On a normal form for intuitionistic propositional logic

by

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§1. *Introduction*

It is well known that one of the problems encountered in automated theorem proving for non classical propositional logics and in particular for intuitionistic logic is the presence of the so called contraction rule which allows inferring from a sequent $M, v, v \Rightarrow w$ the sequent $M, v \Rightarrow w$. From usual calculi for non classical logics this rule can not be eliminated without compromising completeness. But obviously in a calculus with contraction rule deductions may get arbitrarily long. Therefore techniques have to be developped to prevent running into infinite loops during backwards proof search. This tends to make implementations slow and complicated. Here we give a new approach to this problem by introducing a certain kind of normal form theorem for intuitionistic propositional logic. We show that to every intuitionistic sequent s there is a normal form sequent $\mathrm{nf}(s)$ such that

i) if s is deducible in intuitionistic propositional logic, then $\mathrm{nf}(s)$ is deducible without use uf the contraction rule

ii) if s is not deducible, then $\mathrm{nf}(s)$ is not deducible

iii) $\mathrm{nf}(s)$ may be computed form s in polynomial time.

Thus we show that the contraction problem may be completely eliminated by a deterministic algorithm running in polynomial time. Moreover the normal form we produce is of a particularly simple kind and it makes the nature and complexity of deducibility in intuitionistic propositional logic perspicuous. In particular it shows an intimate connection between intuitionistic deductions and computations of alternating Turing machines.

Now it is well known (cf. [1],[2]) that to any sequent s we may construct a sequent cl(s) such that

i) s is deducible in intuitionistic propositional logic if and only if cl(s) is deducible

ii) cl(s) may be computed from s in polynomial time and

iii) cl(s) is of the form $v_1, \dots v_n \Rightarrow v_0$, where v_0 is a propositional variable and all v_i with $i > 0$ are of the form a or $a \to b$ or $(a \wedge b) \to c$ or $(a \to b) \to c$, where a, b and c are propositional variables.

Here we give an even more restrictive kind of normal form by considering so called *well founded* sequents:

To any sequent s we consider a relation $<_s$ on the set of propositional variables of s defined by $a <_s b$ if and only if s contains some implication $a \to b$ or $(a \wedge c) \to b$ or $(c \wedge a) \to b$ or $(c \to a) \to b$. Then s is called well founded if and only if the relation $<_s$ is well founded, i.e. $<_s$ does not contain cycles.

Then our normal form nf(s) is constructed in such a way that

i) s is deducible in intuitionistic propositional logic if and only if nf(s) is deducible

ii) nf(s) may be computed from s in polynomial time

iii) nf(s) is of the form $v_1, \dots v_n \Rightarrow v_0$, where v_0 is a propositional variable and all v_i with $i > 0$ are of the form a or $a \to b$ or $(a \wedge b) \to c$ or $(a \to b) \to c$, where a, b and c are propositional variables and moreover

iv) nf(s) is a well founded sequent.

Since we can show that any deducible well founded sequent is deducible without use of the contraction rule this implies that this normal form fulfills our previous requirements.

We consider a form of Gentzen's sequent calculus LJ for intuitionistic propositional logic (cf. [3]): It has axioms of the form $M, a \Rightarrow a$, where a is a propositional variable and $M, \perp \Rightarrow w$, where $\perp$ is the symbol for absurdity and it has rules

$$\text{E}\wedge \quad \frac{M, u\wedge v, u, v \;\Rightarrow\; w}{M, u\wedge v \;\Rightarrow\; w} \qquad\qquad \text{I}\wedge \quad \frac{M \;\Rightarrow\; u \qquad M \;\Rightarrow\; v}{M \;\Rightarrow\; u\wedge v}$$

$$\text{E}\vee \quad \frac{M, u\vee v, u \;\Rightarrow\; w \qquad M, u\vee v, v \;\Rightarrow\; w}{M, u\vee v \;\Rightarrow\; w} \qquad \text{I}\vee \quad \frac{M \;\Rightarrow\; u}{M \;\Rightarrow\; u\vee v} \qquad \frac{M \;\Rightarrow\; v}{M \;\Rightarrow\; u\vee v}$$

$$\text{E}\rightarrow \quad \frac{M, u\rightarrow v \;\Rightarrow\; u \qquad M, v \;\Rightarrow\; w}{M, u\rightarrow v \;\Rightarrow\; w} \qquad\qquad \text{I}\rightarrow \quad \frac{M, u \;\Rightarrow\; v}{M \;\Rightarrow\; u\rightarrow v}$$

Gentzen's so called contraction rule is implicitly included in this calculus by repeating all left hand side principal formulas in the premisses of the E-rules.

Moreover we consider a form of Gentzen's second type of calculi for intuitionistic propositional logic, viz. his calculus NJ of natural deduction. The form we consider here was originally designed for use as a deductive formalism for certain extended logic programs. (cf. [4],[5]) It uses a restricted language with no $\vee$-symbol and for this language it has the same axioms as LJ and it has a single multipremiss rule of the form

$$\text{N} \quad \frac{M, v_1' \;\Rightarrow\; b_1 \quad \ldots \quad M, v_m' \;\Rightarrow\; b_m \quad M \;\Rightarrow\; b_{m+1} \quad \ldots \quad M \;\Rightarrow\; b_n}{M, v \;\Rightarrow\; a}$$

where v is the formula $(v_1 \rightarrow b_1 \,\wedge\, \ldots \,\wedge\, v_m \rightarrow b_m \,\wedge\, b_{m+1} \,\wedge\, \ldots \,\wedge\, b_n) \;\rightarrow\; a$ and v_i' results from v_i by replacing any $\wedge$ with a ',' and it has the explicit contraction rule

$$\text{C} \quad \frac{M, v, v \;\Rightarrow\; w}{M, v \;\Rightarrow\; w}$$

It is well known that for sequents of the restricted language both calculi LJ and NJ are equivalent.

§2. *Construction of the Normal Form*

Now to construct our normal form $\mathrm{nf}(s)$ we use a common approach requiring introduction of new propositional variables to abbreviate complex subformulas of our given sequent s: We use new propositional variables b_n indexed by natural numbers n, new propositional variables l_v, indexed by the subformulas v occurring in our sequent s and new propositional variables $r_{v,n}$, indexed by natural numbers and by the subformulas of s.

Here l_v is meant as indicating presence of the subformula v in the antecedent of some sequent and all its predecessors on a branch of our derivation and $r_{v,n}$ is meant as indicating presence of the formulas v on the right hand side of the n-th sequent of some branch of our derivation. Moreover the variable b_n is understood as saying that the sequent represented by the l_v and by $r_{v,n}$ is deducible by a deduction of length at most n.

Now let $s = v_1, \ldots, v_n \Rightarrow v_0$ be a sequent, k be the number of subformulas of s, t a natural number and let $\varphi(s,t)$ be the union of the two sets $\varphi_{\mathrm{var}}(s,t)$ and $\varphi_{\perp}(s,t)$, where $\varphi_{\mathrm{var}}(s,t)$ is the set of all formulas $l_a \wedge r_{a,i} \rightarrow b_i$ where a is a propositional variable of s and i is a natural number smaller than t and $\varphi_{\perp}(s,t)$ is the set of all formulas $l_{\perp} \rightarrow b_i$. Then $\varphi_{\mathrm{var}}(s,t)$ and $\varphi_{\perp}(s,t)$ express the fact that any sequent which has as its right hand side a propositional variable which also occurs on the left hand side, or which has $\perp$ on its left hand side, i.e. which is an axiom of our calculus LJ, is provable by a deduction of length at most i for any i.

Moreover let $\chi(s,t)$ be the union of the sets $\chi_R(s,t)$ defined as follows:

$\chi_{I\wedge}(s,t)$ is the set of all formulas $(r_{u,i} \rightarrow b_i \wedge r_{v,i} \rightarrow b_i \wedge r_{u\wedge v,i+1}) \rightarrow b_{i+1}$, where u, v and $u \wedge v$ are subformulas of s and i is a natural number smaller than t. This set of formulas indicates that any pair of sequents with equal left hand sides M and right hand sides u resp. v provable by deductions of length at most i constitute a deduction of the sequent $M \Rightarrow u \wedge v$ of length at most $i + 1$.

Similarly

$\chi_{E\wedge}(s,t)$ is the set of formulas $((l_u \wedge l_v \wedge r_{w,i}) \rightarrow b_i \wedge l_{u\wedge v} \wedge r_{w,i+1}) \rightarrow b_{i+1}$,

$\chi_{I\vee}(s,t)$ is the set of formulas $(r_{u,i} \rightarrow b_i \wedge r_{u\vee v,i+1}) \rightarrow b_{i+1}$ resp. $(r_{v,i} \rightarrow b_i \wedge r_{u\vee v,i+t}) \rightarrow b_{i+1}$,

$\chi_{E\vee}(s,t)$ is the set $((l_u \wedge r_{w,i}) \rightarrow b_i \wedge (l_v \wedge r_{w,i}) \rightarrow b_i \wedge l_{u\wedge v} \wedge r_{w,i+1}) \rightarrow b_{i+1}$,

$\chi_{I\rightarrow}(s,t)$ is the set of all formulas $((l_u \wedge r_{v,i}) \rightarrow b_i \wedge r_{u\rightarrow v,i+1}) \rightarrow b_{i+1}$

and $\chi_{E\rightarrow}(s,t)$ is the set of all formulas $((l_v \wedge r_{w,i}) \rightarrow b_i \wedge r_{u,i} \rightarrow b_i \wedge l_{u\rightarrow v} \wedge r_{w,i+1}) \rightarrow b_{i+1})$. Then

Let $\psi(s,t)$ be the formula $(l_{v_1} \wedge \ldots \wedge l_{v_n} \wedge r_{v_0,t}) \rightarrow b_t$ and

let $\rho(s,t)$ be the sequent $(l_{v_1}, ldots, l_{v_n}, r_{v_0,t}, \varphi(s,t), \chi(s,t) \;\Rightarrow\; b_t$ and let finally

nf(s) be $\rho(s,k^2)$.

§3. *Properties of the normal form*

We use a trivial

Lemma:

a) Any LJ-deduction of a sequent $M, u \wedge v \;\Rightarrow\; w$ may be transformed into an LJ-deduction of the sequent $M, u, v \;\Rightarrow\; w$ of smaller or equal length.

b) Any LJ-deduction of a sequent $M \;\Rightarrow\; u \rightarrow v$ may be transformed into an LJ-deduction of the sequent $M, u \;\Rightarrow\; v$ of smaller or equal length.

c) Any LJ-deduction of a sequent $M \;\Rightarrow\; w$ may be transformed into an LJ-deduction of the sequent $M, N \;\Rightarrow\; w$ of smaller or equal length.

(For a proof of this lemma cf. [6])

Then we show

Theorem 1:

If a sequent s has an LJ-deduction of length t, then any sequent $\omega(s,t') = \varphi(s,t'), \chi(s,t') \;\Rightarrow\; \psi(s,t')$, where $t' \geq t$ has an LJ-deduction.

Proof: For $t = 1$ the sequent s has a deduction of length t iff it is an axiom, i.e. iff either its right hand side is a variable which also occurs on the left hand side or its left hand side contains the formula $\perp$. In the first case the formula $\psi(s,t')$ is of the form $(l_a \wedge \ldots \wedge l_{v_i} \wedge \ldots \wedge r_{a,t'}) \rightarrow b_{t'}$ and the set $\varphi(s,t')$ contains the formula $l_a \wedge r_{a,t'} \rightarrow b_{t'}$. Therefore the sequent $\varphi(s,t') \Rightarrow \psi(s,t')$ has a deduction of length 4. In the second case the formula $\psi(s,t'))$ is of the form $(l_\perp \wedge \ldots \wedge l_{v_i} \wedge \ldots \wedge r_{v_0,t'}) \rightarrow b_{t'}$ and the set $\varphi(s,t')$ contains the formula $l_\perp \rightarrow b_{t'}$. Thus the sequent $\varphi(s,t') \Rightarrow \psi(s,t')$ has a deduction of length 3.

If the sequent s has a deduction of length $t > 0$, then it is the conclusion of some inference and its premisses have deductions of length smaller than t. We consider cases according to the form of the last inference leading to s:

If this inference is an application of $I\wedge$, then $\psi(s,t')$ is of the form $(l_{v_1} \wedge \ldots \wedge l_{v_n} \wedge r_{u\wedge v,t'}) \rightarrow b_{t'}$ and $\chi(s,t')$ contains the formula $((r_{u,t'-1} \rightarrow b_{t'-1}) \wedge (r_{v,t'-1} \rightarrow b_{t'-1}) \wedge r_{u\wedge v,t'}) \rightarrow b_{t'}$. Now for the two premisses s_0 and s_1 of s we have $\psi(s_0,t'-1) = (l_{v_1} \wedge \ldots \wedge l_{v_n} \wedge r_{u,t'-1}) \rightarrow b_{t'-1}$ and $\psi(s_1,t'-1) = (l_{v_1} \wedge \ldots \wedge l_{v_n} \wedge r_{v,t'-1}) \rightarrow b_{t'-1}$ and by the induction hypothesis both sequents $\varphi(s_i,t'-1), \chi(s_i,t'-1) \Rightarrow \psi(s_i,t'-1)$ have LJ-deductions and thus by our lemma the sequents $\varphi(s_0,t'-1), \chi(s_0,t'-1), l_{v_1}, \ldots, l_{v_n}, r_{u,t'-1} \Rightarrow b_{t'-1}$ and $\varphi(s_1,t'-1), \chi(s_1,t'-1), l_{v_1}, \ldots, l_{v_n}, r_{v,t'-1} \Rightarrow b_{t'-1}$ are LJ-deducible and therefore by applications of I$\rightarrow$ the sequents $\varphi(s_0,t'-1), \chi(s_0,t'-1), l_{v_1}, \ldots, l_{v_n} \Rightarrow r_{u,t'-1} \rightarrow b_{t'-1}$ and $\varphi(s_1,t'-1), \chi(s_1,t'-1), l_{v_1}, \ldots, l_{v_n} \Rightarrow r_{v,t'-1} \rightarrow b_{t'-1}$ are deducible and by two applications of I$\wedge$ and an application of the lemma the sequent $\varphi(s,t'), \chi(s,t'), l_{v_1}, \ldots, l_{v_n}, r_{u\wedge v,t'} \Rightarrow (r_{u,t'-1} \rightarrow b_{t'-1}) \wedge (r_{v,t'-1} \rightarrow b_{t'-1}) \wedge r_{u\wedge v,t'}$ has an LJ-deduction and by an application of E$\rightarrow$ the sequent $\varphi(s,t'), \chi(s,t'), l_{v_1}, \ldots, l_{v_n}, r_{u\wedge v,t'} \Rightarrow b_{t'}$ is deducible and thus by applications of E$\wedge$ and I$\rightarrow$ the sequent $\omega(s,t') = \varphi(s,t'), \chi(s,t') \Rightarrow \psi(s,t')$ has an LJ-deduction.

If the sequent s results from s_0 and s_1 by an application of E$\rightarrow$, then $\psi(s,t')$ is of the form $(l_{u\rightarrow v} \wedge l_{v_1} \wedge \ldots \wedge l_{v_n} \wedge r_{v_0,t'}) \rightarrow b_{t'}$ and $\chi(s,t')$ contains the formula $((l_v \wedge r_{v_0,t'-1} \rightarrow b_{t'-1}) \wedge (r_{u,t'-1} \rightarrow b_{t'-1}) \wedge l_{u\rightarrow v,t'} \wedge r_{v_0,t'}) \rightarrow b_{t'}$. Furthermore $\psi(s_0,t'-1)$ and $\psi(s_1,t'-1)$ are of the form $(l_{u\rightarrow v} \wedge l_{v_1} \wedge \ldots \wedge l_{v_n} \wedge r_{u,t'-1}) \rightarrow b_{t'-1}$ resp. $(l_{u\rightarrow v} \wedge l_v \wedge l_{v_1} \wedge \ldots \wedge l_{v_n} \wedge r_{v_0,t'-1}) \rightarrow b_{t'-1}$. Thus by the induction hypothesis and by the lemma the sequents

$\varphi(s_0, t'-1), \chi(s_0, t'-1), l_{u\to v}, l_v, l_{v_1}, \ldots, l_{v_n}, r_{v_0,t'-1} \Rightarrow b_{t'-1}$ and $\varphi(s_1, t'-1), \chi(s_1, t'-1), l_{u\to v}, l_{v_1}, \ldots, l_{v_n}, r_{u,t'-1} \Rightarrow b_{t'-1}$ are LJ-deducible. Thus the sequents $\varphi(s_0, t'-1), \chi(s_0, t'-1), l_{u\to v}, l_{v_1}, \ldots, l_{v_n} \Rightarrow (l_v \wedge r_{v_0,t'-1}) \to b_{t'-1}$ resp. $\varphi(s_1, t'-1), \chi(s_1, t'-1), l_{u\to v}, l_{v_1}, \ldots, l_{v_n} \Rightarrow r_{u,t'-1} \to b_{t'-1}$ are deducible, too. Therefore by applications of I$\wedge$ and our lemma the sequent $\varphi(s, t'), \chi(s, t'), l_{u\to v}, l_{v_1}, \ldots, l_{v_n}, r_{v_0,t'} \Rightarrow (r_{u,t'-1} \to b_{t'-1}) \wedge (l_v \wedge r_{v_0,t'-1}) \to b_{t'-1} \wedge l_{u\to v} \wedge r_{v_0,t'}$ has an LJ-deduction and by an application of E$\to$ the sequent $\varphi(s, t'), \chi(s, t'), l_{u\to v}, l_{v_1}, \ldots, l_{v_n}, r_{v_0,t'} \Rightarrow b_{t'}$ is deducible and by applicatons of E$\wedge$ and I$\to$ the sequent $\omega(s, t)$ is deducible.
All other cases are treated in similar ways.

On the other hand we also have

Theorem 2:

If a sequent $\omega(s, t)$ has an NJ-deduction of length $t' \leq t$, then the sequent s has an LJ-deduction.

Proof: Let s be the sequent $v_0, \ldots, v_n \Rightarrow v_0$; then we show: if the sequent $\omega'(s, t) = l_{v_0}, \ldots, l_{v_n}, r_{v_0,t}, \upsilon(t), \varphi(s, t), \chi(s, t) \Rightarrow b_t$, where $\upsilon(t)$ is a set of formulas of the form $r_{w,t''}$ or $q \to b_{t'\prime}$ with $t'' > t$ has an NJ-deduction of length $t' \leq t$, then s has an LJ-deduction.
No sequent $\omega'(s, t)$ is an axiom. But if such a sequent has a deduction of length 2, then either $\varphi(s, t)$ contains a formula $(l_a \wedge r_{a,t}) \to b_t$ and a equals both v_0 and one of the v_i with $0 < i$ or $\varphi(s, t)$ contains the formula $l_\perp \to b_t$ and one of the v_i with $0 < i$ is $\perp$. In both cases the sequent s is an LJ-axiom.
If $\omega'(s, t)$ has a deduction of length $t' > 2$, then $\chi(s, t)$ contains some formula $p = ((A \to b_{t-1}) \wedge C) \to b_t$ or $p = ((A \to b_{t-1}) \wedge (B \to b_{t-1}) \wedge C) \to b_t$, where A, B and C are conjunctions of propositional variables different from the b_i and the sequents $r_0 = A', l_{v_0}, \ldots, l_{v_n}, r_{v_0,t}, \varphi(s, t), \chi(s, t) \Rightarrow b_{t-1}$ and $r_2 = l_{v_0}, \ldots, l_{v_n}, r_{v_0,t}, \varphi(s, t), \chi(s, t) \Rightarrow C$ resp. the sequents r_0, r_2 and $r_1 = B', l_{v_0}, \ldots, l_{v_n}, r_{v_0,t}, \varphi(s, t), \chi(s, t) \Rightarrow b_{t-1}$ are NJ-deducible, where A' resp. B' result from A resp. B by replacing any $\wedge$ with a ','. Now to the

sequents r_0 and r_1 the induction hypothesis applies and we distinguish cases according to the form of p: If p is in $\chi_{I\wedge}$, then A' is $r_{u,t-1}$, B' is $r_{v,t-1}$ and C is $r_{u\wedge v,t}$. But the formula C does not occur as right hand side of any implication of r_2, therefore, as r_2 is deducible, it must occur atomic on the left hand side of r_2, i.e. it must be $r_{v_0,t}$. Therefore the right hand side of the sequent s is $u \wedge v$. Moreover r_0 and r_1 are of the form $\omega'(s_0, t-1)$ resp. $\omega'(s_1, t-1)$, where s, s_0 and s_1 have the same left hand sides and s_0 and s_1 have right hand sides u resp. v. Thus s_0 and s_1 are the premisses of an application of I$\wedge$ leading to s; and since by the induction hypothesis both s_i are LJ-deducible, s is LJ-deducible, too.
If p is in $\chi_{E\rightarrow}$, then A' is $l_v, r_{w,t-1}$, B' is $r_{u,t-1}$ and C is $l_{u\rightarrow v} \wedge r_{w,t}$. The formulas $l_{u\rightarrow v}$ and $r_{w,t}$ must again occur on the left hand side of r_2. Thus the sequent s right hand side w and a formula $u \rightarrow v$ on its left hand side. Moreover r_0 and r_1 are of the form $\omega'(s_0, t-1)$ resp. $\omega'(s_1, t-1)$, where s_0 is s with an additional formula v on its left hand side and s_1 is s with its right hand side replaced by u. But s_0 and s_1 are LJ-deducible by the induction hypothesis and s is deducible from them by an application of E$\rightarrow$. Thus s itself is LJ-deducible.
All remaining cases are treated in a similar manner.

These two theorems show that the sequent $\omega(s, t)$ is deducible in intuitionistic propositional logic if and only if s is deducible and by our lemma the same holds for $\rho(s, t)$.
Now let s' be $\rho(s, t)$, then it is obvious that $a <_{s'} b$ iff either a is some propositional variable l_v or $r_{v,n}$ and b is some propositional variable b_i or a is b_i and b is b_{i+1}. Therefore our relation s' is well founded and so is $\rho(s, t)$.
On the other hand we have the well known

Observation:

If a sequent s is deducible in intuitionistic propositional logic and s has k subformulas, then s has a deduction of length at most k^2

Proof: When going backwards from the conclusion of an LJ-inference to one of its premisses we can only add some subformula of the conclusion to the left hand side or we can replace the right hand with some subformula of the conclusion. Thus before adding a new formula to the left hand side on some branch of a deduction we can only change the right hand side k many times without producing some redundant sequent. Also we can only add k formulas to the left hand side without producing redundant sequents. Therefore any non redundant branch of an LJ-deduction of s must be of length at most k^2.

Therefore for any sequent s which has k subformulas s is deducible if and only if $\rho(s, k^2) = \mathrm{nf}(s)$ is deducible. But clearly k depends polynomially from the size of s. Thus this definition fulfills our initial requirement that $\mathrm{nf}(s)$ shall be computable from s in polynomial time.
It remains to be seen that if $\mathrm{nf}(s)$ is deducible by NJ, than it is deducible without use of the contraction rule: This follows from the

Lemma:

If a well founded sequent $s = M, u \to v \;\Rightarrow\; w$ is deducible by NJ, where $v <_s w$ does not hold, then the sequent $M \;\Rightarrow\; w$ is deducible.

Proof: The formula $u \to v$ is only needed in a deduction of s if at some stage of the backwards construction of a deduction of s the formula v occurs as right hand side of some sequent. This, however, is not possible since consecutive right hand side variables is such a deduction have to obey the $<_s$-relation.

So when considering a maximal application of the contraction rule together with a preceding application of the N-rule. i.e. a pair of inferences leading e.g. from a sequent $M, (a \to b) \to c, a \;\Rightarrow\; b$ to the sequent $M, (a \to b) \to c, (a \to b) \to c \;\Rightarrow\; c$ to the sequent $M, (a \to b) \to c \;\Rightarrow\; c$ we see that b is smaller in the ordering associated with our sequent than c. Therefore c can not be smaller than b and according to the lemma the formula $(a \to b) \to c$ may be dropped from our first sequent. Thus this application of the contraction rule may be eliminated.

Thus we see that our normal form nf(s) for an arbitrary sequent s fulfills all our initial requirements. This shows that the complexity of theorem proving in intuitionistic propositional logic may be drastically reduced by only considering sequents of the elementary form we have called well founded sequents

§4. *References*

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