5th Assignment Time Series

1. You want to construct the exact likelihood function of an AR(2) process

   $$Y_t = c + \phi_1 Y_{t-1} + \phi_2 Y_{t-2} + \varepsilon_t \qquad \varepsilon_t \sim N(0, \sigma^2) \text{ and i.i.d.}$$

   a) Write down the joint density of the first two observations $f_{Y_1,Y_2}(y_1, y_2)$

   b) Using the conditional density of the third observation $f_{Y_3|Y_2,Y_1}(y_3|y_2, y_1)$ write down the joint density of the first three observations $f_{Y_1,Y_2,Y_3}(y_1, y_2, y_3)$

2. Are the following $MA(q)$ processes invertible?

   $$Y_t = c - 0.9\varepsilon_{t-1} + \varepsilon_t$$
   $$Y_t = c + \varepsilon_{t-1} + \varepsilon_t$$
   $$Y_t = c + 1.2\varepsilon_{t-1} + \varepsilon_t$$
   $$Y_t = c + (1 + 0.7L + 0.4L^2)\varepsilon_t$$
   $$Y_t = c + (1 + 0.2L + 0.4L^2)\varepsilon_t$$

3. a) Write down the joint density of the first three observations of the $MA(3)$ process

   $$Y_t = c + 0.3\varepsilon_{t-1} + 0.2\varepsilon_{t-2} - 0.1\varepsilon_{t-1} + \varepsilon_t \qquad \varepsilon_t \sim N(0, \sigma^2) \text{ and } \varepsilon_t \text{ i.i.d. } N(0, \sigma^2)$$

   b) Suppose you want to set up the conditional likelihood function of this process. You condition on pre-sample values $\varepsilon_0, \varepsilon_{-1}, \varepsilon_{-2}$. Write down the first three elements of the conditional likelihood function.

   $f_{Y_1|\varepsilon_0=0,\varepsilon_{-1}=0,\varepsilon_{-2}=0} =$
   $f_{Y_2|Y_1,\varepsilon_0=0,\varepsilon_{-1}=0,\varepsilon_{-2}=0} =$
   $f_{Y_3|Y_1,Y_2,\varepsilon_0=0,\varepsilon_{-1}=0,\varepsilon_{-2}=0} =$

   c) Which condition has to hold in order to make the Conditional Maximum-Likelihood work?

4. You have succeeded in providing Maximum-Likelihood estimates of the parameters of an $ARMA(2,2)$ process.

   $$(1 - L\phi_1 - L\phi_2)Y_t = c + (1 + \theta_1 L + \theta_2 L^2)\varepsilon_t \qquad \varepsilon_t \sim \ i.i.d.N(0, \sigma^2)$$

   The (conditioned) Maximum-Likelihood estimates are

   $$\begin{aligned} \hat{c} &= 0.2 & \hat{\theta}_1 &= 0.2 \\ \hat{\phi}_1 &= 0.6 & \hat{\theta}_2 &= -0.1 \\ \hat{\phi}_2 &= 0.1 & \hat{\sigma}^2 &= 0.8 \end{aligned}$$

The value of the log likelihood function evaluated at these estimates is -1432.6.

Suppose you want to test the null hypothesis

$$H_0 : \theta_1 = 0.5 \text{ against } H_A : \theta_1 \neq 0.5$$
and $$H_0 : \theta_1 = 0 \text{ against } H_A : \theta_1 \neq 0$$

Perform and interpret the appropriate tests.

An estimate of the variance-covariance matrix of the estimates $\hat{\theta} = (\hat{c}, \hat{\phi}_1, \hat{\phi}_2, \hat{\theta}_1, \hat{\theta}_2, \hat{\sigma}^2)$ is given by

$$\widehat{Var}(\hat{\theta}) = \left[-\frac{\partial^2 lnL(\theta)}{\partial\theta\partial\theta'}\bigg|_{\hat{\theta}} \right]^{-1} = \begin{bmatrix} 0.007 & \cdots & & & & \vdots \\ 0.001 & 0.005 & & & & \\ 0.002 & 0.001 & 0.003 & & & \\ 0.003 & 0.002 & 0.001 & 0.01 & & \\ 0.001 & 0.003 & 0.004 & 0.001 & 0.002 & \\ 0.001 & 0.0001 & 0.0001 & 0.0001 & 0.00002 & 0.0001 \end{bmatrix}$$

$\theta = (c, \phi_1, \phi_2, \theta_1, \theta_2, \sigma^2)'$

You have also estimated an $ARMA(2,0)$ i.e. an $AR(2)$ model. The estimation of this restricted model yields a log likelihood value equal to -1434.3.
Compute and interpret a likelihood ratio statistic to test the hypothesis that the restrictions implied by the $ARMA(2,0)$ specification are correct. Here the $ARMA(2,2)$ specification is the unrestricted model, the $ARMA(2,0)$ is the restricted model.

As another alternative you have estimated an $MA(2)$ model. The log likelihood evaluated at the maximum likelihood estimates is -1442.2. Perform a test of the $ARMA(2,2)$ specification against the MA(2) model.