1	0
13	3.57	23	0	0	29	3.65	21	1	1
14	3.26	25	0	1	30	4	23	1	1
15	3.53	26	0	0	31	3.1	21	1	0
16	2.74	19	0	0	32	2.39	19	1	1

Source: Spector, L. and M. Mazzeo, Probit Analysis and Economic Education. In: Journal of Economic Education, 11, 1980, pp.37-44

Application – Variables

- Grade

  Dependent variable. Indicates whether a student improved his grades after the new teaching method PSI had been introduced (0 = no, 1 = yes).

- PSI

  Indicates if a student attended courses that used the new method (0 = no, 1 = yes).

- GPA

  Average grade of the student

- TUCE

  Score of an intermediate test which shows previous knowledge of a topic.

Application – Estimation

- Estimation results of the model (output from Stata):

```
. probit  grade psi tuce gpa

Iteration 0:   log likelihood =  -20.59173
Iteration 1:   log likelihood = -13.315851
Iteration 2:   log likelihood = -12.832843
Iteration 3:   log likelihood = -12.818826
Iteration 4:   log likelihood = -12.818803

Probit estimates                                  Number of obs   =         32
                                                  LR chi2(3)      =      15.55
                                                  Prob > chi2     =     0.0014
Log likelihood = -12.818803                       Pseudo R2       =     0.3775
```

grade	Coef.	Std. Err.	z	P>|z|	[95% Conf.	Interval]
psi	1.426332	.595037	2.40	0.017	.2600814	2.592583
tuce	.0517289	.0838901	0.62	0.537	-.1126927	.2161506
gpa	1.62581	.6938818	2.34	0.019	.2658269	2.985794
_cons	-7.45232	2.542467	-2.93	0.003	-12.43546	-2.469177

Application – Discussion

- ML estimator: Parameters were obtained by maximization of the log likelihood function.
  Here: 5 iterations were necessary to find the maximum of the log likelihood function (-12.818803)
- Interpretation of the estimated coefficients:
  - Unlike in OLS, estimated coefficients cannot be interpreted as the quantitative influence of the rhs variables on the probability that the lhs variable takes on the value one.
  - This is due to non-linearity and using the standard normal distribution for normalisation.

Coefficients and marginal effects

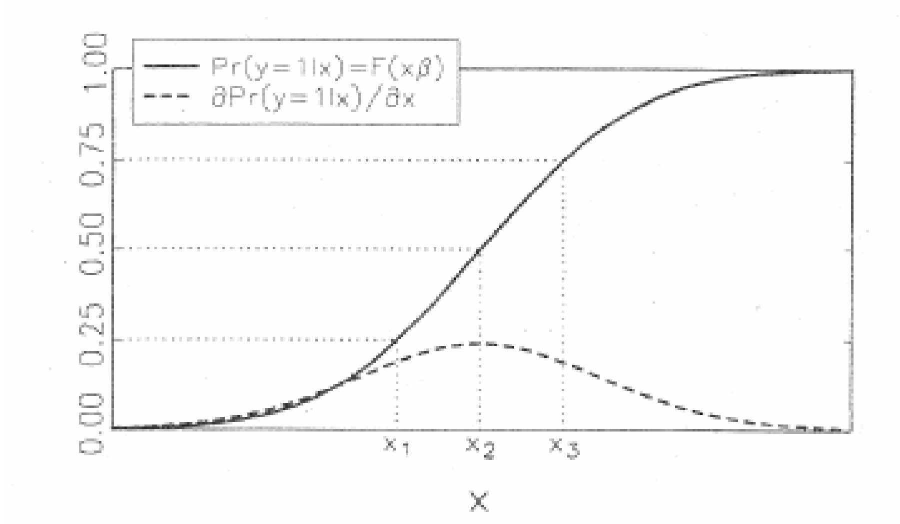
- The marginal effect of a rhs variable is the effect of an infinitesimal change (dummy variables: unit change) of this variable on the probability P(Y = 1|X = x), given that all other rhs variables are constant:

$$\frac{\partial P(y_i = 1 \mid x_i)}{\partial x_i} = \frac{\partial E(y_i \mid x_i)}{\partial x_i} = \varphi(x_i^{'}\beta)\beta$$

- Recap: The slope parameter of the linear regression model measures directly the marginal effect of the rhs variable on the lhs variable.

Coefficients and marginal effects

- The marginal effect depends on the value of the rhs variable.
- Therefore, there exists an individual marginal effect for each person of the sample:



Coefficients and marginal effects – Computation

- Two different types of marginal effects can be calculated:
  - Average marginal effect
    Stata command: `margin`

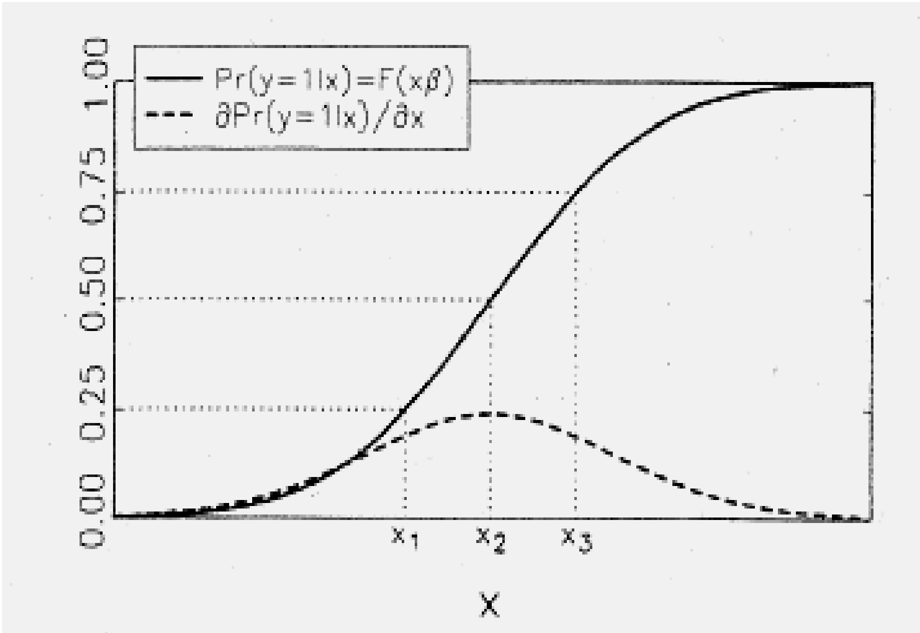
Marginal effects on Prob(grade==1) after probit

grade	Coef.	Std. Err.	z	P>|z|	[95% Conf.	Interval]
gpa	.3637883	.1129461	3.22	0.001	.1424181	.5851586
tuce	.011476	.0184085	0.62	0.533	-.024604	.047556
psi	.3737518	.1399912	2.67	0.008	.0993741	.6481295

  - Marginal effect at the mean:
    Stata command: `mfx compute`

Coefficients and marginal effects – Computation

- Principle of the computation of the average marginal effects:



- Average of individual marginal effects

Coefficients and marginal effects – Computation

- Computation of average marginal effects depends on type of rhs variable:
  - Continuous variables like TUCE and GPA:

$$AME = \frac{1}{n}\sum_{i=1}^{n}\varphi(x_i^{'}\beta)\beta$$

  - Dummy variable like PSI:

$$AME = \frac{1}{n}\sum_{i=1}^{n}\left[\Phi(x_i^{'}\beta \mid x_i^k = 1) - \Phi(x_i^{'}\beta \mid x_i^k = 0)\right]\beta$$

Coefficients and marginal effects – Interpretation

- Interpretation of average marginal effects:
  - Continuous variables like TUCE and GPA:
    A change of TUCE or GPA of size 1 changes the probability that the lhs variable takes the value one by X%.
  - Dummy variable like PSI:
    A change of PSI from zero to one changes the probability that the lhs variable takes the value one by X%.

Coefficients and marginal effects – Interpretation

Variable	Estimated marginal effect	Interpretation
GPA	0.364	If the average grade of a student goes up by size 1, the probability for the variable grade taking the value one rises by 36.4%.
TUCE	0.011	As with GPA,with an increase of 1.1%.
PSI	0.374	If the dummy variable changes from zero to one, the probability for the variable grade taking the value one rises by 37.4%.

Coefficients and marginal effects – Significance

- Significance of a coefficient: test of the hypothesis whether a parameter is significantly different from zero.
- The decision problem is similar to the t-test, whereas the probit test statistic follows a standard normal distribution. The z-value is equal to the estimated parameter divided by its standard error.
- Stata computes a p-value which shows directly the significance of a parameter:

	z-value	p-value	Interpretation
GPA :	3.22	0.001	***significant***
TUCE:	0,62	0,533	***insignificant***
PSI:	2,67	0,008	***significant***

Coefficients and marginal effects

- Only the average of the marginal effects is displayed.
- The individual marginal effects show large variation:

Descriptive statistics for individual marginal effects

	Mean	SD	Min	Max
gpa	0.36379	0.21358	0.06783	0.64807
tuce	0.01148	0.00687	0.00209	0.02063
psi	0.37375	0.12878	0.06042	0.51959

Stata command: `margin, table`

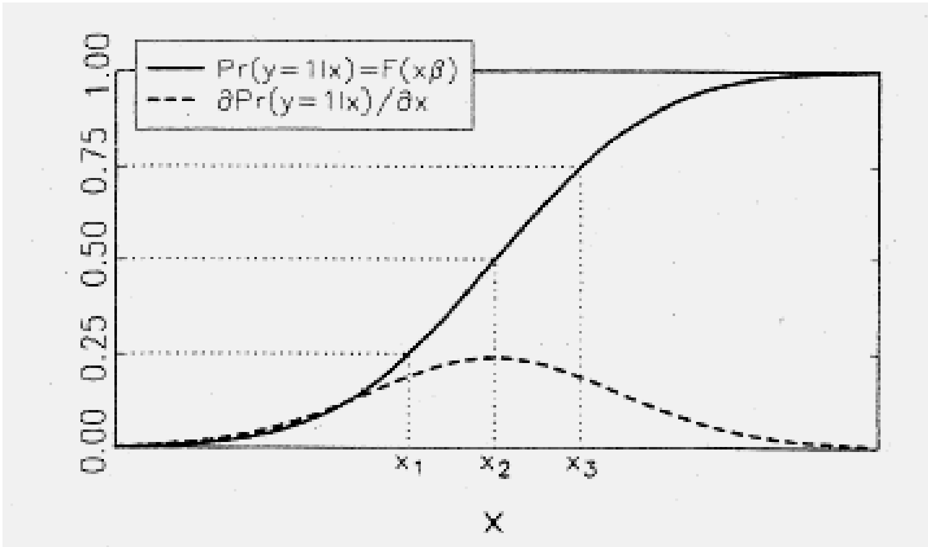
Coefficients and marginal effects

- Variation of marginal effects may be quantified by the confidence intervals of the marginal effects.
- In which range one can expect a coefficient of the population?
- In our example:

	Estimated coefficient	Confidence interval (95%)
GPA:	0,364	- 0,055 - 0,782
TUCE:	0,011	- 0,002 - 0,025
PSI:	0,374	0,121 - 0,626

Coefficients and marginal effects

- What is calculated by `mfx`?
- Estimation of the marginal effect at the sample mean.



Sample mean

Goodness of fit

- Goodness of fit may be judged by McFaddens Pseudo R^2.
- Measure for proximity of the model to the observed data.
- Comparison of the estimated model with a model which only contains a constant as rhs variable.
  - $\ln \hat{L}(M_{Full})$: Likelihood of model of interest.
  - $\ln \hat{L}(M_{Intercept})$: Likelihood with all coefficients except that of the intercept restricted to zero.
  - It always holds that $\ln \hat{L}(M_{Full}) \geq \ln \hat{L}(M_{Intercept})$

Goodness of fit

- The Pseudo R² is defined as:

$$PseudoR^2 = R^2_{McF} = 1 - \frac{\ln \hat{L}(M_{Full})}{\ln \hat{L}(M_{Intercept})}$$

- Similar to the R² of the linear regression model, it holds that $0 \leq R^2_{McF} \leq 1$

- An increasing Pseudo R² may indicate a better fit of the model, whereas no simple interpretation like for the R² of the linear regression model is possible.

Goodness of fit

- R^2_{McF} increases with additional rhs variables. Therefore, an adjusted measure may be appropriate:

$$PseudoR^2_{adjusted} = \overline{R}^2_{McF} = 1 - \frac{\ln \hat{L}(M_{Full}) - K}{\ln \hat{L}(M_{Intercept})}$$

- Further goodness of fit measures: R^2 of McKelvey and Zavoinas, Akaike Information Criterion (AIC), etc. See also the Stata command `fitstat`.

Hypothesis tests

- Likelihood ratio test: possibility for hypothesis testing, for example for variable relevance.
- Basic principle: Comparison of the log likelihood functions of the unresticted model ($\ln L_U$) and that of the resticted model ($\ln L_R$)
- Test statistic: $LR = -2\ln\lambda = -2(\ln L_R - \ln L_U) \quad \chi^2(K)$

$$\lambda = \frac{L_R}{L_U} \qquad 0 \leq \lambda \leq 1$$

- The test statistic follows a χ^2 distribution with degrees of freedom equal to the number of restrictions.

Hypothesis tests

- Null hypothesis: All coefficients except that of the intercept are equal to zero.
- In the example: LR $\chi^2(3) = 15,55$
- Prob > chi2 = 0.0014
- Interpretation: The hypothesis that all coefficients are equal to zero can be rejected at the 1 percent significance level.

The Logit model

- Binary dependent variable: $y = \begin{cases} 1 \\ 0 \end{cases}$
- Let $P(y_i = 1 \mid x) = F(x_i^{'}\beta)$
  (as in the case of Probit)
- In the Logit model, F(.) is given the particular functional form:

$$P(y_i = 1) = \frac{\exp(x_i^{'}\beta)}{1 + \exp(x_i^{'}\beta)}$$

- The model is called Logit because the residuals of the latent model are assumed to be distributed standard logistic.

Notation and statistical foundations – distibutions

- Standard logistic distribution:

$$f(x) = \frac{e^x}{\left(1+e^x\right)^2}, \mu = 0, \sigma^2 = \frac{\pi^2}{3}$$

- Exponential distribution:

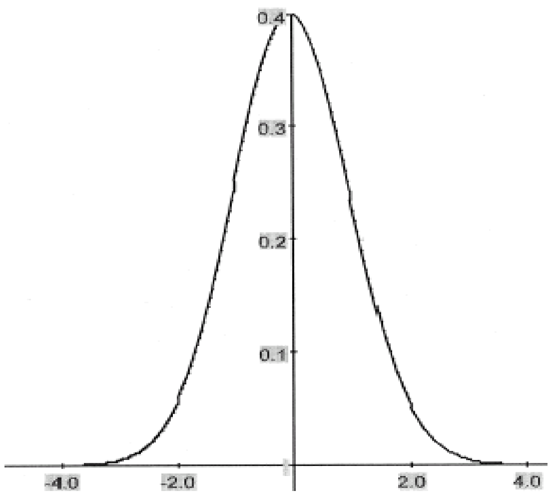
$$f(x) = \begin{cases} \frac{1}{\theta} e^{-\frac{x}{\theta}}, x \geq 0 \\ 0, x \leq 0 \end{cases}, \theta > 0, \mu = \theta, \sigma^2 = \theta^2$$

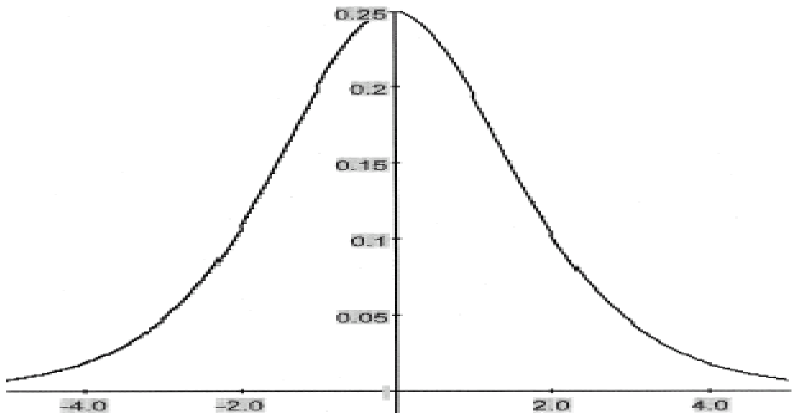
- Poisson distribution:

$$f(x) = \frac{e^{-\theta}\theta^x}{x!}, \mu = \theta, \sigma^2 = \theta$$

PDF Probit vs. Logit

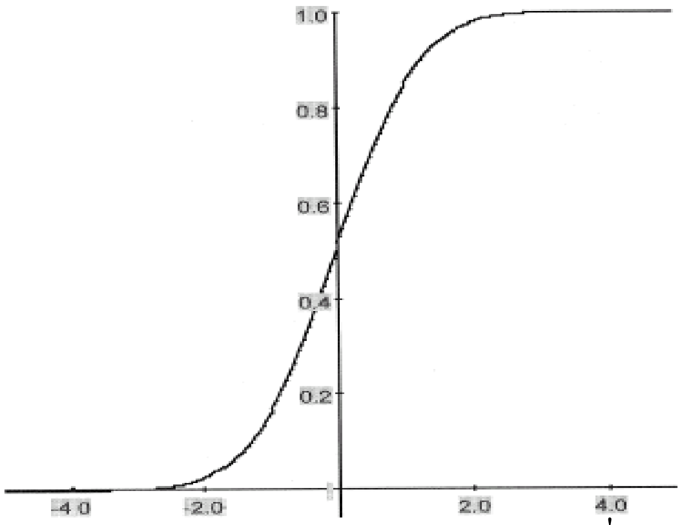
- PDF of Probit: PDF of Logit:



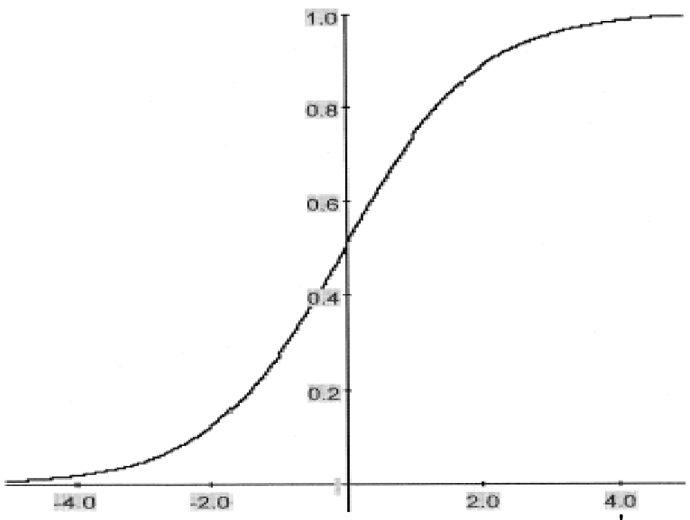


CDF Probit vs. Logit

- F(z) lies between zero and one
- CDF of Probit: CDF of Logit:



$z = x_i^{'}\beta$



$z = x_i^{'}\beta$

Estimation output

The Logit model is implemented in all major software packages, such as Stata:

```
. logit  grade psi tuce gpa

Iteration 0:   log likelihood =  -20.59173
Iteration 1:   log likelihood = -13.496795
Iteration 2:   log likelihood = -12.929188
Iteration 3:   log likelihood = -12.889941
Iteration 4:   log likelihood = -12.889633
Iteration 5:   log likelihood = -12.889633

Logit estimates                                   Number of obs   =         32
                                                  LR chi2(3)      =      15.40
                                                  Prob > chi2     =     0.0015
Log likelihood = -12.889633                       Pseudo R2       =     0.3740

------------------------------------------------------------------------------
       grade |      Coef.   Std. Err.      z    P>|z|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
         psi |   2.378688   1.064564     2.23   0.025       .29218    4.465195
        tuce |   .0951577   .1415542     0.67   0.501    -.1822835    .3725988
         gpa |   2.826113   1.262941     2.24   0.025     .3507938    5.301432
       _cons |  -13.02135   4.931325    -2.64   0.008    -22.68657    -3.35613
------------------------------------------------------------------------------
```

Coefficient magnitudes

Coefficient Magnitudes differ between Logit and Probit:

	Probit	Logit
gpa	1,626	2,826
tuce	0,052	0,095
psi	1,426	2,379

This is due to the fact that in binary models, the coefficients are identified only up to a scale parameter

Coefficient magnitudes

- Coefficient magnitudes can be made comparable by standardizing with the variance of the errors:

  - with logarithmic distribution: Var=$\pi^2/6$
  - with standard normal distribution: Var=1

- approximative conversion of the estimated values using

$$\frac{1}{\sqrt{\pi^2/6}} \approx 0.61$$

Marginal effects

For interpretation we have to calculate the <u>marginal effects</u> of the estimated coefficients (as in the Probit case)

```
5 . margin, table   (AKA margeff)

  Marginal effects on Prob(grade==1) after logit
```

grade	Coef.	Std. Err.	z	P>|z|	[95% Conf.	Interval]
gpa	.3682795	.1088308	3.38	0.001	.1549751	.581584
tuce	.0122101	.0177941	0.69	0.493	-.0226656	.0470859
psi	.3575152	.1420034	2.52	0.012	.0791936	.6358367

Interpretation of the marginal effects analogous to the Probit model