The Geometry of Static Spacetimes

Carla Cederbaum
Duke University / Albert Einstein Institute

1. Setup

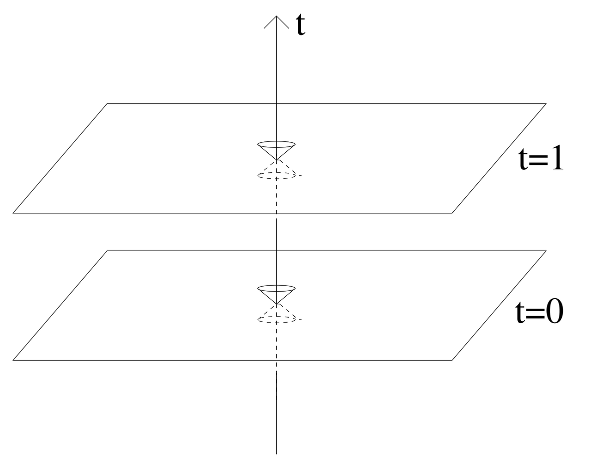
Static isolated general relativistic systems have been studied from a number of perspectives including their regularity, compactification and asymptotic considerations, symmetry classifications, construction of explicit solutions etc. They serve as models of static stars and black holes. Here, we present a new geometric approach to the study of static isolated systems and their physical properties for which we suggest the name **geometrostatics** [C1,C2].

Static space-times are Lorentzian space-times possessing a timelike Killing vector field X – i. e. $\nabla_{(\alpha}X_{\beta)} = 0$ – that is hypersurface-orthogonal, i. e. $X_{[\alpha}\nabla_{\beta}X_{\gamma]} = 0$. They generically possess a $3+1$-decomposition with vanishing shift vector. In this *canonical* decomposition, the *canonical* lapse function is time-independent and coincides with the Lorentzian length of the time-like Killing vector field. The spacelike time-slices orthogonal to the time-like Killing vector field are isometric and have vanishing extrinsic curvature. Their induced Riemannian metric is time-independent. We will subsequently identify all canonical time-slices (M^3, g).

Generic static space-times (M^4, ds^2) can be canonically decomposed via $X = \partial_t$ into

$$M^4 = \mathbb{R} \times M^3 \quad \text{and} \quad ds^2 = -N^2c^2dt^2 + g$$

with induced Riemannian metric g and **lapse function** $N := \frac{1}{c}\sqrt{-ds^2(\partial_t, \partial_t)} > 0$.



Here, c is the speed of light. Outside the support of the matter variables, Einstein's equations reduce to the **Vacuum Static Metric Equations**

$$N\,{}^g\mathrm{Ric} = {}^g\nabla^2 N \quad \text{and} \quad {}^g\triangle N = 0 \tag{1}$$

Here, ${}^g\mathrm{Ric}$ is the Ricci curvature tensor of the metric g, ${}^g\triangle N$ is the curvilinear Laplacian and ${}^g\nabla^2 N$ the curvilinear Hessian (symmetric second covariant derivatives) of N.

2. Regularity and Asymptotics

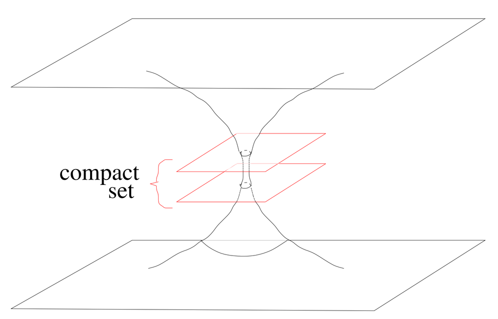
It is useful to study the system (1) in **wave-harmonic coordinates**, i. e. local coordinates (x^i) on (M^3, g) satisfying

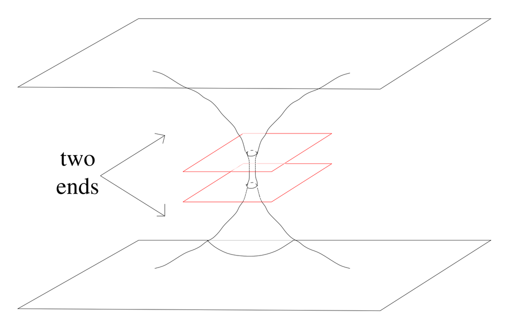
$$^{ds^2}\Box\, x^i = 0, \tag{2}$$

where $^{ds^2}\Box$ is the d'Alembert or wave operator with respect to $ds^2 = -N^2c^2dt^2 + g$.
In wave-harmonic coordinates, the vacuum static metric equations (1) are *elliptic* and therefore have **locally real analytic** solutions (g_{ij}, N) [MzH]. We consider static space-times that are **asymptotically flat** in the sense that the Riemannian manifold (M^3, g) consists of a compact set $K \subset M^3$ and one (or several) asymptotically flat ends $E \subset M^3$. On the end E, there are global coordinates (x^i) such that

$$g_{ij} = \delta_{ij} + \mathcal{O}\left(\frac{1}{r}\right) \quad \text{and} \quad N = 1 + \mathcal{O}\left(\frac{1}{r}\right) \tag{3}$$

where $r := \sqrt{(x^1)^2 + (x^2)^2 + (x^3)^2}$. In other words, the Riemannian metric is asymptotically Euclidean and the lapse function decays like in Minkowski.





Static asymptotically flat space-times satisfying the (vacuum) static metric equations (1) automatically **asymptotically decay like the spherically symmetric Schwarzschild solutions** [KM]

$$N = 1 - \frac{mG}{rc^2} + \mathcal{O}\left(\frac{1}{r^2}\right) \quad \text{and} \quad g_{ij} = \left(1 + \frac{2mG}{rc^2}\right)\delta_{ij} + \mathcal{O}\left(\frac{1}{r^2}\right) \tag{4}$$

in wave-harmonic asymptotically flat coordinates. Here, G is the gravitational constant, c the speed of light, and m is the ADM-mass of the slice (M^3, g).

3. Pseudo-Newtonian Gravity

The asymptotic decay (4) of the lapse function N resembles the asymptotic decay of the Newtonian potential U in the classical Newtonian theory of gravity. The similarity becomes more prominent if we make a change of variables (which is frequently used in the literature):

$$\gamma := N^2 g \quad \text{and} \quad U := c^2 \log N \tag{5}$$

We suggest to call these new variables **pseudo-Newtonian potential** U and **pseudo-Newtonian metric** γ, respectively [C1,C2]. The vacuum static metric equations (1) transform into the **vacuum pseudo-Newtonian equations**

$$^{\gamma}\mathrm{Ric} = \frac{2}{c^4}\, dU \times dU \quad \text{and} \quad {}^{\gamma}\triangle U = 0. \tag{6}$$

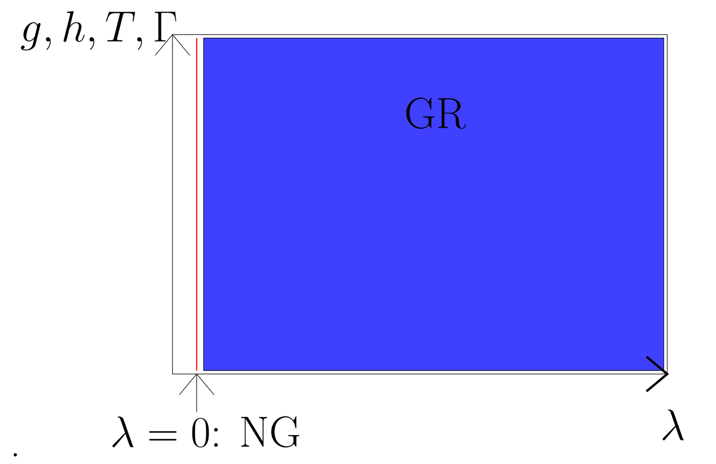
The asymptotic decay (4) can be transformed accordingly. Comparing these equations and decay conditions to the governing equation of vacuum static Newtonian Gravity, $\triangle U = 0$ and the well-known decay for the Newtonian potential, we obtain

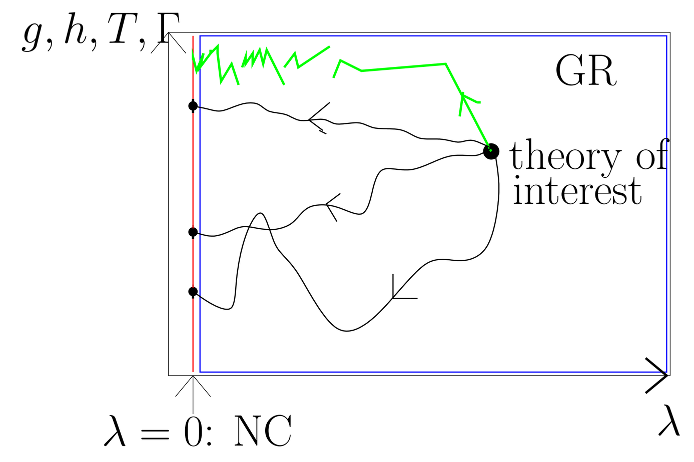
Newtonian Gravity	**Pseudo-Newtonian Gravity**
$^{\delta}\mathrm{Ric} = 0$	$^{\gamma}\mathrm{Ric} = \frac{2}{c^4}\, dU \times dU$
$^{\delta}\triangle U = 0$	$^{\gamma}\triangle U = 0$
$U = -\frac{mG}{r} + \mathcal{O}\left(\frac{1}{r^2}\right)$	$U = -\frac{mG}{r} + \mathcal{O}\left(\frac{1}{r^2}\right)$
$\delta_{ij} = \delta_{ij}$	$g_{ij} = \delta_{ij} + \mathcal{O}\left(\frac{1}{r^2c^4}\right)$

Here, δ denotes the flat background metric of Newtonian Gravity. As we work in three spatial dimensions, the equation $^{\delta}\mathrm{Ric} = 0$ is equivalent to δ being the flat background metric of Newtonian physics and can thus be added to the ordinary vacuum Newtonian equation $^{\delta}\triangle U = 0$. In the pseudo-Newtonian variables (γ, U), static space-times thus resemble static Newtonian gravitating systems even more than in the geometrostatic variables (g, N).

4. Newtonian Limit

On a formal level, the vacuum pseudo-Newtonian equations (6) converge to the vacuum Newtonian equation(s) $^{\delta}\mathrm{Ric} = 0$ and $^{\delta}\triangle U = 0$ as $c \to \infty$. This can be made rigorous with the help of Ehlers' frame theory [E]. In Ehlers' frame theory, General Relativity and Newtonian Gravity (or rather Newton-Cartan Theory) appear as disjoint regimes of a common framework parametrized by $\lambda := c^{-2}$ or $\lambda := 0$ in the Newtonian case. The Lorentzian metric ds^2 as well as the Newtonian potential then appear as derived variables of two tensor fields g, h and an affine connection Γ. The Newtonian limit is hence not defined for a single relativistic system but *for a whole family of systems*. The choice of this family is by no means unique as the figure below illustrates.





Frame theory is a geometric (coordinate invariant) theory which has not yet been widely studied. We suggest notions of Killing vectors, staticity, pseudo-Newtonian metric/potential, and asymptotic flatness within frame theory [C1]. With these notions, we prove the following theorem in [C1].

Theorem 1. *Let $(M^3, g(\lambda), N(\lambda))$ be a family possessing a Newtonian limit (M^3, δ, U_N) as $\lambda \to 0$. Then the pseudo-Newtonian variables behave such that*

$$\gamma_{ij}(\lambda) \to \gamma_{ij}(0) = \delta_{ij} \tag{7}$$
$$U(\lambda) \to U(0) = U_N \tag{8}$$

as $c \to \infty$ in wave-harmonic/Galilean coordinates (x^i).

5. Mass and Center of Mass

How do physical properties behave along the Newtonian limit? Does, for example, ADM-mass converge to Newtonian mass as $\lambda \to 0$? We can answer this question – and the corresponding question for the ADM-center of mass to the affirmative [C1].

Theorem 2. *Let $(M^3, g(\lambda), N(\lambda))$ be a family of static spacetimes possessing a Newtonian limit (M^3, δ, U_N) as $\lambda \to 0$. Then the ADM-mass $m_{ADM}(\lambda)$ and the ADM-center of mass $\vec{z}_{ADM}(\lambda)$ converge to the Newtonian mass $m_{ADM}(0) = m_N$ and center of mass $\vec{z}_{ADM}(0) = \vec{z}_N$ as $\lambda \to 0$. The latter convergence assumes the use of wave-harmonic/Galilean coordinates. Moreover, the CMC-center of mass [HY] coincides with the ADM-center of mass and thus converges to the Newtonian center of mass, too.*

This theorem relies on our definition of pseudo-Newtonian mass and center of mass [C1]:

Definition 1. *Let (M^3, γ, U) be a pseudo-Newtonian system. Let Σ be a closed 2-surface in M^3. Let ν be the γ-outer unit normal to and $d\sigma$ is the γ-surface measure on Σ. We define the* **pseudo-Newtonian mass** *and the* **pseudo-Newtonian center of mass** *of Σ by*

$$m_{PN}(\Sigma) := \frac{1}{4\pi G}\int_{\Sigma} \frac{\partial U}{\partial \nu}\, d\sigma \quad \text{and} \quad \vec{z}_{PN}(\Sigma) := \frac{1}{4\pi G m_{PN}}\int_{\Sigma} \left(\frac{\partial U}{\partial \nu}\vec{x} - U\frac{\partial \vec{x}}{\partial \nu}\right) d\sigma \tag{9}$$

where $\vec{x}$ is the vector of asymptotically flat wave-harmonic (or γ-harmonic) coordinates.

By the Laplace equation in (1), both m_{PN} and $\vec{z}_{PN}$ are in fact *independent of* Σ if the surface Σ encloses the support of the matter. Abbreviating $\vec{z} := \vec{z}_{PN}$, we obtain an improvement of (4) as well as a result on the Newtonian limit of mass and center of mass [C1].

$$U = -\frac{mG}{r} - \frac{mG\,\vec{z}\cdot\vec{x}}{r^3} + \mathcal{O}\left(\frac{1}{r^3}\right), \tag{10}$$

Theorem 3. *On any surface Σ enclosing the support of the matter, we have*

$$m_{PN}(\Sigma) = m_{ADM} \quad \text{and} \quad \vec{z}_{PN}(\Sigma) = \vec{z}_{ADM} = \vec{z}_{CMC} \tag{11}$$

We have thus localized ADM-mass and center of mass in the static setting.

For more results on geometrostatic systems, for example a discussion of test body behavior and of photon spheres, please see [C1].

6. References

[C1] C. Cederbaum, *The Newtonian Limit of Geometrostatics.* PhD thesis, FU Berlin (2011).

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