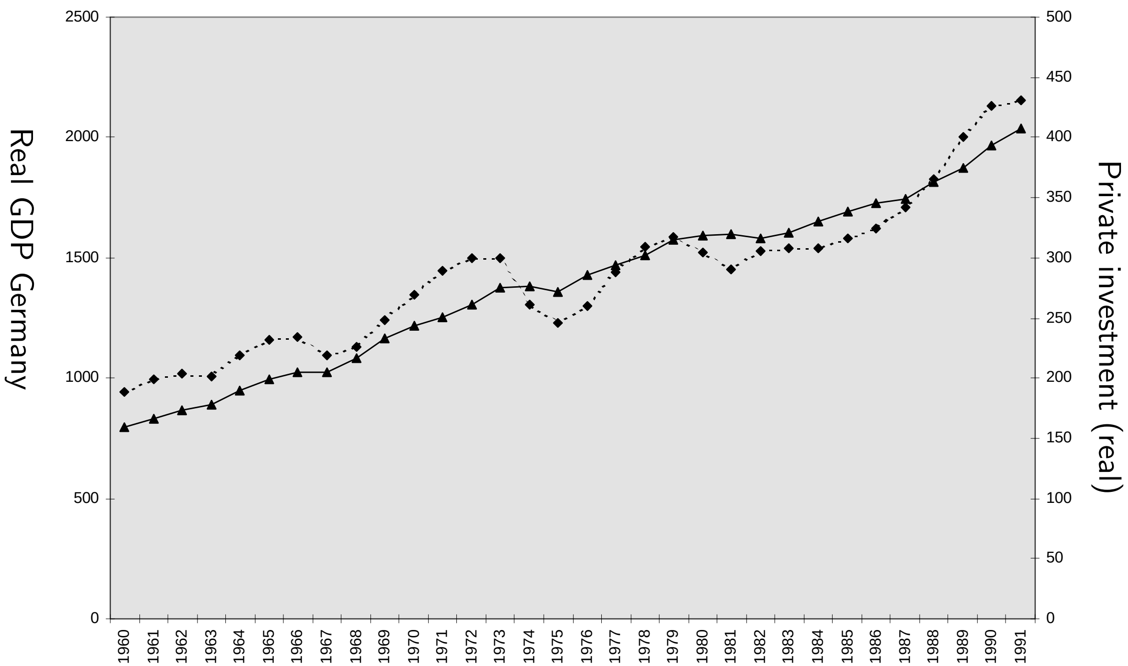
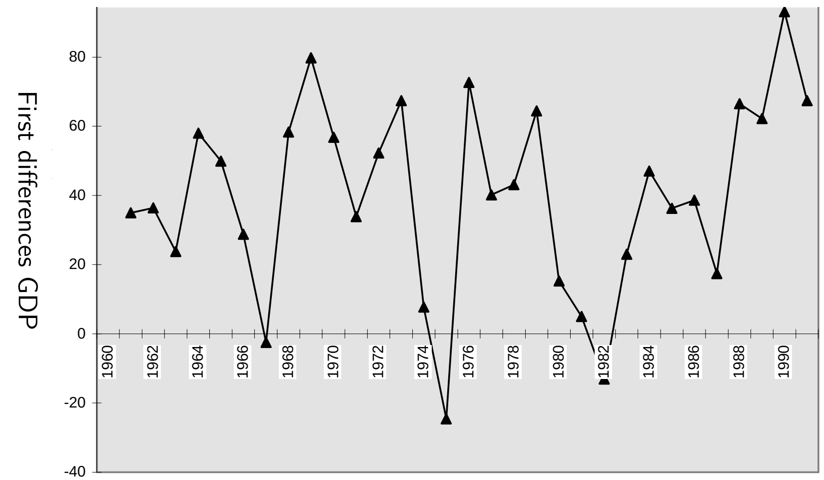
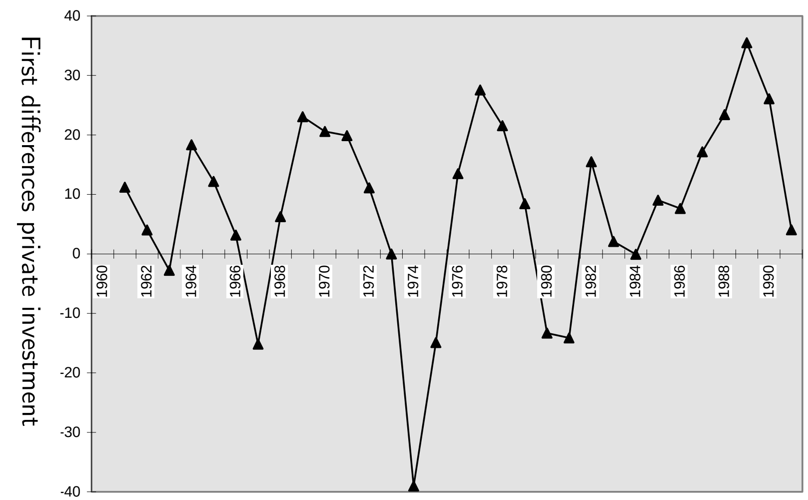
Basics and applications of cointegration analysis
Hamilton (1994) Ch. 19 (parts), Enders (1995), Ch. 6. (parts)

Many time series in economics and finance look like realizations of nonstationary stochastic processes







Ordinary Least Squares (OLS) regression using nonstationary time series is hazardous!

Applying OLS to macro time series yields

- small t-values

- high R^2

- positively autocorrelated residuals

Granger and Newbold

Simulation of independent random walk processes

$$y_{1t} = y_{1t-1} + u_{1t}$$

$$y_{2t} = y_{2t-1} + u_{2t}$$

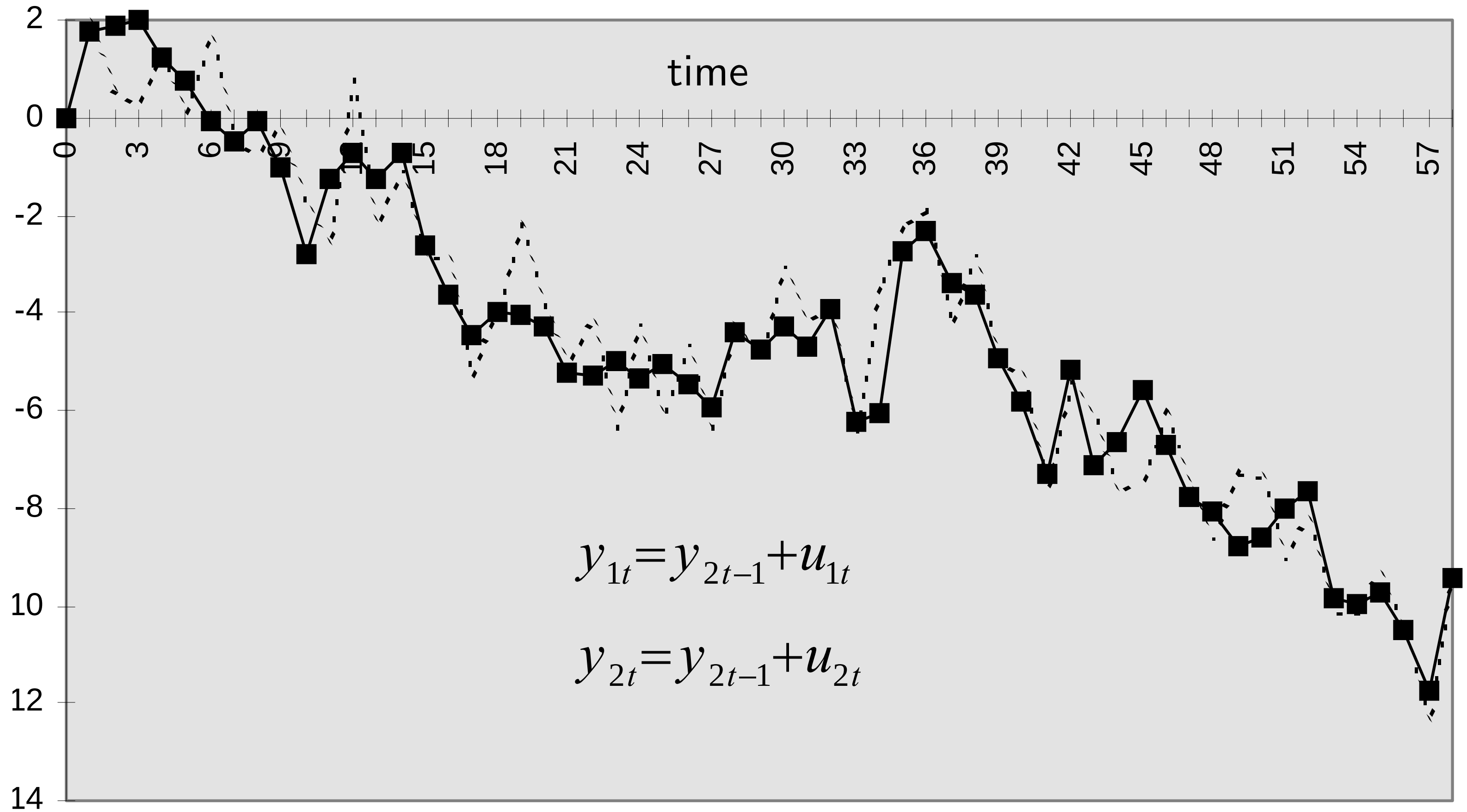
M

$$y_{1t} = \alpha + \gamma_1 y_{2t-1} + \gamma_2 y_{3t-1} + \ldots . e_t$$

Multiple linear regression

Result: Regression yields too often to a rejection of the (correct) null hypothesis that slope parameters are zero.

Cointegration: Graphical illustration



Cointegration: An economic interpretation

Long run equilibrium relation of economic time series

Possibility of short term deviations from equilibrium

Economic mechanisms move system to equilibrium

Examples:

Term structure of interest rates

Stock prices of assets traded on different markets

Purchase power parity between two countries

Consumption and Income

Cointegration: A definition

Long run equilibrium relation of economic time series

n x 1 vector of time series $y_t = (y_{1t}, y_{2t}, y_{3t}, .., y_{nt})'$ is cointegrated if each series is

- nonstationary (integrated of order one)

- there exists (at least one) linear combination $a'y_t$ which produces a stationary process

Bivariate example

$$y_{1t} = \gamma y_{2t} + u_{1t}$$
$$y_{2t} = y_{2t-1} + u_{2t}$$

$$y_{1t} - y_{1t-1} = \gamma u_{2t} + u_{1t} - u_{1t-1}$$
$$y_{2t} - y_{2t-1} = \Delta y_{2t} = u_{2t}$$

Linear combination $(y_{1t} - \gamma y_{2t}) = u_{1t}$ stationary

$(y_{1t} - \gamma y_{2t})$ Cointegrating relation $a = (1, -\gamma)'$ cointegrating vector

Cointegration: An economic example Purchase Power Parity (PPP)

No transaction costs and free trade

P_t^S Index of price level Switzerland (CHF per good)
P_t^U Index of price level USA ($ per good)
S_t Exchange rate (Dollar/CHF)

$$P_t^U = S_t P_t^S$$

$$p_t^U = s_t + p_t^S$$

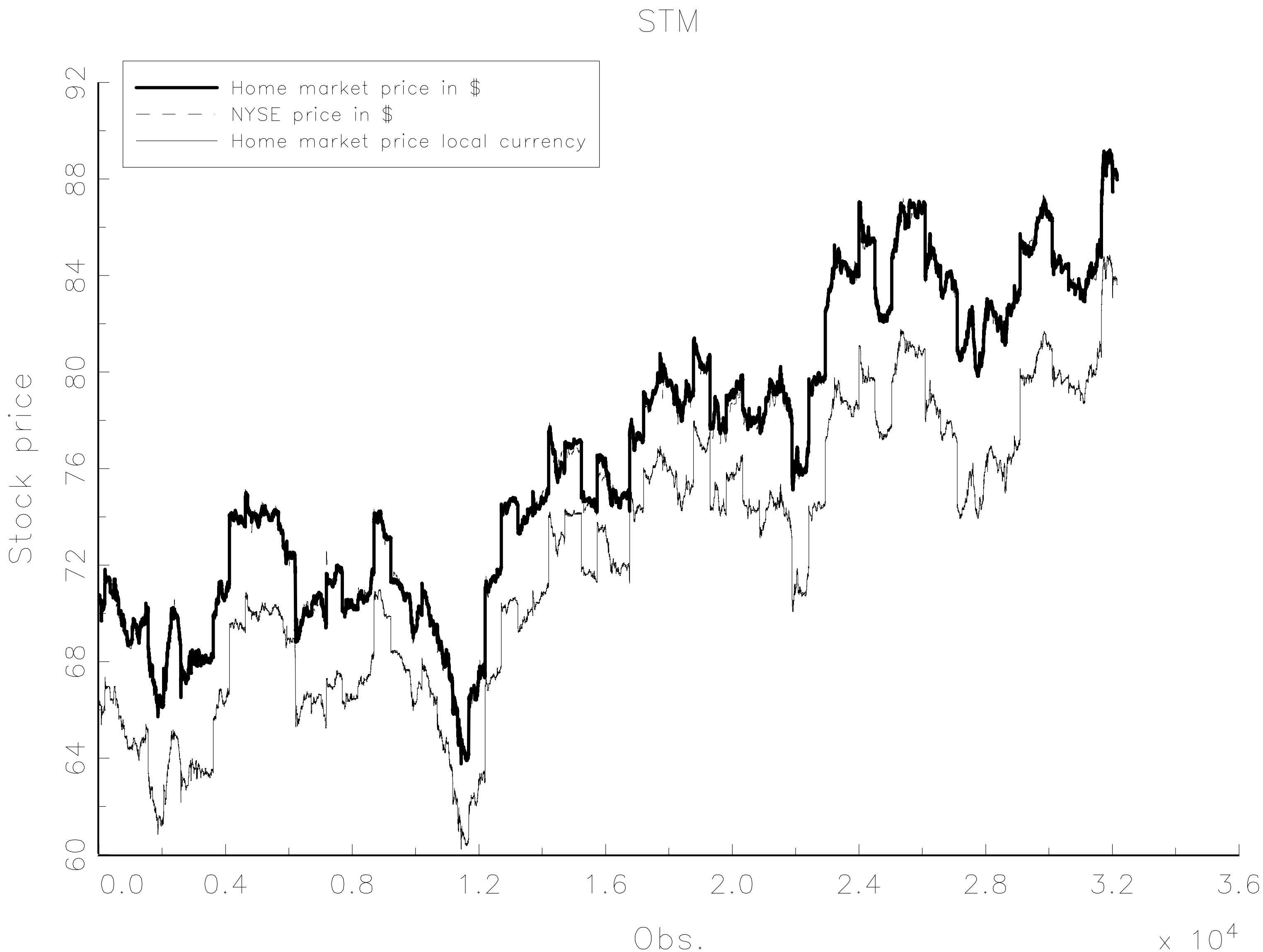
$$p_t^U - p_t^S - s_t = 0$$

Weaker version of PPP

$$z_t \equiv p_t^U - s_t - p_t^S$$

$\{z_t\}$ stationary stochastic process

Purchase power parity in the real world: Assets traded on parallel markets



The appropriate econometric specification to model dynamics of cointegrated time series: The equilibrium correction model

The bivariate case

$$\Delta y_{1t} = a_0 + \gamma_1 \left(y_{1t-1} - \beta_0 - \beta_1 y_{2t-1} \right) + a_{11} \Delta y_{1t-1} + a_{12} \Delta y_{2t-1} + more\ lags + u_{1t}$$
$$\Delta y_{2t} = b_0 + \gamma_2 \left(y_{1t-1} - \beta_0 - \beta_1 y_{2t-1} \right) + a_{21} \Delta y_{1t-1} + a_{22} \Delta y_{2t-1} + more\ lags + u_{2t}$$

Multivariate Case: Number of cointegrating relations?

Engle and Granger have proposed a method to estimate the parameters of a cointegrated system (1)

OLS estimation of ECM not feasible

$$\Delta y_{1t} = a_0 + \gamma_1 \left(y_{1t-1} - \beta_0 - \beta_1 y_{2t-1} \right) + a_{11} \Delta y_{1t-1} + a_{12} \Delta y_{2t-1} + more\ lags + u_{1t}$$

$$\Delta y_{2t} = b_0 + \gamma_2 \left(y_{1t-1} - \beta_0 - \beta_1 y_{2t-1} \right) + a_{21} \Delta y_{1t-1} + a_{22} \Delta y_{2t-1} + more\ lags + u_{2t}$$

Assume n variables, $h=1$ cointegrating relation (e.g. PPP $n=3$ $h=1$)

Step 1:

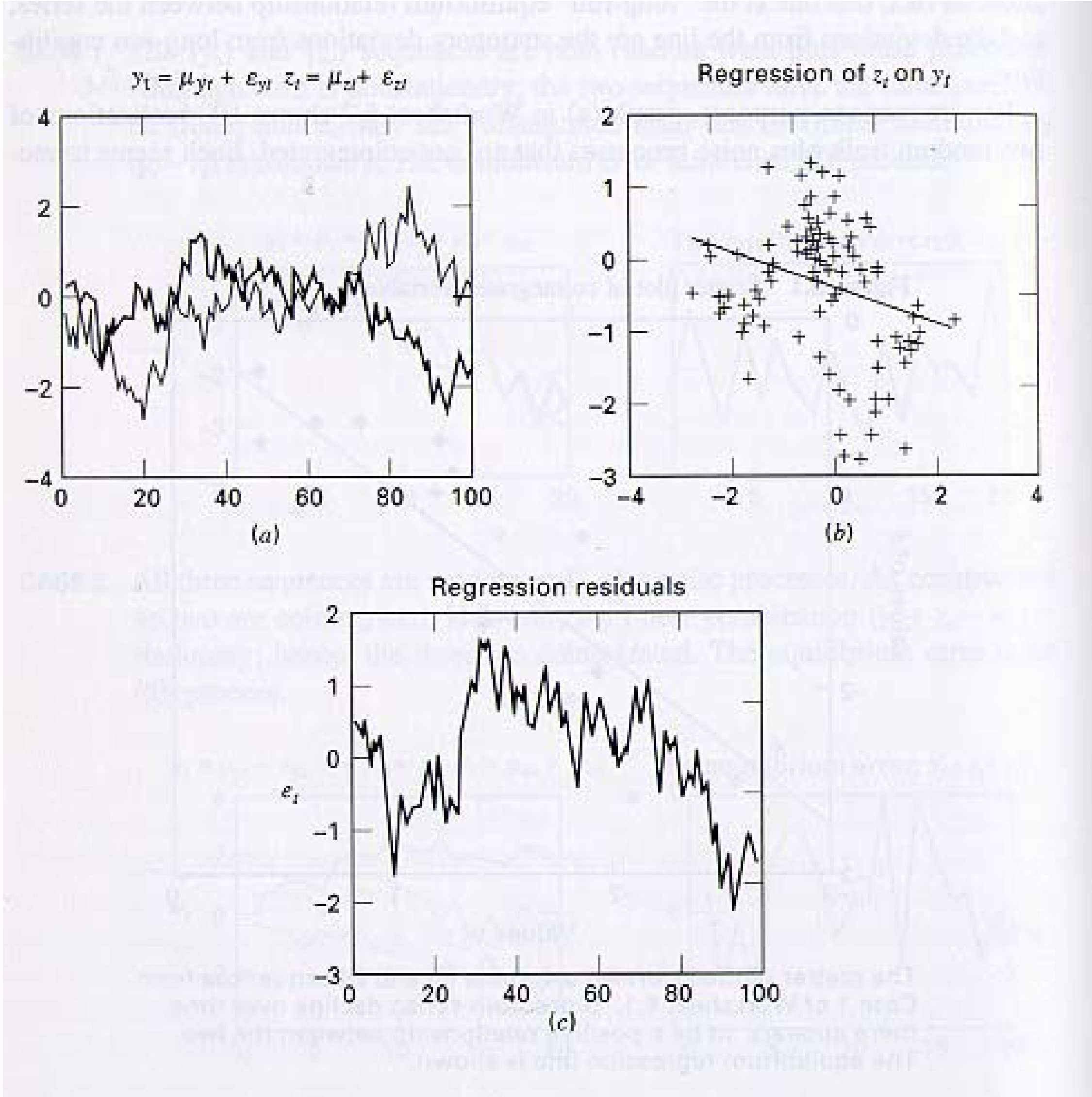
Test whether each of n variables is integrated of order one
(I(1), non-stationary, unit root process, first difference yields stationary series).

Standard tests: Dickey-Fuller and Perron tests. Null hypothesis: series non-stationary. Distribution of test statistic under Null: Non-standard, obtained by simulations.
Critical values (quantiles) tabulated.

If null hypothesis rejected (given α) for all series: cointegration rejected. Model with VAR.
If for some variables null rejected (given α) for others not: cointegration hypothesis rejected. Exclude variables from cointegrating relation.

If null hypothesis (non-stationarity) maintained (α) proceed to step 2

Illustration: Regression when no cointegration present



Engle and Granger have proposed a method to estimate the parameters of a cointegrated system (2)

Step 2:

Impose normalization. Put one variable LHS, others on RHS. Run a regresssion:
exemplary: n=2

$$y_{1t} = \beta_0 + \beta_1 y_{2t} + \varepsilon_t$$

Back out $\hat{\beta}_{0,}\hat{\beta}_1$

and $\hat{\varepsilon}_t = y_{1t} - \hat{\beta}_0 - \hat{\beta}_1 y_{2t}$

If y_{1t}, y_{2t} cointegrated => $\varepsilon_t = y_{1t} - \beta_0 - \beta_1 y_{2t}$ stationary

Test nonstationarity of series $\hat{\varepsilon}_t = \hat{\beta}_0 - y_{1t} - \hat{\beta}_1 y_{2t}$ using stationarity tests

Residual series based on estimated parameters: Different distribution of test statistic: Use correct critical value tables!

If null hypothesis of non-stationarity $\hat{\varepsilon}_t$ rejected. Proceed to step 3.

Engle and Granger have proposed a method to estimate the parameters of a cointegrated system (3)

Step 3

Replace in ECM

$$\Delta y_{1t} = a_0 + \gamma_1 \left(y_{1t-1} - \beta_0 - \beta_1 y_{2t-1} \right) + a_{11} \Delta y_{1t-1} + a_{12} \Delta y_{2t-1} + more\ lags + u_{1t}$$

$$\Delta y_{2t} = b_0 + \gamma_2 \left(y_{1t-1} - \beta_0 - \beta_1 y_{2t-1} \right) + a_{21} \Delta y_{1t-1} + a_{22} \Delta y_{2t-1} + more\ lags + u_{2t}$$

$$y_{1t} - \beta_0 - \beta_1 y_{2t}$$

by

$$\hat{\varepsilon}_t = y_{1t} - \hat{\beta}_0 - \hat{\beta}_1 y_{2t}$$

$$\Delta y_{1t} = a_0 + \gamma_1 \hat{\varepsilon}_{t-1} + a_{11} \Delta y_{1t-1} + a_{12} \Delta y_{2t-1} + more\ lags + u_{1t}$$

$$\Delta y_{2t} = b_0 + \gamma_2 \hat{\varepsilon}_{t-1} + a_{21} \Delta y_{1t-1} + a_{22} \Delta y_{2t-1} + more\ lags + u_{2t}$$

Estimate parameters by OLS. Regression with only stationary variables on both sides.

Engle and Granger have proposed a method to estimate the parameters of a cointegrated system (4)

Step 4 Innovation accounting

Plot impulse response functions iterating on

$$\Delta y_{1t} = \hat{a}_0 + \hat{\gamma}_1\left(y_{1t-1} - \hat{\beta}_0 - \hat{\beta}_1 y_{2t-1}\right) + \hat{a}_{11}\Delta y_{1t-1} + \hat{a}_{12}\Delta y_{2t-1} + more\ lags + u_{1t}$$

$$\Delta y_{2t} = \hat{b}_0 + \hat{\gamma}_2\left(y_{1t-1} - \hat{\beta}_0 - \hat{\beta}_1 y_{2t-1}\right) + \hat{a}_{21}\Delta y_{1t-1} + \hat{a}_{22}\Delta y_{2t-1} + more\ lags + u_{2t}$$

As for SVAR: Possible contemporaneous correlation of u_{1t}, u_{2t}

To plot the impulse response function:

Cholesky Decomposition

Ordering of variables impacts results

Problems of E&G method

With n variables up to n-1 cointegrating relations may exist

Conclusion step 2 may depend on ordering

Johansen method for ML estimation of Gaussian cointegrated systems.

The work horse to test for non-stationarity: Dickey-Fuller tests
Basics: Unit Root Processes vs. Trend stationary processes

Two types of non-stationarity

$$y_t = \mu + y_{t-1} + u_t \qquad \text{(A)}$$

$$y_t = \alpha + \beta \cdot t + u_t \qquad \text{(B)}$$

(A) Special case of

$$y_t = \mu + \phi y_{t-1} + u_t$$

$|\phi| < 1$ $\qquad$ $|\phi| > 1$ $\qquad$ $|\phi| = 1$

$$y_t = \phi y_{t-1} + u_t$$
$$= \phi u_{t-1} + \phi^2 u_{t-2} + \phi^3 u_{t-3} + \ldots.. + \phi^t u_0 + \phi^{t+1} y_{-1} + u_t$$

The work horse to test for non-stationarity: Dickey-Fuller tests
Basics: Unit Root Processes vs. Trend stationary processes

$$y_t = \mu + \phi_1 y_{t-1} + \phi_2 y_{t-2} + .. + \phi_p y_{t-p} + u_t$$

Explosive? Stationary? Permanent Effects (Unit root)?

$$y_t = f^1 u_{t-1} + f^2 u_{t-2} + f^3 u_{t-3} + ... + f^t u_0 + y_{-1}^{t+1} + u_t$$

$$F = \begin{pmatrix} \phi_1 & \phi_2 & \phi_3 & \Lambda & \phi_{p-1} & \phi_p \\ 1 & 0 & 0 & \Lambda & 0 & 0 \\ 0 & 1 & & & 0 & 0 \\ M & M & M & M & & \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

Compute p eigenvalues of F

absolute value largest root = 1: unit root process

for p=2

$$\left| \begin{pmatrix} \phi_1 & \phi_2 \\ 1 & 0 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} \right| = 0$$

$$\lambda^2 - \phi_1 \lambda - \phi_2 = 0$$

The work horse to test for non-stationarity: Dickey-Fuller tests
Basics: unit root processes vs. trend stationary processes

Two types of non-stationarity

$$y_t = y_{t-1} + u_t \quad \text{or} \quad y_t = \mu + y_{t-1} + u_t \qquad \text{(A)}$$

$$y_t = \alpha + \beta \cdot t + u_t \qquad \text{(B)}$$

(A) Special case of

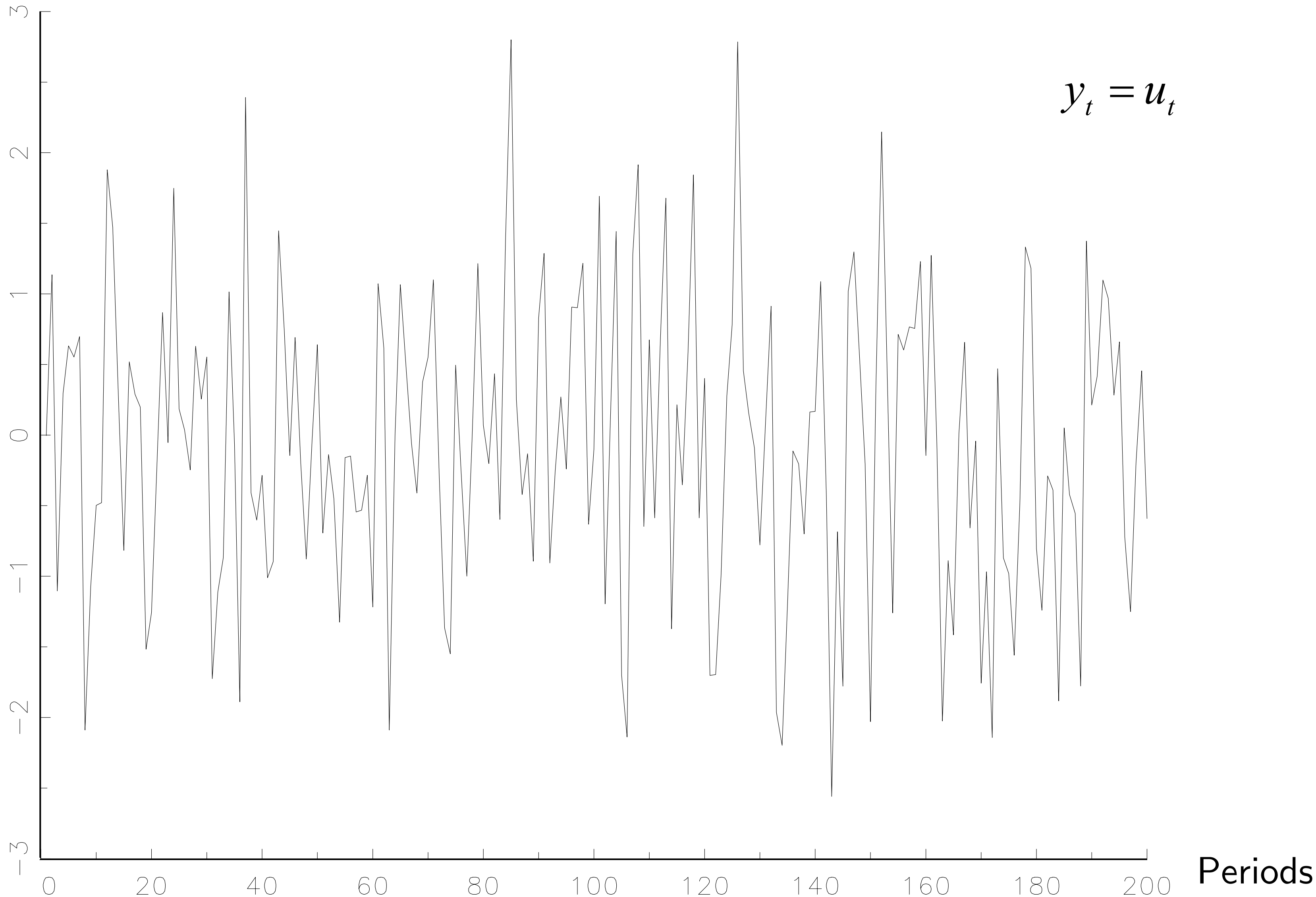
$$y_t = \mu + \phi y_{t-1} + u_t$$

$|\phi| < 1$ | $|\phi| > 1$ | $|\phi| = 1$

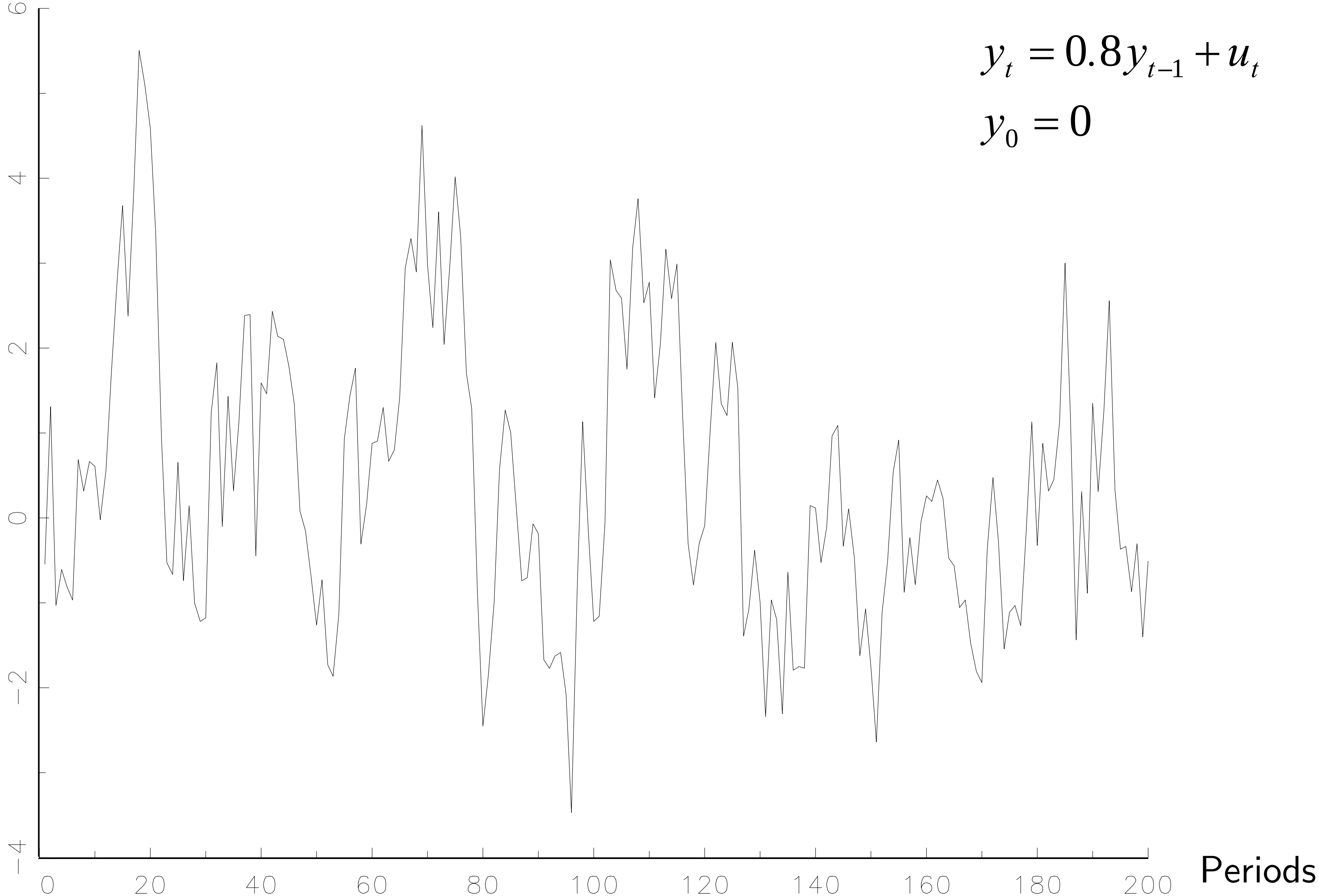
With $\mu = 0$

$$y_t = \phi y_{t-1} + u_t$$
$$= \phi u_{t-1} + \phi^2 u_{t-2} + \phi^3 u_{t-3} + \ldots.. + \phi^t u_0 + \phi^{t+1} y_{-1} + u_t$$

Realization of a white noise process



Realization of a stationary process (autoregressive process of order one)

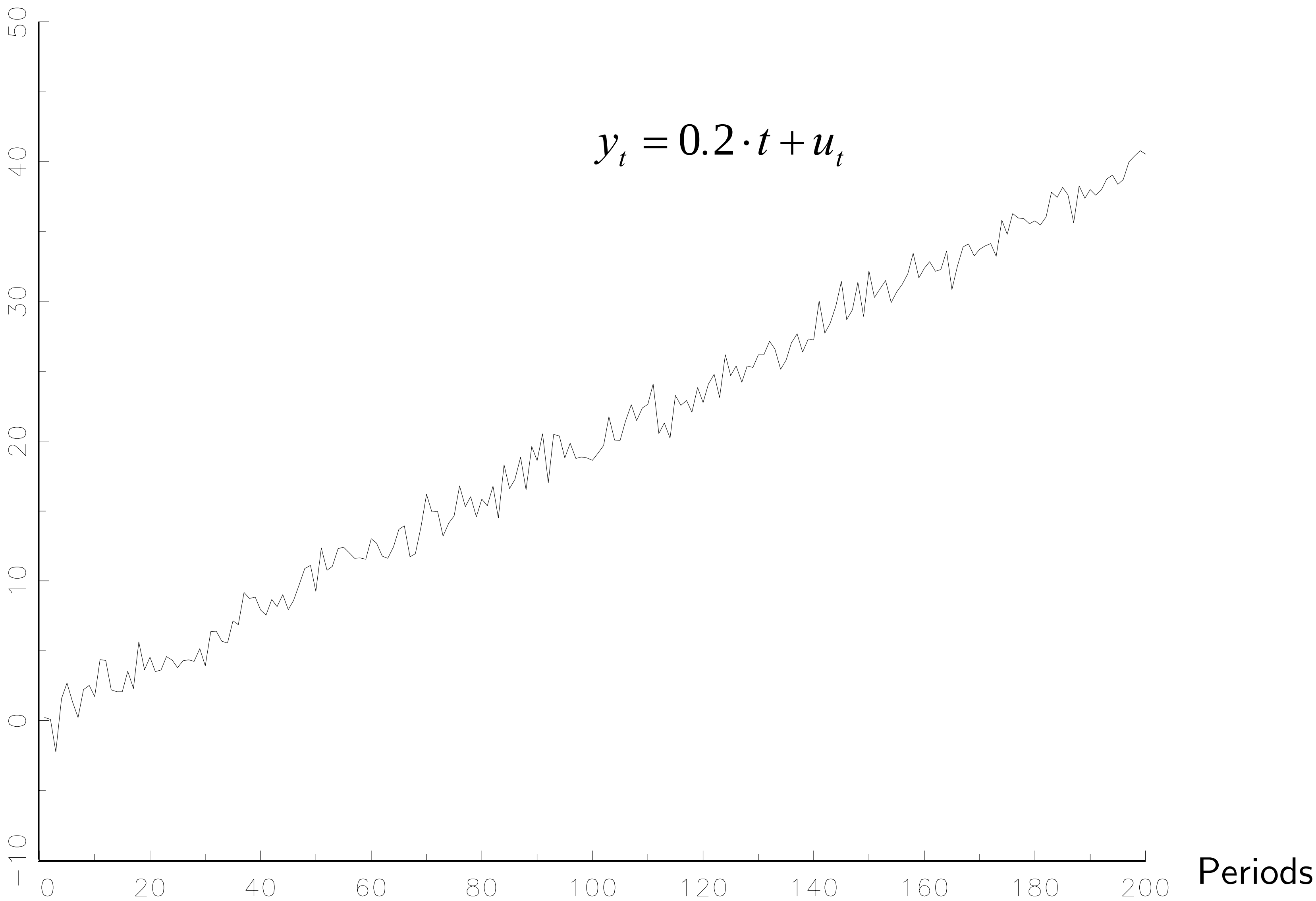


Realization of a random walk without drift

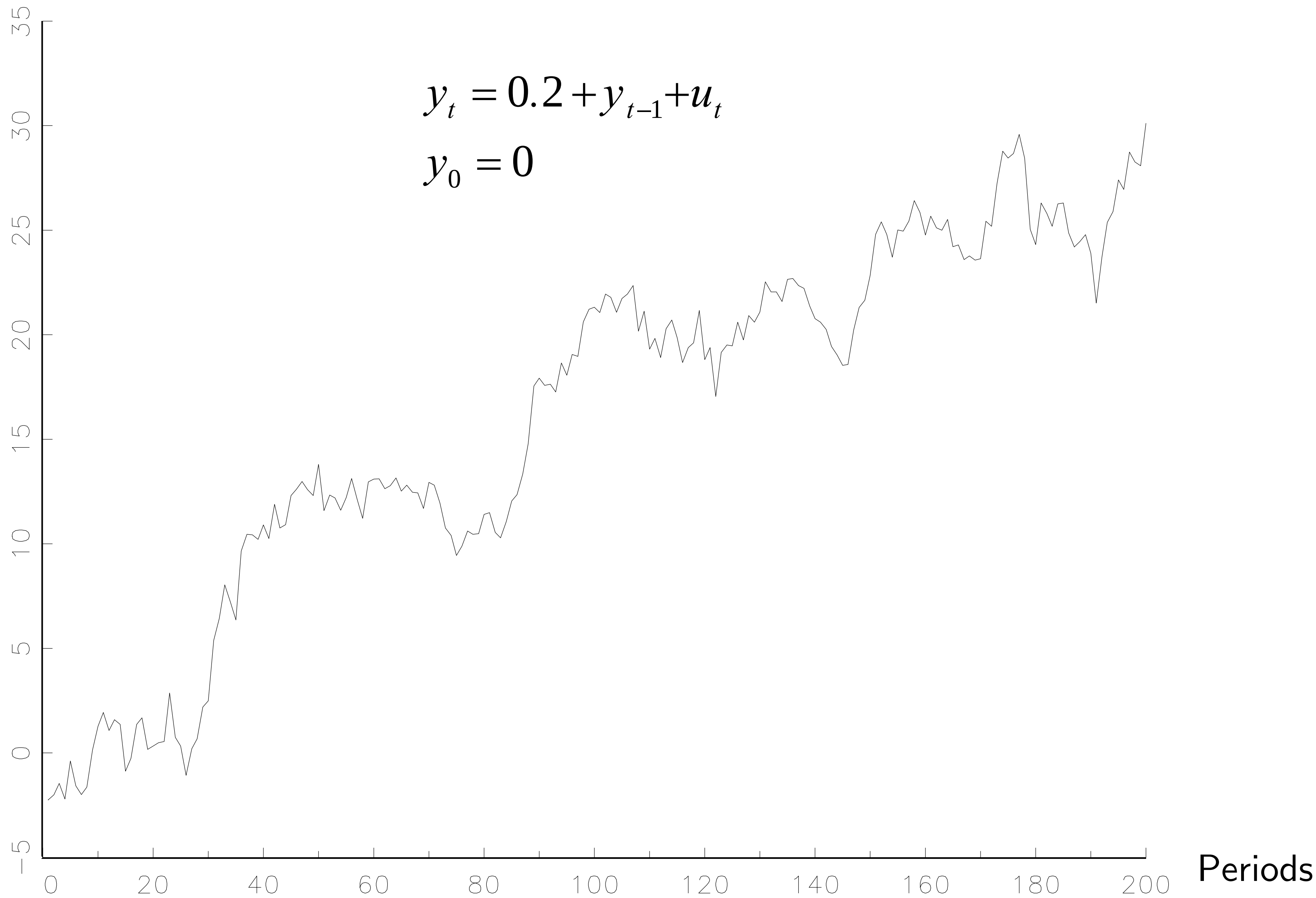
$$y_t = y_{t-1} + u_t$$

$$y_0 = 0$$

Realization of a trend-stationary processes



Realization of a random walk with drift



The work horse to test for non-stationarity: Dickey-Fuller tests

Basic idea. Test whether $a_1 = 1$ in

$$y_t = a_1 y_{t-1} + u_t$$

Run a regression,back out $\hat{a}_1, s.e.(\hat{a}_1)$

Calculate t-statistic $\tau = \frac{\hat{a}_1 - 1}{s.e.(\hat{a}_1)}$

distribution of τ under the null: non-standard. Obtained by simulations. Refer to tables (e.g. in Hamilton)

Equivalent (and usually done)

$$y_t - y_{t-1} = \Delta y_t = (a_1 - 1) y_{t-1} + u_t$$
$$= \gamma y_{t-1} + u_t$$

$$\tau = \frac{\hat{\gamma}}{s.e.(\hat{\gamma})}$$

The work horse to test for non-stationarity: Dickey-Fuller test statistics

Related tests. Look at your data! Estimated models:

$$y_t = a_0 + a_1 y_{t-1} + u_t \qquad\qquad y_t = a_0 + a_1 y_{t-1} + a_2 t + u_t$$

$$\Delta y_t = a_0 + \gamma y_{t-1} + u_t \qquad\qquad \Delta y_t = a_0 + \gamma y_{t-1} + a_2 t + u_t$$

Test whether $a_1 = 1$ resp. $\gamma = 0$

Run regression, back out $\hat{\gamma}$, $s.e.(\hat{\gamma})$

Calculate t-statistic

$$\tau_\mu = \frac{\hat{\gamma}}{s.e.(\hat{\gamma})} \qquad\qquad \tau_\tau = \frac{\hat{\gamma}}{s.e.(\hat{\gamma})}$$

both have under the null hypothesis $\gamma = 0$ non-standard distributions: look up correct quantile table!!

Critical values (quantiles) for Dickey-Fuller test statistics

STATISTICAL TABLES

Table A **Empirical Cumulative Distribution of τ**

Probability of a Smaller Value **Sample Size**	**0.01**	**0.025**	**0.05**	**0.10**	**0.90**	**0.95**	**0.975**	**0.99**
No Constant or Time ($a_0 = a_2 = 0$)				τ				
25	−2.66	−2.26	−1.95	−1.60	0.92	1.33	1.70	2.16
50	−2.62	−2.25	−1.95	−1.61	0.91	1.31	1.66	2.08
100	−2.60	−2.24	−1.95	−1.61	0.90	1.29	1.64	2.03
250	−2.58	−2.23	−1.95	−1.62	0.89	1.29	1.63	2.01
300	−2.58	−2.23	−1.95	−1.62	0.89	1.28	1.62	2.00
∞	−2.58	−2.23	−1.95	−1.62	0.89	1.28	1.62	2.00
Constant ($a_2 = 0$)				τ_μ				
25	−3.75	−3.33	−3.00	−2.62	−0.37	0.00	0.34	0.72
50	−3.58	−3.22	−2.93	−2.60	−0.40	−0.03	0.29	0.66
100	−3.51	−3.17	−2.89	−2.58	−0.42	−0.05	0.26	0.63
250	−3.46	−3.14	−2.88	−2.57	−0.42	−0.06	0.24	0.62
500	−3.44	−3.13	−2.87	−2.57	−0.43	−0.07	−0.24	0.61
∞	−3.43	−3.12	−2.86	−2.57	−0.44	−0.07	0.23	0.60
Constant + time				τ_τ				
25	−4.38	−3.95	−3.60	−3.24	−1.14	−0.80	−0.50	−0.15
50	−4.15	−3.80	−3.50	−3.18	−1.19	−0.87	−0.58	−0.24
100	−4.04	−3.73	−3.45	−3.15	−1.22	−0.90	−0.62	−0.28
250	−3.99	−3.69	−3.43	−3.13	−1.23	−0.92	−0.64	−0.31
500	−3.98	−3.68	−3.42	−3.13	−1.24	−0.93	−0.65	−0.32
∞	−3.96	−3.66	−3.41	−3.12	−1.25	−0.94	−0.66	−0.33

Source: This table was constructed by David A. Dickey using Monte Carlo methods. Standard errors of the estimates vary, but most are less than 0.20. The table is reproduced from Wayne Fuller. *Introduction to Statistical Time Series.* (New York: John Wiley). 1976.