Bachelor Thesis in Economics

The Birth of Credit Ratings
The Z-Score Model and the Concept of Discriminant Analysis

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"Prediction is very difficult, especially of the future."
NILS BOHR

1 Introduction

Accurate measurement of credit risk is essential to the functioning of financial markets due to the fact that, from an economic point of view, the main task of financial intermediaries is to take deposits and to grant credit. Therefore, the stability of an individual institution exposed to credit risk and ultimately the stability of the whole system is determined by the expertise in credit risk management and regulation in this area. The current financial market turbulences have put the subject of credit risk to the very top of the agenda in business and politics. Until now, no final judgement has been made about the interaction of the different factors that have caused the crisis, but credit risk is undoubtedly at the very centre of the turmoil with the excessive accumulation of sub-prime lending as the focal point.
The fundamental question of credit risk is straightforward: Is a debtor going to repay his debt? Finding an answer is much more complex. Many exogenous and endogenous factors influence the ability to repay debt, and the challenge is to take all these factors into account to pass a judgement on the creditworthiness of an individual or an institution. For a very long time this was done solely by the personal judgment of the creditor. Bankers used information about the individual characteristics of a borrower, such as reputation, existing collateral and income, to reach a basically subjective assessment of the credit risk involved in a transaction [5, p.1722]. Subsequently, models were developed for corporations that used accounting figures to derive a still subjective credit scoring. The creditor analysed key financial accounting ratios and figures of potential borrowers and the change of these indicators over time and compared them with industry averages [5, p.1723]. Later came models that used more advanced statistical techniques in order to increase the accuracy of credit risk measurement and to restrict the subjective impact of the decision maker. These models have dominated credit risk research for many years and they still play an important role today [5, p.1723]. EDWARD I. ALTMAN, Professor of Finance at New York University, developed the Z-Score model [1] that applies discriminant analysis to predict corporate bankruptcy of manufacturing firms in the United States. It was published in 1968 and has since then influenced subsequent research and practice thoroughly.
This work investigates the structure of ALTMAN's Z-Score model and the impact that the model had on subsequent research. In particular, it should be assessed whether further applications of similar models generally support ALTMAN's results or if objections were raised against the applicability of the model. Thus, the question put forward is: Is the model relevant to the present-day decision maker? To inquire this research question, section two illustrates the statistical theory and presents ALTMAN's model, section three compares the results for the Z-Score model with those from other publications and the final section summarises the comparative analysis and it is attempted to assess the relevance of the model and to discuss critique.

2 Altman's Z-Score Model

2.1 Univariate Ratio Analysis

Before Altman published the Z-Score model, academics primarily applied financial ratio analysis to assess the performance of a business enterprise. Financial ratio analysis in this context means that financial ratios and figures from balance sheets and income statements are discussed separately with a final overall judgement of an analyst. Studies from this period[1] concluded that firms experiencing financial distress, or in the worst of all cases file for bankruptcy and discontinue business, publish significantly different figures than healthy ongoing companies [1, p.590]. These studies evinced the potential of ratio analysis as predictors of bankruptcy but also raised the question, which ratio is the most effective one as to be used in the prediction of business failure. According to Altman, it was due to this predicament, that on the one hand ratio analysis was successful, but on the other hand did not present a consistent picture, why academics moved away from ratio analysis as an analytical technique. Although academics had developed this objection towards using financial ratio analysis, the method was widely used by practitioners [1, p.589]. The contribution of the author was to master this opposition between theory and practice by taking the formerly univariate approach to a multivariate level. He proposed a linear combination of variables in order to make it possible to examine them simultaneously rather than individually. For this purpose he used discriminant analysis.

2.2 Discriminant Analysis

2.2.1 Two-Group Discriminant Analysis

Discriminant analysis is a multivariate statistical technique applied to "best" separate two or more classes of objects and is thus an appropriate method to analyse the relationship between a categorical dependent variable and metrical independent variables. It can be used in descriptive terms to separate observations in a sample or to predict the class for future observations [17, p.5]. In contrast to univariate methods, where differences between groups with respect to only one variable are analysed, discriminant analysis separates groups in an n-dimensional space. Thus, n variables are examined simultaneously using a linear discriminant function, which assigns a discriminant score Z as a linear combination of the values of n variables to each observation. The comparison of the total value of Z with a cutoff-value determines the assignment to a particular group [9, p.245].

The first step is to define the groups, which are bankrupt and non-bankrupt firms in Altman's Z-Score model. Any definition for the groups is feasible as long as they are mutually exclusive and exhaustive and every observation can be assigned to only one group. Then, data are collected for both groups and n variables. Before discriminant analysis can be used for prediction,

[1] For instance, Smith and Winakor (1935), Merwin (1942), Hickman (1958) and Beaver (1967).

the discriminant function has to be derived such that it distinguishes best between the *a priori* groups of the sample. The discriminant function is given by

$$
\begin{aligned}
Z &= v_1X_1 + v_2X_2 + \cdots + v_nX_n \qquad (1)\\
&= \text{discriminant score .}\\
v_1, v_2, \ldots, v_n &= \text{discriminant coefficients}\\
X_1, X_2, \ldots, X_n &= \text{independent variables}
\end{aligned}
$$

The objective of discriminant analysis is to derive the discriminant coefficients such that

$$v_1, v_2, \ldots, v_n := \underset{v_1,v_2,\ldots,v_n}{\text{argmax}}\,\{\lambda\}\ , \qquad (2)$$

where

$$\lambda = \frac{\sum_{g=1}^{G} I_g(\bar{Z}_g - \bar{Z})^2}{\sum_{g=1}^{G}\sum_{i=1}^{I_g}(Z_{gi} - \bar{Z}_g)^2} = \frac{\text{between-group sum of squares}}{\text{within-group sum of squares}} = \frac{SS_b}{SS_w} \qquad (3)$$

is the discriminant criterion, $\bar{Z}_g$ is the mean value of discriminant scores in group g (centroid), $\bar{Z}$ is the overall mean value of discriminant scores, Z_{gi} is the ith discriminant score in group g, I_g is the number of observations in group g and G is the number of groups. Although the two-group case is discussed here and $G = 2$, the formulas are presented in general terms for G groups. Thus, this procedure maximises the variation of the centroids around the overall mean of discriminant scores and minimises the variation of the discriminant scores around the centroids within each group simultaneously in terms of their squared differences. To derive the actual values of the discriminant coefficients, matrix notation[2] is used and λ can be written as

$$\lambda = \frac{\mathbf{v}'\mathbf{B}\mathbf{v}}{\mathbf{v}'\mathbf{W}\mathbf{v}}\ , \qquad (4)$$

where

$$
\begin{aligned}
\mathbf{v} &= \text{vector of discriminant coefficients } v_j \quad j = 1, \ldots, n\ ,\\
B_{jr} &= \sum_{g=1}^{G} I_g(\bar{X}_{jg} - \bar{X}_j)(\bar{X}_{rg} - \bar{X}_r) \quad j, r = 1, \ldots, n\ , \qquad (5)\\
W_{jr} &= \sum_{g=1}^{G}\sum_{i=1}^{I_g}(X_{jgi} - \bar{X}_{jg})(X_{rgi} - \bar{X}_{rg}) \quad j, r = 1, \ldots, n\ , \qquad (6)
\end{aligned}
$$

B_{jr} and W_{jr} are the elements of the $n \times n$ matrices $\mathbf{B}$ and $\mathbf{W}$, X_{jgi} is the sample value of observation i in group g for variable j and $\bar{X}_{jg}$ is the mean value of variable j in group g.

[2]See Appendix A for the derivation of the matrix notation used in (4)-(6).

Vector differentiation yields the following first order condition for the discriminant coefficients:

$$\frac{\delta\lambda}{\delta\mathbf{v}} = \frac{2[(\mathbf{Bv})(\mathbf{v}'\mathbf{Wv}) - (\mathbf{v}'\mathbf{Bv})(\mathbf{Wv})]}{(\mathbf{v}'\mathbf{Wv})^2} \overset{!}{=} \mathbf{0}\ . \tag{7}$$

The numerator and denominator is then divided by $(\mathbf{v}'\mathbf{Wv})$, and under the definition of γ as the maximum value of λ, the first order condition is transformed into a more familiar form:

$$(\mathbf{W}^{-1}\mathbf{B} - \gamma\mathbf{E})\mathbf{v} = \mathbf{0}\ , \tag{8}$$

where $\mathbf{E}$ is the identity matrix and $\mathbf{W}$ is assumed to be regular. Therefore, it is obvious that the highest eigenvalue of the nonsymmetric matrix $\mathbf{W}^{-1}\mathbf{B}$ and the corresponding eigenvector $\mathbf{v}$ need to be calculated. In the two-group case there is only one eigenvalue because one of the roots of the characteristic function is zero. Consequently, only this particular eigenvalue needs to be found [26, p.278]. Furthermore, the eigenvector is not defined by the absolute but rather by the relative values of its elements. For that reason, given the eigenvector, the absolute values of v_j are not unique but only their ratios have a unique solution. Thus, the discriminant criterion does not change for multiples of the discriminant coefficients because both the between-group sum of squares and the within-group sum of squares increase by an equal factor if the coefficients are multiplied by the same value.
With a specific discriminant function, the sample can be reclassified and future observations can be assigned to one of the *a priori* groups based on a cutoff-value, which is usually the weighted mean of the two centroids. Moreover, statistical decision methods can be used to apply certain desirable criteria to the assignment procedure [26, p.255].

2.2.2 Assumptions of Discriminant Analysis

As with all multivariate techniques, discriminant analysis is based on a number of assumptions. Two key assumptions exist. Multivariate normality of the independent variables is assumed as well as the equality of covariance matrices of the two groups. More specifically, it is assumed that the vector $(X_{11}, \ldots, X_{j1}, \ldots, X_{n1}, X_{12}, \ldots, X_{j2}, \ldots, X_{n2})'$, where X_{jg} is variable j in group g, is multivariate normally distributed with $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$. The vector $\boldsymbol{\mu}$ contains the two vectors $\boldsymbol{\mu}_1$ and $\boldsymbol{\mu}_2$, where the elements of $\boldsymbol{\mu}_1$ and $\boldsymbol{\mu}_2$ are the expected values of the n variables in group one and two, respectively [26, p.263].
The assumption of multivariate normality is necessary for the significance tests presented in the following section. If the assumption is violated, the tests lose validity. Violations of the covariance equality assumption affect the significance tests and classification results. Remedies may be possible through transformations of the data to reduce the disparities of the matrices.
Mixed evidence exists concerning the sensitivity of discriminant analysis to contraventions of these assumptions, but a number of sources suggest that the method seems to be quite robust against violations [26, p.264].

2.2.3 Test Statistics

There are various test statistics that can be applied in the context of discriminant analysis. Two main test statistics are used very frequently and will be discussed below. They form the basis for an assessment of the significance of a discriminant analysis approach.
The first test is referred to as a univariate test and is equivalent to an independent sample t-test if there are only two groups. It is used to evaluate the significance of the variables X_j.
A general version of this test is based on Wilks' Λ

$$\Lambda = \frac{1}{1+\lambda} = \frac{SS_w}{SS_t} \in [0,1] \; , \tag{9}$$

where SS_t is the total sum of squares and $SS_t = SS_b + SS_w$. The significance of the Λ is then assessed by converting it into the F-ratio

$$F = \left(\frac{1-\Lambda}{\Lambda}\right)\left(\frac{I_1 + I_2 - p - 1}{p}\right) \; , \tag{10}$$

where p is the number of variables that is tested for. Under the null hypothesis

$$H_0 : \boldsymbol{\mu}_1 = \boldsymbol{\mu}_2 \; , \tag{11}$$

where the elements of $\boldsymbol{\mu}_1$ and $\boldsymbol{\mu}_2$ are the expected values of the n variables in group one and two, respectively, the ratio is F-distributed with p and $I_1 + I_2 - p - 1$ degrees of freedom and the null hypothesis is rejected for small values of Λ. If a univariate test is conducted and $p = 1$, then the test is equal to the square of the independent sample t-test with the null hypothesis $H_0 : \mu_{j1} = \mu_{j2}$, where μ_{j1} and μ_{j2} are the expected values of variable j in group one and two, respectively [26, p.250]. It is important to stress that this test compares the different variables separately and no judgement is made about the whole profile of variables.
Once the discriminant function is derived, it is still unknown how well the function is able to discriminate between observations of the groups. To test the significance of the discriminant function the Wilks' Λ test statistic

$$\chi^2 = -\left[N - \frac{n+G}{2} - 1\right] \ln \Lambda \tag{12}$$

is used with the null hypothesis $H_0 : \boldsymbol{\mu}_1 = \boldsymbol{\mu}_2$, where $N = \sum_{g=1}^{G} I_g$. Under the null hypothesis, the Wilks' Λ test statistic is asymptotically χ^2-distributed with $n(G-1)$ degrees of freedom [26, p.252]. This test statistic takes into account the multivariate interrelation of the variables and thus tests the whole profile of n variables simultaneously. Equivalently, the significance of the discriminant function can be assessed by the F-ratio in equation 10 for $p = n$ variables.[3]

[3] Various test statistics such as the t-statistic, F-statistic and Hotellings T^2 are special cases of the Wilks' Λ test statistic [26, p.252].

2.3 Development of the Model

2.3.1 The Sample

ALTMAN's intention is to "attempt an assessment of [...] the quality of ratio analysis as an analytical technique" to predict corporate bankruptcy using discriminant analysis [1, p.589] and therefore to distinguish between bankrupt and non-bankrupt firms on the basis of financial accounting figures. The data consist of 66 publicly listed manufacturing corporations. 33 of them form the bankrupt group. These firms applied for bankruptcy proceedings under Chapter X of the National Bankruptcy Act in a period covering 20 years. For the bankrupt firms a paired sample is chosen on a stratified random basis from non-bankrupt firms that were still in existence in 1966. Table 1 summarizes the basic features of the data set.

Group	Sample size	Asset range	Asset mean	Time covered
Bankrupt	$N = 33$	\$0.7 – \$25.9 million	\$6.4 million	1946-1965
Non-bankrupt	$N = 33$	\$1 – \$25 million[4]	\$9.6 million	1946-1965

Table 1: Characteristics of the initial sample of 66 manufacturing firms [1, p.593]

The data for 22 different variables in the initial sample are taken from financial statements one reporting period prior to bankruptcy with an average lead time to bankruptcy of seven and one half months. ALTMAN chooses 22 variables to be considered in his model that showed univariate discriminating ability in past studies [1, p.594].

2.3.2 The Discriminant Function

Using an iterative computer program,[5] the following discriminant function is processed:

$$
\begin{aligned}
Z &= 0.012X_1 + 0.014X_2 + 0.033X_3 + 0.006X_4 + 0.999X_5 \ . && (13)\\
X_1 &= \text{Working capital/Total assets}\\
X_2 &= \text{Retained Earnings/Total assets}\\
X_3 &= \text{Earnings before interest and taxes/Total assets}\\
X_4 &= \text{Market value equity/Book value of total debt}\\
X_5 &= \text{Sales/Total assets}
\end{aligned}
$$

On the basis of four different criteria, five from the 22 variables are chosen for the discriminant function [1, p.594]. "The function [...] does the best job among the alternatives which include numerous computer runs analyzing different ratio-profiles" although the "profile finally established did not contain the most significant, amongst the twenty-two original ones, measured independently" [1, p.594]. It can be seen very clearly how the multivariate is different to the

[4]Range of the asset values for the stratification process. Range in the actual sample might differ.

[5]The program was developed by COOLEY and LOHNES.

univariate perspective. If a variable is able to distinguish between groups significantly, it does not mean that the contribution to a multivariate approach is necessarily as noteworthy.

2.3.3 Discriminating Ability of the Model

In his paper, Altman uses a test for the individual discriminating ability of the variables, which he calls "F"-test, that is, the fraction of "the difference between the average values of the ratios in each group [and] the variability (or spread) of values of the ratios within each group" [1, p.596]. The F-distribution is used to conduct the test. The exact test statistic is not specified in the paper, but it is probably an independent sample t-test as presented in equation 10.

Variable	**Sample Mean Bankrupt Group**	**Sample Mean Non-Bankrupt Group**	**F Ratio**
X_1	-6.1%	41.4%	32.60*
X_2	-62.6%	35.5%	58.86*
X_3	-31.8%	15.3%	26.56*
X_4	40.1%	247.7%	33.26*
X_5	150.0%	190.0%	2.84

*Significant at the 0.1% level with $F_{1,60}$

Table 2: Results for the test of the individual significance of the variables [1, p.596]

The results in Table 2 strongly imply significant differences between the groups for variables X_1 to X_4. Thus, the null hypothesis, that the expected value of each variable is equal for the two groups, can be rejected at the 0.1% significance level and correspondingly the hypothesis that they come from the same distribution. It is clear from these results that the first four variables are included in the model as they contain high discriminating ability on a univariate level. Altman includes the fifth variable for another reason elaborated below.

It is not sufficient to only test for univariate significance because a variable might contribute to the overall discriminating ability in combination with other variables, even if it does not show significant univariate discriminating ability. For this purpose Altman uses standardised discriminant coefficients, defined as the discriminant coefficient multiplied by the standard deviation of the particular variable. The higher the standardised discriminant coefficient, the stronger "the contribution of the variable to the total discriminating power of the function" [1, p.596]. Because X_5 has the second highest contribution to the discriminating power, it becomes clear that the variable should be included in the model. To assess the overall discriminating ability of the discriminant function, the author applies an "F"-test, which is probably the equivalent to Wilks' Λ test statistic, with the "null hypothesis that the observations come from the same population" [1, p.598]. The F-distribution is used with five and 60 degrees of freedom. Again, the exact test statistic is not mentioned in the paper. From the data $F = 20.7$ is calculated, which indicates high significance and therefore the null hypothesis can be rejected at the 1% significance level [1, p.598].

Variable	Standardised coefficient	Ranking
X_1	3.29	5
X_2	6.04	4
X_3	9.89	1
X_4	7.42	3
X_5	8.41	2

Table 3: Relative contribution of the variables to the overall discriminating power of the model [1, p.597]

In conclusion, the aforementioned results from the test statistics support the notion that the ratio profiles are different for bankrupt and non-bankrupt firms, in particular when analysed simultaneously. Consequently, using the discriminant function to predict bankruptcy for future observations is expected to show significantly better results than a random choice.

2.3.4 Descriptive and Predictive Ability of the Model

The preceding part discussed the discriminating ability of the model. Obviously, this is just the first step. Lachenbruch suggests that "discriminant analysis is concerned with the problem of assigning an unknown observation to a group with a low error rate" [20, p.1]. Thus, the point of discriminant analysis is not to discriminate between the groups in the original sample but to use the discriminant function to predict the group assignment for future observations. In the context of the Z-Score model, Altman tests if a newly observed firm is predicted to be part of the bankrupt or non-bankrupt group using the Z-Score function. For the purpose of testing how well the model can predict the bankruptcy of firms, the author uses accuracy matrices illustrated in Table 4. Clearly, Type I and II errors should be small and the correct classifications should preponderate if the function is to be accepted for prediction. The accuracy should significantly differ from a random assignment to groups, which is an expected accuracy of 50% in this model with equal group sizes. Altman chooses six different samples to test the descriptive and predictive ability starting with the initial sample.

	Predicted group	
Actual group	Bankrupt	Non-bankrupt
Bankrupt	Hit	Type I error
Non-Bankrupt	Type II error	Hit

Table 4: General structure of the accuracy matrix [1, p.599]

Initial sample: Typically, the initial sample is tested although this is clearly not a strict measure for the predictive ability but rather a descriptive measure for how well the discriminant function is able distinguish between the groups in the original sample. Using the discriminant function, the 66 observations from the initial sample are reclassified. Table 5 shows that 95% of all observations are classified correctly. This result is "encouraging" [1, p.599] but an outcome that would be expected because the sample was used to construct the Z-Score function.

Actual group	Predicted group Bankrupt	Non-bankrupt
Bankrupt	31 (94%)	2 (6%)
Non-Bankrupt	1 (3%)	32 (97%)

Table 5: Accuracy matrix for the initial sample [1, p.599]

Initial sample two years prior to bankruptcy: In addition, Altman uses data from two years prior to the bankruptcy for the firms of the initial sample, whereas the initial data are from one reporting period prior to bankruptcy. Evidently, the accuracy is reduced because of the larger time lag. Table 6 indicates that eleven out of 65 observations (17%) are not correctly classified. Still, 72% of the bankrupt and 94% of the non-bankrupt manufacturers are classified correctly. Interestingly, the prediction of non-bankruptcy seems to be more accurate in this case. According to the author, these results are evidence that bankruptcy can be predicted up to two years prior to bankruptcy [1, p.600].

Actual group	Predicted group Bankrupt	Non-bankrupt
Bankrupt	23 (72%)	9 (28%)
Non-bankrupt	2 (6%)	31 (94%)

Table 6: Accuracy matrix for the initial sample two years prior to bankruptcy [1, p.600]

Subsets of the initial sample: Because the initial sample of 66 firms is used to calculate the discriminant function, the resulting accuracy might be biased upward owing to the fact that sampling errors or a search bias could exist in the original sample [1, p.600]. To test whether there is a bias in the data, a method suggested by Frank et al. [15], which estimates the parameters of the model using subsets of 16 firms, is employed. Then, the other firms are classified, and a basic t-test as presented in equation 14 is applied. From Table 7 Altman concludes "that any search bias does not appear significant" [1, p.601].

Replication	Correct classifications	Value of t
1	91.2%	4.8*
2	91.2%	4.8*
3	97.0%	5.5*
4	97.0%	4.5*
5	91.2%	4.8*

*Significant at the 0.1% level

Table 7: Classification accuracy for subsets of the initial sample [1, p.601]

$$t = \frac{\text{proportion correct} - 0.5}{\sqrt{\frac{0.5(1-0.5)}{N}}} \tag{14}$$

Secondary sample of bankrupt firms: A new sample of 25 bankrupt firms is chosen with the same asset range as the original sample, and data are collected from one statement prior to bankruptcy. This test is essential to measure the predictive ability of the model. The results in Table 8 are "surprising" [1, p.601] because the accuracy is higher in comparison to the initial sample with 96% correct classifications in the new sample versus 94% in the initial sample. This clearly suggests that the model is useful for the prediction of bankruptcy.

	Predicted group	
Actual group	Bankrupt	Non-bankrupt
Bankrupt	24 (96%)	1 (4%)

Table 8: Accuracy matrix for a secondary sample of bankrupt firms [1, p.601]

Secondary sample of non-bankrupt firms: For the next test ALTMAN chooses 66 firms that encountered earning problems between 1958 and 1961, a time with poor GNP growth. Thus, not simply non-bankrupt firms are selected but also firms that might be likely to be classified as bankrupt firms (Type II error). Table 9 shows that 79% of the firms are correctly classified by the discriminant model. ALTMAN concludes that "the selection process is successful in choosing firms which have showed signs (profitability) of deterioration" [1, p.602]. Taking into account that these firms experienced earning problems, the result is quite convincing as the model is still able to distinguish between bankrupt and non-bankrupt entities under these circumstances.

	Predicted group	
Actual group	Bankrupt	Non-bankrupt
Non-bankrupt	14 (21%)	52 (79%)

Table 9: Accuracy matrix for a secondary sample of non-bankrupt firms [1, p.602]

Long-range predictive accuracy: One last perspective that ALTMAN chooses is the time lag before the date of bankruptcy. The question is whether tendencies of bankruptcy can be identified a long time before the actual event. Data from the initial sample of 33 bankrupt firms are taken from the first to the fifth year prior to bankruptcy. Due to data unavailability, the sample size varies. Noticeably, the reliability of the model decreases with the time lag. "After the second year, the discriminant model becomes unreliable in its predictive ability" [1, p.604].

Years prior to bankruptcy	**Sample size**	**Hits**	**Misses**	**Correct classifications**
1	$N = 33$	31	2	95%
2	$N = 32$	23	9	72%
3	$N = 29$	14	15	48%
4	$N = 28$	8	20	29%
5	$N = 25$	9	16	36%

Table 10: Long-range classification accuracy for the initial sample [1, p.604]

2.4 Summary

ALTMAN's results are very "encouraging" with 94% of the initial sample and 95% of all firms in the bankrupt and non-bankrupt groups correctly classified and similarly good results for secondary samples up to two years prior to bankruptcy [1, p.609]. He argues that "based on the results it is suggested that the bankruptcy prediction model is an accurate forecaster of failure up to two years prior to bankruptcy and that the accuracy diminishes substantially as the lead time increases" [1, p.604]. Clearly, the accuracy of the model is quite convincing. Still, the model is applied to a very narrow sample of publicly listed manufacturing firms in the United States. Beyond doubt, the results suggest that the model can be used to predict bankruptcy, but scepticism is reasonable at this point. To pass a judgement on the research question, further studies need to be reviewed. The next section therefore presents extensions and further results for the Z-Score model as well as a general review of studies that also applied discriminant analysis to the subject of corporate financial distress. A few selected studies will be examined in more detail, which use discriminant analysis to predict business failure, in order to assess whether ALTMAN's results are generally confirmed in different environments and for a variety of entities. The selection of studies is subjective and based on the similarity to the Z-Score model and the possible contribution to the discussion of this work. In particular, the objective of the comparative analysis is to find out whether the model is applicable in different countries, to different company types and in different years.

3 Discriminant Analysis in a Modern Financial Market Environment

3.1 Extensions of the Z-Score Model

After its publication in 1968, the Z-Score model is followed by two key extensions. The reason for these revisions is the fact that the original model is restricted to publicly listed manufacturing firms. To make the model applicable to a wider range of companies, these restrictions needed to be addressed. Therefore, the book value is substituted for the market value in variable X_4 to deal with the first restriction. With the same sample used to develop the Z-Score model, discriminant analysis then derives the Z′-Score model:

$$Z' = 0.717X_1 + 0.847X_2 + 3.107X_3 + 0.420X_4 + 0.998X_5 \ . \tag{15}$$

A further modification is used to remove the focus on manufacturing firms. The "industry-sensitive variable" X_5 (Sales/Total Assets) is removed to "minimize the potential industry effect" [8, p.149]. Again, the book value is used for variable X_4. The Z′′-Score model is given by

$$Z'' = 6.56X_1 + 3.26X_2 + 6.72X_3 + 1.05X_4 \ . \tag{16}$$

The reclassification results for the revised Z'-Score model are less accurate in comparison to the original Z-Score model. 91% of bankrupt firms are classified correctly by the model versus 94% in the original model. The accuracy for non-bankrupt firms is identical with 72%. Reclassification results for the Z''-Score model are not available [8, p.148]. Obviously, generalising the original model comes at a cost and market values should be used when available. The market value seems to contain a higher predictive ability suggesting that the market generates information about the condition of a company not included in accounting figures.

3.2 A Brief History of Subsequent Research

Altman was probably the first academic to apply discriminant analysis to predict business failure. Before that, Durand [12] applied this technique to the evaluation of the creditworthiness of used car loan applicants in 1941 and Myers and Forgy [24] used it to evaluate good and bad instalment loans in 1963. Since then, numerous studies have followed similar approaches.

In 1997, Altman together with Narayanan published a survey of international business failure classification models [4]. They looked at a large number of empirical applications of models from 22 countries.[6] Apart from discriminant analysis, researches also applied multi-nomial logit analysis, probit analysis, recursive partitioning, Bayesian discriminant analysis, survival analysis and neural networks. However, discriminant analysis is still the most popular technique and it seems to be the standard comparison for corporate financial distress prediction models [4, p.2]. Where researchers have not explicitly used it, results where usually compared to those from discriminant analysis. One interesting outcome from the survey is that results from discriminant analysis are in most cases equally successful in comparison to the other abovementioned methods [4, p.2].

In a different article, published in 1998, Altman and Saunders present a brief summary of developments in credit risk measurement for the preceding 20 years [5]. The authors also draw the conclusion that discriminant analysis models have dominated the research in the field of credit risk [5, p.1723]. A selection of articles that deal with models for identifying company and country risk problems can be found in two special issues of the *Journal of Banking and Finance* from 1984 and 1988.

Without going into detail about the particular studies at this point, it can be generally concluded that the application of discriminant analysis to the prediction of business failure has been at the centre of corporate credit risk research since its first renowned publication by Altman. More sophisticated multivariate statistical techniques did not challenge this position fundamentally while still providing a wider set of tools. This finding is supported by the fact that the initial Z-Score model is cited in many finance and statistics textbooks today, 40 years later.[7]

[6] A list of the surveyed studies can be found in Appendix B. See also Altman (2005), Blum (1974), Deakin (1972), Edminster (1972) and Micha (1984).

[7] See Duffie et al. (2003), Gourieroux et al. (2007) and Lando (2004).

3.3 A Survey of Discriminant Analysis in Credit Risk Research

3.3.1 Problem Banks in the United States

In 1975, SINKEY published the article "A Multivariate Statistical Analysis of the Characteristics of Problem Banks" [28] in which he applies discriminant analysis to distinguish between problem and non-problem banks in the United States. A problem bank is a bank that "has violated a law or regulation or engaged in an 'unsafe or unsound' banking practice to such an extent that the present or future solvency of the bank is in question"[8] [28, p.21]. The Federal Deposit Insurance Corporation (FDIC) is responsible for the classification of the banks. The first part of the data set consists of 110 banks that were newly classified as problem banks during 1972 and the first few months of 1973. All observations were non-problem banks during 1969-1971. Each problem bank is then matched with a non-problem bank and the financial figures are taken from year-end balance sheets. The 110 problem banks are representative in terms of size and branching structure for all banks insured by the FDIC [28, p.23]. The author applies quadratic discriminant analysis instead of linear discriminant analysis and uses ten variables in his model.[9] For every year one discriminant function is calculated, and the observations from the initial sample are reclassified. The reclassification results are presented in Table 11.

		Predicted	
	Actual	Problem	Non-problem
1969	Problem	68 (62%)	42 (38%)
	Non-problem	17 (15%)	93 (85%)
1970	Problem	71 (65%)	39 (35%)
	Non-problem	22 (20%)	88 (80%)
1971	Problem	79 (72%)	31 (28%)
	Non-problem	22 (20%)	88 (80%)
1972	Problem	80 (78%)	23 (22%)
	Non-problem	14 (14%)	89 (86%)

Table 11: Accuracy matrix for four subsequent years [28, p.32]

Between 72% in 1970 and 82% in 1972 of all problem and non-problem banks are correctly reclassified and the accuracy is substantially higher for the prediction of non-problem institutions, a result that is also apparent in the *Z*-Score model. Because a secondary sample is not available, an almost unbiased classification method proposed by LACHENBRUCH [15] is applied, which supports the results. A potential sample bias does not appear to be significant [28, p.32]. SINKEY concludes that the "classification results are quite encouraging" and that "the implications are [...] that problem banks appear quite distinct from nonproblem banks" [28, p.33]. Again, the

[8] This definition was introduced by the Federal banking agencies in the United States. See SINKEY [28] for further explanations.

[9] See Appendix C for the specification of the variables. Quadratic discriminant analysis is used when the assumption of equal covariance matrices is not fulfilled. The general approach is very similar to linear discriminant analysis and the results can be interpreted equivalently.

application of discriminant analysis to a very different sample was successful.
These results generally confirm ALTMAN's findings but also add another dimension to it. In this study, discriminant analysis is performed with a very different definition of failure. Not bankruptcy as a worst-case scenario is used but problem and non-problem banks, which is a less rigorous understanding of financial distress. Clearly, the actual distinction between the groups does not appear to be restrictive and various definitions of failure can be used.

3.3.2 Industry-Wide Application in Australia

In 1984, IZAN applied discriminant analysis to the failure of Australian companies [18]. The author analyses a larger sample than most preceding studies with 51 failed and 48 non-failed firms, now from different industries. In contrast to publicly listed manufacturing firms in ALTMAN's model, the groups discussed in this study are more heterogeneous. Failure for this research design means that a company had a receiver or liquidator appointed during the years 1963-1979. From ten candidate ratios, five are selected for the linear discriminant analysis as they work together most successfully in the classification of business failure and data are collected for one year prior to failure.[10] The five variables in the model are significant at the 5% level or better when a univariate F-test is applied [18, p.313]. A test for the whole profile of variables is not conducted. One distinctive aspect of this model is that the ratios are standardised in the sense that every ratio is divided by the respective firms' industry median of the ratio to account for the differences of average ratios across industries.

	Industry-relative Ratios		Raw Ratios	
	Predicted group		**Predicted group**	
Actual group	Failed	Non-failed	Failed	Non-failed
Failed	48 (94%)	3 (6%)	46 (90%)	5 (10%)
Non-failed	5 (10%)	43 (90%)	5 (10%)	43 (90%)

Table 12: Accuracy matrix for the industry relative and raw ratio model [18, p.314]

Table 12 shows that the accuracy of the standardised model for the reclassification of the initial sample is slightly better with 94% correct classifications one year prior to the event of failure for the failed firms versus 90% in the model using the traditional ratios. The overall accuracy for the first model is 92%. A LACHENBRUCH bias test [19] indicates that the results are not responsive to a sample bias. The industry-relative analysis also shows good classification precision for larger time lags with an overall accuracy of 82% and 76% for all failed and non-failed firms and 75% and 64% for the sample of failed firms for the second and third year prior to the failure date, respectively. These results indicate that the industry-relative approach works well and that it can be applied more confidently to broad-industry samples. This conclusion is supported by a secondary sample of ten failed firms with 100% and 70% correct classifications in year one and two prior to bankruptcy, respectively [18, p.316].

[10] See Appendix C for the specification of the variables.

Again, the results from this study support the view that Z-Score type models are successful in predicting financial distress of companies. In particular, Izan demonstrates how the model can be applied to an industry-wide sample. It should also be noted that the applicability does not seem to be affected by time as it could be expected since average ratio measures shift over time.

3.3.3 Korean Failed Companies

Both aforementioned studies apply discriminant analysis to predict corporate financial distress of firms in 'developed countries,'[11] but a relevant question is whether a model similar to the Z-Score model can be applied to 'developing countries.' In 1995, Altman, Kim and Eom constructed a distress classification model for Korean companies [3]. Korea qualified as a 'developing country' in this context. The first part of the sample consists of 34 publicly listed industrial and trading companies with an asset range of \$13 million and \$296 million. Either technical insolvency, which occurs when credit of a company is no longer accepted, or liquidation is the definition of failure and whatever occurs first is taken as the date of failure. A sample of 61 non-failed firms is assembled by a random one to one pairing procedure. 22 variables are considered for the model and four chosen based on different test results [3, p.238]. All four variables[12] are significant at a 1% level for a univariate t-test [3, p.326]. Two models are derived, a $K1$ model using the book value of equity for variable four and a $K2$ model, where the market value of equity is used. This procedure is equivalent to the Z- and Z'-Score models discussed earlier. In the following, only the results for the more general $K1$ model are presented owing to the fact that the differences between the results for the two models are only moderate and that the $K1$ model can be applied to a wider range of companies.

Years prior to bankruptcy	**Sample size**	**Hits**	**Misses**	**Correct classifications**
1	$N = 34$	33	1	97%
2	$N = 34$	30	4	88%
3	$N = 33$	23	10	70%
4	$N = 32$	16	16	50%
5	$N = 16$	11	5	69%

Table 13: Long-range classification accuracy for the distressed group [3, p.240]

Table 13 shows the reclassification results for the initial sample for different time lags using the same discriminant function based on the data from one year prior to failure. These results are more accurate than those from Altman's Z-Score model. For the first year, the model is extremely accurate with only one misclassified observation. Obviously, the accuracy diminishes as the time prior to distress increases although not as rapidly as in the Z-Score context. The classification accuracy of non-failed firms ranges between 77% and 93% in the years 1988

[11] See Appendix B for a definition developed and developing countries.

[12] See Appendix C for the specification of the variables.

to 1992, an equivalently convincing degree of accuracy [3, p.242]. The $K2$ model showed comparably strong precision.
Two important remarks have to be made regarding this model. First, discriminant analysis is successfully applied to a 'developing country' and second, the argument, that the time horizon in which the model is used does not affect the applicability, is supported.

3.4 More Recent Results for the Z-Score Model

In a recent paper from 2002, Altman discusses the Z-Score model and its relevance concerning the recent developments in credit risk measurement [2].

Year prior to bankruptcy	**1969-1975** $N = 86$	**1976-1995** $N = 110$	**1997-1999** $N = 120$
1	82%	85%	94%
2	68%	75%	74%

Table 14: Accuracy matrix for subsequent samples [2, p.18]

One important point of his analysis is the testing of the Z-Score model for subsequent bankrupt firm samples. Table 14 shows that the model is correct in classifying between 82% and 94% of the observations for the first year prior to bankruptcy and the accuracy decreases for the second year. Based on data from one reporting period prior to bankruptcy, the accuracy ranges between 80% and 90% [2, p.18]. From the other studies discussed in section 3.3, published in the years 1975, 1984 and 1995, it was concluded that the model maintained its ability to predict financial distress regardless of the time frame it was applied in. The results mentioned above confirm this view. Another outcome is that, although the initial sample used to derive the Z-Score function is from before 1968, it can still be applied successfully in later years without the need to recalculate the discriminant function on the basis of the new samples.

3.5 Summary

In conclusion, it can be said that Z-Score type models have proofed to be very successful in a variety of contexts. Applications to different company types, diverse countries and three consecutive decades have produced similarly convincing results. Clearly, the Z-Score model illustrates high accuracy for later samples of bankrupt firms. Models that use discriminant analysis to predict financial distress of business enterprises thus generally qualify as instruments for prediction of these problems today. Nonetheless, the fact that the research design across studies significantly differs raises a problem: Which approach should be used for prediction? Owing to the fact that the discussed models are purely empirical, this question cannot be answered satisfactorily. The lack of a theoretical underpinning damps the enthusiasm for the model. No hypothesis can be derived from economic theory regarding the general validity of the results. Therefore, the applicability depends on empirical circumstances.

4 Conclusion

This work attempted to investigate whether discriminant analysis can be applied to predict corporate financial distress and consequently to pass a judgement on the relevance of the technique to the present-day decision maker.
Generally, the work of ALTMAN can be seen as "pathbreaking" [25, p.318] because it has initiated one major strand of empirical research in the field of credit risk measurement. More specifically, the results from ALTMAN and other academics discussed in this work suggest that discriminant analysis can be confidently applied to predict corporate financial distress with a lead time up to two years prior to the event. A large number of applications similar to the initial Z-Score model proofed to be highly accurate in very diverse settings. From this perspective it is suggested that the research question should be approved of.
Still, the analysis has also revealed a critical weakness of the model, namely the lack of an "explicit and well-developed theory" [25, p.316]. The question why certain ratios are successful in predicting bankruptcy or other forms of financial distress when analysed simultaneously is not based on a theoretical argument, neither by ALTMAN nor by any of the other authors mentioned in this work. Though the success of the research indicates a strong underlying regularity, the objections remain, particularly because it is not self-evident under which circumstances the model works well and has a high predictive ability and in what cases discriminant functions, that have been calculated, are less accurate. This question cannot be answered with regard to the empirical orientation. Ultimately, the applicability depends on circumstances although many studies suggest that the model is robust against variations of the research design. "The danger is that the models [...] derived will not predict well when confronted with new data" [25, p.325] one reason why "these models do not command full professional acceptance" [25, p.316].
Thus, the overall conclusion from this thesis is two-sided. On the one hand ALTMAN revolutionised credit risk measurement with his well-known Z-Score model by introducing a multivariate empirical framework, but on the other hand he did not contribute extensively to the theoretical understanding of credit risk. Yes, the model works well, but it is not really understood why and when it does, which is a dilemma that remains unsolved.

Apendix

Appendix A

Proof for the equality of λ in sum- and matrix-notation for $n = 2$:

$$\lambda = \frac{\sum_{g=1}^{G} I_g(\bar{Z}_g - \bar{Z})^2}{\sum_{g=1}^{G}\sum_{i=1}^{I_g}(Z_{gi} - \bar{Z}_g)^2} = \frac{\mathbf{v'Bv}}{\mathbf{v'Wv}}$$

$$\mathbf{B} = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix} \text{ and } B_{12} = B_{21}$$

$$\mathbf{W} = \begin{pmatrix} W_{11} & W_{12} \\ W_{21} & W_{22} \end{pmatrix} \text{ and } W_{12} = W_{21}$$

$$\begin{aligned}
\mathbf{v'Bv} &= v_1^2 B_{11} + v_1 v_2(B_{12} + B_{21}) + v_2^2 B_{22} \\
&= v_1^2 B_{11} + 2v_1 v_2 B_{12} + v_2^2 B_{22} \\
&= v_1^2 \sum_{g=1}^{G} I_g(\bar{X}_{1g} - \bar{X}_1)^2 + 2v_1 v_2 \sum_{g=1}^{G} I_g(\bar{X}_{1g} - \bar{X}_1)(\bar{X}_{2g} - \bar{X}_2) \\
&\quad + v_2^2 \sum_{g=1}^{G} I_g(\bar{X}_{2g} - \bar{X}_2)^2 \\
&= \sum_{g=1}^{G} I_g(v_1\bar{X}_{1g} - v_1\bar{X}_1)^2 + 2\sum_{g=1}^{G} I_g(v_1\bar{X}_{1g} - v_1\bar{X}_1)(v_2\bar{X}_{2g} - v_2\bar{X}_2) \\
&\quad + \sum_{g=1}^{G} I_g(v_2\bar{X}_{2g} - v_2\bar{X}_2)^2 \\
&= \sum_{g=1}^{G} I_g[(v_1\bar{X}_{1g} - v_1\bar{X}_1) + (v_2\bar{X}_{2g} - v_2\bar{X}_2)]^2 \\
&= \sum_{g=1}^{G} I_g[(v_1\bar{X}_{1g} + v_2\bar{X}_{2g}) - (v_1\bar{X}_1 + v_2\bar{X}_2)]^2 \\
&= \sum_{g=1}^{G} I_g(\bar{Z}_g - \bar{Z})^2
\end{aligned}$$

since

$$\begin{aligned}
\bar{Z}_g &= v_1\bar{X}_{1g} + v_2\bar{X}_{2g} \\
\bar{Z} &= v_1\bar{X}_1 + v_2\bar{X}_2
\end{aligned}$$

$$
\begin{aligned}
\mathbf{v'Wv} &= v_1^2 W_{11} + v_1 v_2 (W_{12} + W_{21}) + v_2^2 W_{22} \\
&= v_1^2 W_{11} + 2 v_1 v_2 W_{12} + v_2^2 W_{22} \\
&= v_1^2 \sum_{g=1}^{G} \sum_{i=1}^{I_g} (X_{1gi} - \bar{X}_{1g})^2 + 2 v_1 v_2 \sum_{g=1}^{G} \sum_{i=1}^{I_g} (X_{1gi} - \bar{X}_{1g})(X_{2gi} - \bar{X}_{2g}) \\
&\quad + v_2^2 \sum_{g=1}^{G} \sum_{i=1}^{I_g} (X_{2gi} - \bar{X}_{2g})^2 \\
&= \sum_{g=1}^{G} \sum_{i=1}^{I_g} (v_1 X_{1gi} - v_1 \bar{X}_{1g})^2 + 2 \sum_{g=1}^{G} \sum_{i=1}^{I_g} (v_1 X_{1gi} - v_1 \bar{X}_{1g})(v_2 X_{2gi} - v_2 \bar{X}_{2g}) \\
&\quad + \sum_{g=1}^{G} \sum_{i=1}^{I_g} (v_2 X_{2gi} - v_2 \bar{X}_{2g})^2 \\
&= \sum_{g=1}^{G} \sum_{i=1}^{I_g} [(v_1 X_{1gi} - v_1 \bar{X}_{1g}) + (v_2 X_{2gi} - v_2 \bar{X}_{2g})]^2 \\
&= \sum_{g=1}^{G} \sum_{i=1}^{I_g} [(v_1 X_{1gi} + v_2 X_{2gi}) - (v_1 \bar{X}_{1g} + v_2 \bar{X}_{2g})]^2 \\
&= \sum_{g=1}^{G} \sum_{i=1}^{I_g} (Z_{gi} - \bar{Z}_g)^2
\end{aligned}
$$

since

$$
\begin{aligned}
Z_{gi} &= v_1 X_{1gi} + v_2 X_{2gi} \\
\bar{Z}_g &= v_1 \bar{X}_{1g} + v_2 \bar{X}_{2g}
\end{aligned}
$$

The vector differentiation of λ with $n = 2$ is given by:

$$
\begin{aligned}
\frac{\delta \lambda}{\delta \mathbf{v}} &= \frac{\delta}{\delta \mathbf{v}} \left(\frac{\mathbf{v'Bv}}{\mathbf{v'Wv}} \right) \\
&= \frac{\left[\frac{\delta}{\delta \mathbf{v}} (\mathbf{v'Bv}) \right] \mathbf{v'Wv} - \mathbf{v'Bv} \left[\frac{\delta}{\delta \mathbf{v}} (\mathbf{v'Wv}) \right]}{(\mathbf{v'Wv})^2} \\
&= \frac{2[(\mathbf{Bv})(\mathbf{v'Wv}) - (\mathbf{v'Bv})(\mathbf{Wv})]}{(\mathbf{v'Wv})^2} \\
\frac{\delta}{\delta \mathbf{v}} (\mathbf{v'Bv}) &= \begin{pmatrix} \dfrac{\mathrm{d}}{\mathrm{d}v_1} \\ \dfrac{\mathrm{d}}{\mathrm{d}v_2} \end{pmatrix} (\mathbf{v'Bv}) = \begin{pmatrix} \dfrac{\mathrm{d}}{\mathrm{d}v_1} \\ \dfrac{\mathrm{d}}{\mathrm{d}v_2} \end{pmatrix} (v_1^2 B_{11} + 2 v_1 v_2 B_{12} + v_2^2 B_{22}) \\
&= \begin{pmatrix} \dfrac{\mathrm{d}}{\mathrm{d}v_1} [v_1^2 B_{11} + 2 v_1 v_2 B_{12} + v_2^2 B_{22}] \\ \dfrac{\mathrm{d}}{\mathrm{d}v_2} [v_1^2 B_{11} + 2 v_1 v_2 B_{12} + v_2^2 B_{22}] \end{pmatrix} \\
&= \begin{pmatrix} 2 v_1 B_{11} + 2 v_2 B_{12} \\ 2 v_1 B_{12} + 2 v_2 B_{22} \end{pmatrix} = 2\mathbf{Bv} \\
\frac{\delta}{\delta \mathbf{v}} (\mathbf{v'Wv}) &= 2\mathbf{Wv}
\end{aligned}
$$

Appendix B

List of studies discussed in the paper "An International Survey of Business Failure Classification Models" by Altman and Narayanan published in *Financial Markets, Institutions and Instruments* [4, p.4]:

Developed countries	
Japan	Takahashi, Kurokowa and Watase (1979)
	Ko (1982)
Switzerland	Stein (1968)
	Beermann (1976)
	Weinrich (1978)
	Gebhardt (1980)
	Fischer (1981)
	von Stein and Ziegler (1984)
	Baetge, Huss and Niehaus (1988)
England	Taffler and Tisshaw (1977)
	Marais (1979)
	Earl and Marais (1982)
France	Altman, et al. (1973)
	Mader (1975, 1979, 1981)
	Collongues (1977)
	Bontemps (1981)
Canada	Knight (1979)
	Altman and Lavallee (1981)
The Netherlands	Bilderbeek (1977)
	van Frederiklust (1978)
Spain	Briones, Marin and Cueto (1988)
	Fernandez (1988)
Italy	Cifarelli, Corielli and Foriestieri (1988)
	Altman, Marco and Varetto (1994)
Australia	Castagna and Matolcsy (1981)
	Altman and Izan (1983)
	Izan (1984)
Greece	Gloubos and Grammatikos (1988)
	Theodossiou and Papoulias (1988)

Developing countries	
Argentina	Swanson and Tybout (1988)
Brazil	Altman, et al. (1979)
India	Bhatia (1988)
Ireland	Cahill (1981)
South Korea	Altman, Kim and Eom (1995)
Malaysia	Bidin (1988)
Singapore	Ta and Seah (1988)
Finland	Suominen (1988)
Mexico	Altman, Hartzell and Peck (1995)
Uruguay	Pascale (1988)
Turkey	Unal (1988)

The definition of “developing” and “developed” countries differs from the traditional use of the words in academic literature. In the survey the main characteristics of developed countries are: “1) failure prediction models have a long history, 2) corporate financial data are more readily available, 3) failure is easier to identify because of the existence of bankruptcy laws and banking infrastructures, 4) government intervention is somewhat less, but not nonexistent and 5) there is a more sophisticated regulation of companies to protect investors. The developing countries are characterized by the relative absence of the above factors” [4, p.3].

Appendix C

Variables used by Sinkey [28, p.28] in his model discussed in section 3.3.1:

X_1 = (Cash + U.S. Treasury Securities)/Assets

X_2 = Loans/Assets

X_3 = Provision for loan losses/Operational expense

X_4 = Loans/(Capital + Reserves)

X_5 = Operating expense/Operating income

X_6 = Loan revenue/Total revenue

X_7 = U.S. Treasury Securities’ revenue/Total revenue

X_8 = State & local obligations’ revenue/Total revenue

X_9 = Interest paid on deposits/Total revenue

X_{10} = Other expenses/Total revenue

Variables used by Izan [18, p.310] in his model discussed in section 3.3.2:

X_1 = Earnings before interest and taxes (EBIT)/Tangible total assets

X_2 = EBIT/Interest payments

X_3 = Current assets/Current liabilities

X_4 = Funded debt (borrowings)/Shareholder funds

X_5 = Market value of equity/Total liabilities

Variables used by Altman, Kim and Eom [3, p.240] in their $K1$ model discussed in section 3.3.3:

X_1 = log(Total assets)

X_2 = log(Sales/Total assets)

X_3 = Retained earnings/Total assets

X_4 = Book value of equity/Total liabilities

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