EBERHARD KARLS
UNIVERSITÄT
TÜBINGEN

WIRTSCHAFTS- UND
SOZIALWISSENSCHAFTLICHE
FAKULTÄT

Chair of Statistics, Econometrics and Empirical Economics

Dr. Julie Schnaitmann

S414
Advanced Mathematical Methods

Exercises

WS 2021/22

Linear Algebra

Exercise 1 **Eigenvalues**

Derive the characteristic equations for the matrices from exercise a)-c) and determine the eigenvalues.

a) $\mathbf{B} = \begin{pmatrix} 4 & 1 \\ -2 & -0,5 \end{pmatrix}$

b) $\mathbf{C} = \begin{pmatrix} 1 & 2 \\ 3 & -4 \end{pmatrix}$ c) $\mathbf{D} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & -3 \\ 0 & -1 & 3 \end{pmatrix}$

Exercise 2 **Eigenvalues and Eigenvectors**

Given the matrix:

$$\mathbf{A} = \begin{bmatrix} -3 & 2 \\ -2 & 2 \end{bmatrix}$$

a) Calculate the eigenvalues and the respective eigenvectors of $\mathbf{A}$.

b) Use the eigenvalues to calculate the determinant of $\mathbf{A}$.

Exercise 3 **Eigenvalues**

A 3×3 matrix $\mathbf{A}$ has the eigenvalues $\lambda_1 = 1$, $\lambda_2 = 3$ and $\lambda_3 = 4$. Compute the determinant of $\mathbf{A}$, $\text{rg}(\mathbf{A})$, the determinant of $\mathbf{A}^{-1}$ and the eigenvalues of $\mathbf{A}^{-1}$. What can be said about the quadratic form $\mathbf{x}'\mathbf{A}\mathbf{x}$ of the matrix $\mathbf{A}$ for any vectors of $\mathbf{x}$?

Exercise 4 **Eigenvalues**

Find the characteristic vectors of the matrix $\begin{pmatrix} 4 & 2 \\ 2 & 1 \end{pmatrix}$:

EXERCISE 5 **Quadratic Form**

Given the matrix

$$\mathbf{A} = \begin{pmatrix} 2 & -2 \\ 1 & 2 \end{pmatrix} \quad .$$

a) Determine the definiteness of the quadratic form $Q = \boldsymbol{x}'\boldsymbol{A}\boldsymbol{x}$.

b) Explain in two sentences maximum what this means for the graph $\{(x_1, x_2, Q) | Q = (x_1; x_2)\boldsymbol{A}(x_1; x_2)'\}$.

EXERCISE 6 **Quadratic Form**

Write the quadratic form

$$Q = 4x_1^2 + 4x_1x_2 - x_2^2$$

in matrix notation and determine its definiteness.

EXERCISE 7 **Sign definiteness**

Express each quadratic form below as a matrix product involving a *symmetric* coefficient matrix:

a) $q = 3u^2 - 4uv + 7v^2$

b) $q = u^2 + 7uv + 3v^2$

c) $q = 8uv - u^2 - 31v^2$

d) $q = 6xy - 5y^2 - 2x^2$

e) $q = 3u_1^2 - 2u_1u_2 + 4u_1u_3 + 5u_2^2 + 4u_3^2 - 2u_2u_3$

f) $q = -u^2 + 4uv - 6uw - 4v^2 - 7w^2$

Exercise 8 **Sign definiteness**

Given a quadratic form $u'Du$. where D is 2×2, the characteristic equation of D can be written as:

$$\begin{vmatrix} d_{11} - r & d_{12} \\ d_{21} & d_{22} - r \end{vmatrix} = 0 \qquad (d_{12} = d_{21})$$

Expand the determinant; express the roots of this equation by use of the quadratic formula and deduce the following:

a) No imaginary number (a number involving $\sqrt{-1}$) can occur in r_1 and r_2.

b) To have repeated roots, the matrix D must be in the form of $\begin{pmatrix} c & 0 \\ 0 & c \end{pmatrix}$

c) To have either positive or negative smidefiniteness, the determinant of the matrix D must vanish, i.e. $|D| = 0$.

Solution Exercise 1:

a) $\lambda_1 = 3.5$ and $\lambda_2 = 0$

b) $\lambda_1 = 2$ and $\lambda_2 = -5$

c) $\lambda_1 = 4.30278;\ \lambda_2 = 0.69722;\ \lambda_3 = 1$

Solution Exercise 2:

a) Eigenvector for $\lambda_1 = 1$:
$\Rightarrow \begin{pmatrix} a \\ 2a \end{pmatrix} \qquad$ for $a \in \mathbb{R} \setminus \{0\}$

Eigenvector for $\lambda_2 = -2$:
$\Rightarrow \begin{pmatrix} b \\ \frac{1}{2}b \end{pmatrix} \qquad$ for $b \in \mathbb{R} \setminus \{0\}$

b) $\det(\boldsymbol{A}) = -2$

Solution Exercise 4:

$$v_1 = \begin{pmatrix} -\frac{1}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} \end{pmatrix}, \qquad v_2 = \begin{pmatrix} \frac{2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \end{pmatrix}$$

Solution Exercise 5:

a) positive definite

Solution Exercise 6:

$\boldsymbol{Q} = \boldsymbol{x}'\boldsymbol{A}\boldsymbol{x}$ with $A = \begin{pmatrix} 4 & 2 \\ 2 & -1 \end{pmatrix}$
$\boldsymbol{A}$ is indefinite

Solution Exercise 7:

Quadratic form: $q = \mathbf{x}'\mathbf{A}\mathbf{x}$

a)

$$q = \begin{pmatrix} u \\ v \end{pmatrix}' \begin{pmatrix} 3 & -2 \\ -2 & 7 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}$$

b)

$$q = \begin{pmatrix} u \\ v \end{pmatrix}' \begin{pmatrix} 1 & 3.5 \\ 3.5 & 3 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}$$

c)

$$q = \begin{pmatrix} u \\ v \end{pmatrix}' \begin{pmatrix} -1 & 4 \\ 4 & -31 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}$$

d)

$$q = \begin{pmatrix} x \\ y \end{pmatrix}' \begin{pmatrix} -2 & 3 \\ 3 & -5 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

e)

$$q = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}' \begin{pmatrix} 3 & -1 & 2 \\ -1 & 5 & -1 \\ 2 & -1 & 4 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}$$

f)

$$q = \begin{pmatrix} u \\ v \\ w \end{pmatrix}' \begin{pmatrix} -1 & 2 & -3 \\ 2 & -4 & 0 \\ -3 & 0 & -7 \end{pmatrix} \begin{pmatrix} u \\ v \\ w \end{pmatrix}$$

Solution Exercise 8:
Refer to the video.