Advanced Mathematical Methods

WS 2019/20

4 Mathematical Statistics

PD Dr. Thomas Dimpfl

Chair of Statistics, Econometrics and Empirical Economics

EBERHARD KARLS UNIVERSITÄT TÜBINGEN

WIRTSCHAFTS- UND SOZIALWISSENSCHAFTLICHE FAKULTÄT

Outline: Mathematical Statistics

4.8 Joint distributions

4.9 Marginal Distributions

4.10 Covariance and correlation

4.11 Conditional Distributions

4.12 Conditional Moments

Readings

A. Papoulis and A. U. Pillai. *Probability, Random Variables and Stochastic Processes*.
Mc Graw Hill, fourth edition, 2002 Chapter 6

Online References

MIT Course on Probabilistic Systems Analysis and Applied Probability (by John Tsitsiklis)

- Discrete RVs II: Functions of RV, conditional probabilities, specific distribution, total expectation theorem, joint probabilities
  https://www.youtube.com/watch?v=-qCEoqpwjf4
- Discrete RVs III: Conditional distributions and joint distributions continued
  https://www.youtube.com/watch?v=EObHWIEKGjA
- Multiple Continuous RVs: conditional pdf and cdf, joint pdf and cdf
  https://www.youtube.com/watch?v=CadZXGNauY0

4.8 Joint distributions

Definition: Random vector

Assume a probability space $(\Omega, \mathcal{F}, \mathcal{P})$. A vector-valued function $X(\cdot) : \Omega \to \mathbb{R}^n; \omega \mapsto \underline{X}(\omega)$ which attributes to every singleton ω a vector of real numbers $\underline{X}(\omega)$ is called a random vector.

4.8 Joint distributions

Definition: Joint density function

The joint density for two discrete random variables X_1 and X_2 is given as

$$f_{\underline{X}}(x_1, x_2) = \begin{cases} P(X_1 = x_{1i} \cap X_2 = x_{2i}) & \forall i, j \\ 0 & \text{else} \end{cases}$$

Properties:

- $f_{\underline{X}}(x_1, x_2) \geq 0 \quad \forall \quad (x_1, x_2) \in \mathbb{R}^2$
- $\sum_i \sum_j f_{\underline{X}}(x_{1i}, x_{2j}) = 1$

4.8 Joint distributions

Definition: Joint cumulative distribution function

The cdf for two discrete random variables X_1 and X_2 is given as

$$F_{\underline{X}}(x_1, x_2) = P(X_1 \leq x_1 \cap X_2 \leq x_2) = \sum_{x_{1i} \leq x_1} \sum_{x_{2i} \leq x_2} f_{\underline{X}}(x_{1i}, x_{2i})$$

it follows that

$$P(a \leq X_1 \leq b \cap c \leq X_2 \leq d) = \sum_{a \leq x_1 \leq b} \sum_{c \leq x_2 \leq d} f_{\underline{X}}(x_{1i}, x_{2i})$$

4.8 Joint distributions

if X_1 and X_2 are two continuous random variables, the following holds:

$$\begin{aligned} \text{pdf} \qquad f_{\underline{X}}(x_{1i}, x_{2i}) &= \frac{\partial^2 F_{\underline{X}}(x_1, x_2)}{\partial x_1 \partial x_2} \\ \text{cdf} \qquad F_{\underline{X}}(x_1, x_2) &= \int_{-\infty}^{x_1} \int_{-\infty}^{x_2} f_{\underline{X}}(u_1, u_2) du_2 du_1 \end{aligned}$$

4.9 Marginal Distributions

derive the distribution of the individual variable from the joint distribution function
$\rightarrow$ sum or integrate out the other variable

$$f_{X_1}(x_{1i}) = \begin{cases} \sum\limits_{j} f_X(x_{1i}, x_{2j}) & \text{if } X \text{ is discrete} \\ \int\limits_{-\infty}^{\infty} f_X(x_1, x_2) dx_2 & \text{if } X \text{ is continuous} \end{cases}$$

4.9 Marginal Distributions

two random variables are statistically independent if their joint density is the product of the marginal densities:

$$f_X(x_1, x_2) = f_{x_1}(x_1) \cdot f_{x_2}(x_2) \Leftrightarrow X \text{ and } Y \text{ are independent}$$

under independence the cdf factors as well:

$$F_{XY}(x, y) = F_X(x) \cdot F_Y(y)$$

Expectations in a joint distribution are computed with respect to the marginals

4.10 Covariance and correlation

$$Cov[X_1, X_2] = E[(X_1 - E[X_1])(X_2 - E[X_2])]$$

Properties:

- symmetry: $Cov[X_1, X_2] = Cov[X_2, X_1]$
- linear transformation:

$$Y_1 = b_0 + b_1 X_1 \qquad Y_2 = c_0 + c_1 X_2$$
$$\Rightarrow Cov[Y_1, Y_2] = b_1 c_1 Cov[X_1, X_2]$$

- $Cov[X_1, X_2] = \sum_i \sum_j x_{1i} x_{2j} f_X(x_{1i}, x_{2j}) - E[X_1]E[X_2]$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x_1 x_2 f_X(x_1, x_2) dx_2 dx_1 - E[X_1]E[X_2]$$

4.10 Covariance and correlation

Pearson's correlation coefficient

$$\rho_{x_1,x_2} = \frac{Cov(X_1, X_2)}{\sqrt{Var(X_1) \cdot Var(X_2)}} = \frac{\sigma_{x_1,x_2}}{\sigma_{x_1}\sigma_{x_2}}$$

- if X_1 and X_2 are independent, they are also uncorrelated
- attention: uncorrelated does not imply independence!
- exception: normal distribution, characterized by 1st and 2nd moment

4.11 Conditional Distributions

- Distribution of the varibale X_1 given that X_2 takes on a certain value x_1
- Closely related to conditional probabilities:

$$P(X_1 = x_1 | X_2 = x_2) = \frac{P(X_1 = x_1 \cap X_2 = x_2)}{P(X_2 = x_2)}$$

conditional pdf of X_1 given $X_2 = x_2$:

$$f_{X_1|X_2}(x_1|x_2) = \frac{f_{X_1,X_2}(x_1, x_2)}{f_{X_2}(x_2)}$$

4.11 Conditional Distributions

conditional cdf of X_1 given $X_2 = x_2$:

$$P(X_1 = x_1 | X_2 = x_2) = \sum_{x_{1i} \leq X} f_{X_1|X_2}(x_{1i}|x_2) = F_{X_1|X_2}(x_1|x_2)$$

if X_1 and X_2 are independent, the conditional probability and the marginal probability coincide:

$$f_{X_1|X_2}(x_1|x_2) = f_{X_1}(x_1)$$

because

$$f_{X_1 X_2}(x_1, x_2) = f_{X_1}(x_1) \cdot f_{X_2}(x_2)$$

4.11 Conditional Distributions

the joint pdf can be derived from conditional and marginal densities in 2 ways:

$$f_{X_1 X_2} = f_{X_1|X_2}(x_1|x_2) \cdot f_{X_2}(x_2) = f_{X_2|X_1}(x_2|x_1) \cdot f_{X_1}(x_1)$$

4.12 Conditional Moments

$$
\begin{aligned}
E[Y^k|X=x] &= \sum_j y_j^k \cdot \frac{P(X=x \cap Y=y_j)}{P(X=x)} \\
&= \sum_j y_j^k \cdot P(Y=y_j|X=x) \\
&= \sum_j y_j^k \cdot f_{Y|X}(y_j|x) \\
&= \sum_j y_j^k \cdot \frac{f_{XY}(x,y_j)}{f_X(x)} \qquad \text{if } Y \text{ is discrete}
\end{aligned}
$$

$$
E[Y^k|X=x] = \int_{-\infty}^{\infty} y^k \cdot \frac{f_{XY}(x,y)}{f_X(x)} \qquad \text{if } Y \text{ is continuous}
$$

4.12 Conditional Moments

$$Var[Y|X = x] = E_{Y|X}[(Y - E[Y|X = x])^2]$$

$$= \sum_j (y_j - E[Y|X = x])^2 \cdot f_{Y|X}(y_j|x)$$

if Y is discrete

$$Var[Y|X = x] = E_{Y|X}[(Y - E[Y|X = x])^2]$$

$$= \int_{-\infty}^{\infty} (y - E[Y|X = x])^2 \cdot f_{Y|X}(y|x) dy$$

if Y is continuous

4.12 Conditional Moments

Law of total Expectations/ Law of iterated Expectations

$$E[Y] = E_X\left[E[Y|X]\right]$$

$$E_X\left[E_{Y|X}[Y|X]\right] = E[Y] = \int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} y \cdot \frac{f_{XY}(x,y)}{f_X(x)} dy\right] f_X(x) dx$$

$E_{Y|X}$ is a random value as X is a random variable