About the Curry-Howard Correspondence

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Tübingen 8-12/VIII/2022

OUTLINE

The Correspondence

The Curry-Howard Correspondence is a subject with many names: we just mention two of them

- *formulae-as-types*, which occurs in the title of Howard's paper of 1969 (published in 1980), where he was dwelling on the Correspondence noted by Curry in 1934 between some axioms of Intuitionistic (implicational) logic and his own Combinatory logic,
- *Propositions-as-Types*, a name probably originated by Martin-Löf's contributions to Type Theory in the 1970s. This name emphasizes the correspondence between the area of programming languages, where Types live, and logic, the area of relevance of Propositions. Thus, a constructive proof of a proposition A is *related* to a program of the type which is named A after the proposition.

From Correspondence to Isomorphism

In order to speak of an *Isomorphism* it is necessary that structures of the same kind already exist on either side. In fact, people gradually realized that simplifications, or reductions, or normalizations (of proofs) correspond to evaluations (of the terms which name the programs corresponding to the proofs).

Inferential Semantics

Curry-Howard Correspondence largely overlaps the general framework of "Proof-theoretic semantics", an area of research which arose around the seventies of the last century as opposed to the classical model-theoretic explication of the notion of logical consequence. According to the latter, a sentence A is said to follow by logic from a set of sentences Γ if, and only if, every interpretation which makes any sentence of the set Γ true makes A true too.

The new perspective emphasizes the central position of the notion of *inference* for semantics as well. A thumbnail description of this field of ideas must take into account at least three kinds of sources:

Inferential Semantics II

- At first, an important stimulus came through the attempts to provide an explanation for the meaning of the logical constants within Intuitionistic logic; attempts which led to the so called **BHK**-explanation.
- A second main motivation, soon intertwined with the former, was supplied by the investigations originated by Gentzen's in-depth renewal of working in proof theory.

Inferential Semantics III

- A third major role in this area was played by the developments of the "meaning-as-use" point of view in the philosophy of language, leading to an inference (instead of a truth) based approach to the explanation of the meaning of the logical constants. An important part in this area of investigation was produced by the proof-theoretic semantics developed mainly by Dummett and Prawitz, starting from the seventies of the last century, but already present in Popper's logical papers of the late forties of the last century.

Hilbertian Roots

- A common tenet of the inferentialist approach was Hilbert's idea to regard proofs as the *objects* of a specific mathematical inquiry. Since proving is the common practice of any field of mathematical research, the inquiry which focuses on the very same notion of proof is rated to concern every mathematical framework, and so it rightly deserves the name of *proof theory* or *meta*mathematics.
- No doubt, this theme was firmly present from the beginning in Hilbert's investigations (I mean, for instance, the memories *Über die Grundlagen der Logik und der Arithmetik*, of 1904, and *Axiomatisches Denken*, of 1917), and it played a pivotal role in the development of the idea of a proof-theoretic semantics.
- However, I think that it is also interesting to trace this idea itself back to E. Husserl's philosophical investigations.

Husserlian Roots

As is well-known, Husserl was active at the University of Göttingen, from 1891 to 1916, and he was there closely connected with mathematicians and physicists (being a mathematician himself). Afterwards he moved to Freiburg, where in 1929 he published *Formale und Transzendentale Logik*, just one year after his forced retirement from the University. Husserl wrote this work in a few months between 1928 and 1929, immediately before devoting himself to the preparation of the lectures “Cartesian Meditations”, to be given in 1929 in Paris.

Husserlian Roots II

In *Formale und Transzendentale Logik* he resumed the theme of a layered presentation of logic provided in the *IV. Logische Untersuchung. Grammatik und Intentionalität* (in the second volume of the *Logische Untersuchungen*, 1900-1901), and this theme underwent a further development which provided a three-layered frame for the fundamental structures of traditional Logic in the book of 1929: *Formenlehre, Konsequenzlogik* and *Wahrheitslogik*.

Husserlian Roots III

In §§ 24-25 of the book Husserl emphasizes that the step from *Konsequenzlogik* to *Wahrheitslogik* involves the emergence of a cognitive motivation. Differently from what characterizes the first two layers, in the third one judgements are no longer considered as syntactic buildings to be analysed in their construction or morphology (as within the *Formenlehre*) and in their mutual deductive relations (as within the *Konsequenzlogik*), but as tools to reach the knowledge of the object.

Husserlian Roots IV

In this way, once made explicit –we could dare to say: once *formalized*–, a judgement may find a *semantic* realization, or satisfaction (it can be *erfüllt*). In the previous § 19 he had already said that:

> *Jetzt sind von vornherein die Urteile nicht als bloße Urteile gedacht, sondern als von einem Erkenntnisstreben durchherrschte, als Meinungen, die sich zu erfüllen haben, die nicht Gegenstände für sich sind im Sinne der Gegebenheiten aus bloßer Deutlichkeit, sondern Durchgang zu den erzielenden "Wahrheiten" selbst. (Husserl, E. [1969], p.70).*

Translation

"Now the judgments are thought of from the very beginning, not as mere judgments, but as judgments pervaded by a dominant *cognitional striving*, as meanings that have to become *fulfilled*, that are not objects by themselves, like the data arising from mere distinctness, but passages to the "truths" themselves that are to be attained." (Husserl, E. [1974], p. 65)

Husserlian Roots V

The *Konsequenzlogik*, defined as "die Wissenschaft von den *möglichen Formen wahrer Urteile*" (Husserl, E. [1969], p. 54), is the layer in which we speak "about" propositions; in the *Wahrheitslogik* we instead speak "in" or "through" them, making an effort toward knowledge.

To elaborate this change of perspective, Husserl emphasizes the close link between Mathematics, considered as Formal Ontology (*i.e.*, a theory of the *Gegenstand-überhaupt* or *Etwas-überhaupt*, pp. 77-78), and a Formal Theory of Assertions. A link which "*führt notwendig auf eine formale apophantische "Mathematik"*" (p.77). Thus, he stresses that Formal Ontology and Formal Apophantik, although directed toward different subjects, have to be developed in close connection and cannot be disentangled.

Husserlian Roots VI

In order to support his thesis, Husserl focuses firstly on the direction from *Gegenstand* to *Urteilen*, and reminds us that

> *Schließlich treten doch alle Formen von Gegenstanden, alle Abwandlungsgestalten des Etwas-überhaupt in der formalen Apophantik selbst auf, wie ja wesensmäßig Beschaffenheiten (Eigenschaften und relative Bestimmungen), Sachverhalte, Verbindungen, Beziehungen, Ganze und Teile, Mengen, Anzahlen und welche Modi der Gegenständlichkeit sonst, in concreto und ursprünglich expliziert, für uns als wahrhaft seiende oder möglicherweise seiende nur sind als in Urteilen auftretende (Husserl, E. [1969], p. 79).*

Translation

"Ultimately *all the forms of objects*, all the derivative formations of anything-whatever, do make their appearance *in formal apophantics itself*; since indeed, as a matter of essential necessity, determinations (properties and relative determinations), predicatively formed affair-complexes, combinations, relationships, wholes and parts, sets, cardinal numbers, and all the other modes of objectivity, *in concreto* and explicated originaliter, have being for us –as truly existent or possibly existent modes– only as making their appearance in judgements." (Husserl, E. [1974], p. 79).

Husserlian Roots VII

Then, moving to the opposite direction, Husserl emphasizes that declarative statements, *i.e.* predicative judgements, can be treated as members of the Formal Ontology, that is as *objects*. This is an insight which Leibniz first realized by acknowledging that propositional forms can be treated within the realm of mathematical deductive procedures and that

> *man mit ihnen ebenso "rechnen" kann wie mit Zahlen, Größen usw (Husserl, E [1969], p. 77).*

Husserlian Roots VIII

The link highlighted between Formal Ontology, the science of the "object in general", and Formal Apophantik, the science of the possible forms of judgements, lets us know that Husserl was fully aware of the idea of a *correspondence "judgements-objects"*: an idea that can be detailed as follows:

1. To "judge" only means to judge about "objects" and properties of objects, so that objects exist for us only in so far as they occur in judgements.

   > *Indessen man braucht sich nur daran zu erinnern, daß Urteilen soviel heißt wie über Gegenstände urteilen, von ihnen Eigenschaften aussagen oder relative Bestimmungen; so muß man merken, daß formale Ontologie und formale Apophantik trotz ihrer ausdrücklich verschiedenen Thematik doch sehr nahe zusammengehören müssen und vielleicht untrennbar sind. (Husserl, E. [1969], p. 83).*

2. Every "judgement", in turn, can become an "object", falling in this way within the framework of Formal Ontology.

Translation

"Nevertheless one need only remind oneself that judging is the same as judging about *objects*, predicating *properties* of them, or *relative determinations*; taking this into consideration, one cannot fail to note that formal ontology and formal apophantics, despite their expressly different themes, must be very intimately related and are perhaps inseparable". (Husserl, E. [1974], pp.78-79).

Husserlian Roots IX

Previous points are of course reminiscent of Leibniz's attempts to build a *Mathesis Universalis*. Some comments are here opportune:

- Actually, the first point consists in turning, by *nominalization*, a modification of the category *Gegenstand-überhaupt*, like for instance "*S* is a cardinal number", into a part of a judgement, as "the cardinality of *S*", by adopting the perspective of *praedicatum inest subjecto*.
- The second point recalls Leibniz's attempts to codify sentences, starting a combinatorial calculus to the end of getting (an enumeration of) the true propositions (of a given area of inquiry).

Husserlian Roots X

The latter point itself, however, also foresees the forthcoming (two years later) Gödel's arithmetization of the metatheory, which provides a numerical code for any syntactic entity. Compared with Leibniz's attempt to develop a new theory, called *Logica* (or *Ars*) *Inveniendi*, when Gödel supplied a numerical code to all sorts of syntactic entities, by following the steps of their formation, he didn't mean to make it possible to calculate with them as rather to find a way to merge the metatheory into the theory. In one sense, however, the codes provided for a proof can be seen as a first example of "proof-objects", a perspective which will be enlightened by Kleene in 1945 introducing the idea of *realizability* as a tool to extract, or make explicit, the algorithmic content of *constructive* proofs, according to their informal semantics then become widely known under the name of **BHK**-interpretation.

Church and Curry

More or less in that same period, that is, during the years between the late twenties and the early thirties, the field of mathematical logic was enhanced by two new formalisms, λ-*calculus* and *combinatory logic*, invented, respectively, by A. Church and H.B. Curry. Actually, combinatory logic was invented in the early twenties by M.I. Schönfinkel, but it is well known that they got their theories independently. Both theories were bound to play a major role in the late sixties. In this regard, here there are some basic points which are worth to be reminded:

1. Even though they were developed apparently whitout being aware of each other, they share a common root in the discipline of the abstraction and substitution procedures originated by Frege and amply investigated by Russell in the first chapter of *Principia Mathematica*.

Church and Curry II

2. They were centered on the study of function abstraction and function application, where functions are understood not as sets of ordered pairs but as rules of correspondence.
3. As is well known, both systems were discovered to be inconsistent by Kleene and Rosser in 1934, by deriving a form of the Richard's paradox within both systems.
4. As Russell before them, both Church and Curry reacted by exploiting the notion of *type*, which was thought of as a *syntactic decoration* that restricts the formation of terms so that functions may only be applied to appropriate arguments.

Bernays and Gentzen

5. Incidentally, both Church and Curry visited Göttingen in 1928-29, and there Curry completed his dissertation, making most of the work with P. Bernays. This is relevant since Schönfinkel lectured in Göttingen in 1920 on combinatory logic, producing a paper which was published in 1924, and later, in 1928, a paper on the *Entscheidungsproblem der mathematischen Logik* was published as the result of joint work of Bernays with Schönfinkel.
6. Thus, one could say that visiting Göttingen, both Curry and Church had good opportunities to develop and improve their new proposals. This complex set of relationships highlights the relevance of Bernays in that historical context, keeping moreover into account that Bernays was also the major reference point for G. Gentzen's work.

Bernays and Gentzen II

7. And it is pertinent to stress that Gentzen's *Natural Deduction Calculus* is the formalism where the Intuitionistic –or, better, *Minimal*– base flavour was the natural framework to study correspondences with the λ-*calculus*, one of its nearest kin.
8. Actually, the latter remark is something we may state in retrospect, after the emergence of a new notion of constructivity, the so-called "propositions-as-types" notion of constructivity – our current subject–, which contributed to making the idea to treat proofs as objects *explicit*, an idea that was *implicit* in Gentzen's and Prawitz's Normalization Theorems (respectively, of 1934-5 and of 1965).

Hilbert and Husserl

10. This was an idea that, in hindsight, we could venture to frame within the scope of Husserl's idea, earlier discussed, of a *correspondence "judgements-objects"*. It is opportune to note that in this way things are pushed a step further than in Hilbert's idea, previously mentioned, to regard proofs as the *objects of a specific mathematical inquiry*: to put it briefly, in the new framework proofs become *mathematical objects in itself*.

Curry

Focussing in the early thirties on Hilbert's style of formalization, the axiomatic procedures, Curry noted an interesting but possibly merely coincidental structural similarity between the implicative fragment of Heyting's axiomatic system of Intuitionistic logic:

$$A \supset A$$
$$A \supset (B \supset A)$$
$$(A \supset (B \supset C)) \supset ((A \supset B) \supset (A \supset C))$$

and his primitive combinators $\mathbf{I}, \mathbf{K}, \mathbf{S}$, where the combinator $\mathbf{I}$ is the *identity* operator such that for any x we have $\mathbf{I}x = x$. $\mathbf{K}$ is the *constancy* operator such that $\mathbf{K}xy = x$. So, $\mathbf{K}x$ is the constant function which for any argument y returns x as value. $\mathbf{S}$ is the *distribution* operator such that applied to x, y, z returns $xz(yz)$ as value.

Formulae-as-types

This remark, which surely deserves to be considered as the beginning of the *formulae-as-types*, or *propositions-as-types* framework, was published in 1934.

This area of investigation evolved into the project to understand, or to extract, the computational content of proofs focusing on the "space of formal proofs" as a mathematical structure in its own right, instead of mainly pursuing the meta-theoretical features of a formal theory (consistency, soundness, completeness, decidability, . . .).

The logico-computational content of the Cut-Elimination Theorem, and the same is true of the Normalization Theorem, is clear: they pave the way for a treatment of *proofs as objects* on which one can algorithmically operate by a simplification calculus whose steps consist in the systematic eliminations of irrelevant propositions.

Formulae-as-types II

According to the *proof*-centered setting, the notion to clarify was

"*F* is a construction that proves the proposition *A*"

and this is a task that one should accomplish by reasoning *in terms of the logical form of A*.
In relation to the same notion of construction, however, there was a sort of tension consisting in the fact that in the **BHK**-explanation the same idea seemed to occur both at the meta- and at the object-level. Typically, in fact, the clause for implication: "*F* is a construction that proves the proposition $A \supset B$", is formulated by stating that "for any α, *if* α is a construction that proves *A*, *then* $F(\alpha)$ is a construction that proves *B*".

Formulae-as-types III

However, the form of circularity involved in defining the meaning of "$\supset$" (and an analogous remark could be made for other logical constants) by exploiting at the meta-level the clause *"if . . . then . . . "* must not cause too much trouble. It is appropriate, to this end, to realize that a sensible representation of mathematical practice has to envisage a three-layered framework; that is, a framework consisting of

1. informal, or pre-formal, mathematics,
2. (informal) axiomatic theories, and
3. formal theories.

Formulae-as-types IV

Concerning this representation it is to be emphasized that no phase deletes the previous ones; all of them, so to speak, co-exist. And this means that nobody starts doing mathematics building on a *tabula rasa*, but everyone doing mathematics shares a group of first-level proof procedures and theoretical constructions, including a good informal understanding of implication, which may act as a foundation for its formal description at the second or third level.

However, the idea that the **BHK**-clauses provide an implicit definition of the class of constructive proofs which the interpretation refers to met serious problems in virtue of the fact that the clauses for $\supset$ contain quantifiers which are intended to range over the class of all constructive proofs, potentially inclusive of those which figure in the proof conditions of yet more complex formulas.

Formulae-as-types V

Here we face a new form of *circularity* which is due to the fact that the class of constructive proofs forms a potentially infinite totality which however should not be regarded as inductively generated, in virtue of the occurrence of the universal quantifier over proofs in the previously mentioned clauses.

One thing was clear: the fact that the Intuitionistic explications of the meaning of logical constants involve constructions (the previous α and β) and constructions operating on constructions (the *F*); that is, *functionals* acting on other functionals. Where a functional was intended to be given not through a set of ordered pairs but when we have been given not only the action or process it performs but also its *type* (domain and counterdomain).

Formulae-as-types VI

A further important step forward was accomplished when somebody started to realize that the term "construction" may have different meanings. Adopting the now widespread proposal which was explicited in the early '80s by G. Sundholm in [1983], it is opportune to keep at least the two following notions of "construction" separate:

1. the notion of a construction-as-a-*process*
2. the notion of a construction-as-an-*object*-got-carrying-out-a-*process*-of-construction.

Equipped with this distinction, we may observe that since Heyting was reasoning in terms of the first notion, he didn't consider the proofs of a given proposition as constitutive of a set of objects, or, to use a word then become familiar with this meaning, a *type*.

Howard

The second perspective, which is more reminiscent of Husserl's point of view, and in which constructions are regarded as mathematical objects, the so called *proof-objects*, promised to be very helpful (also in treating the first form of circularity). This perspective was chosen by W. Howard –and then by many other scholars– who in 1969 developed Curry's remark of 1934 by assigning to each basic combinator **I**, **K**, **S**, as its *type*, the formula stating the relevant implicational axiom of Heyting's propositional calculus, getting:

$$\mathbf{I}^{A \supset A}$$
$$\mathbf{K}^{A \supset (B \supset A)}$$
$$\mathbf{S}^{(A \supset (B \supset C)) \supset ((A \supset B) \supset (A \supset C))}$$

The Basic Idea

The basic idea is that for each proposition in a given logic there is a type of objects, the proofs of that proposition, so that *a proposition* is to be considered *as the type of its proofs*.

In other words, a type, or set, simply is the type, or set, of proofs of the proposition that *labels* that type, and, reciprocally, a proposition is just the type, or set (labeled by that same proposition), of its proofs. Howard was able to extend the correspondence between the combinators and the implicational fragment of intuitionistic logic to first-order predicate logic.

Simply-typed λ-calculus

Two facts are noteworthy.

1. The first is that Howard, to develop Curry's remark of 1934, judged opportune to stress the analogous correspondence between derivations in (Intuitionistic) natural deduction calculus and terms in simply-typed λ-calculus. This perspective helped to extend the correspondence propositions-types to that between proofs (of a proposition) and programs (named by a term of the corresponding type), and finally to that between simplifications, or reductions, (of proofs) and evaluations (of the terms which name the programs). An extension which got an Isomorphism out of a Correspondence.

Resolving Circularity

2. Secondly, it is remarkable that the *Curry*–Howard Correspondence pivots on the close link with *Church*'s simply-typed λ-calculus, developed around the same time for an unrelated purpose, but the label lacks any reference to the inventor of the λ-calculus.

The new perspective allowed Howard to correctly frame the tension that we previously emphasized. He wonders how to formulate the notion that "F is a construction of $A \supset B$", for instance, in order to avoid circularity, and he answers by taking that notion to mean:

(*) F is assigned the type $A \supset B$ according to the way F is built up;
i.e., the way in which F is constructed.

Resolving Circularity II

In other words, "F is a construction of $A \supset B$ *by construction*".
In his own words (reported in Wadler, P. [2015]):

> *What was new was the thought (*) plus the recognition that Curry's idea provided the way to implement (*). I got this basic insight in the summer of 1966. Once I saw how to do it with combinators, I wondered what it would look like from the viewpoint of the lambda calculus, and saw, to my delight, that this corresponded to the intuitionistic version of Gentzen's sequent calculus.*

Resolving Circularity III

- A necessary step to develop such a perspective was to provide a formalization of the notion "c is a construction which proves proposition A", in short "$c : A$".
- This, in turn, presupposes that proofs occur as parts of the language. Thus, what is necessary within the framework of the "propositions-as-types" correspondence is a language which is able to *speak* about proofs, and not only able to provide the items for constructing a proof.

Resolving Circularity IV

- This language will have expressions like $c : A$, meaning that c denotes a proof of proposition A, and $x : A$, meaning that an hypothetical proof of A is available. Consequently, the language of the "calculus of proofs" will contain variables $x, y, \ldots$, an operator λ binding variables, and an operation Ap of application.
- Instead of formulas, or propositions, premises and conclusions in the arguments built within this language will be *judgements* of the form $c : A$, indicating that the term c has type A.

Resolving Circularity V

We noted that in defining, for instance, implication, $\supset$, we exploit the *meta* level "*if* ... *then* ...", expressed by the line of inference, in order to define the *object* level formula $A \supset B$, here on the left:

$$\dfrac{\begin{matrix}[A] \\ \vdots \\ B\end{matrix}}{A \supset B} \qquad \dfrac{\begin{matrix}[x : A] \\ \vdots \\ b : B\end{matrix}}{\lambda x.b : A \supset B}$$

whereas in the figure on the right we exploit again implication at the *meta* level (since we assume that *if* the terms above the inference line are well typed, *then* also the term below is well typed), but at the *object* level we have a function having type $A \supset B$, because if it receives as input a value of type A it returns a value of type B. This is what Howard means when he says that "F is a construction of $A \supset B$ *by construction*" or "definition".

More examples: **K** and **S**

May be interesting to see the decoration of the derivations of the formulas corresponding to the combinators **K**:

$$\dfrac{\dfrac{\begin{array}{c}[b : B] \\ [a : A]\end{array}}{\lambda b.a : B \supset A}}{\lambda ab.a : A \supset (B \supset A)}$$

and **S**:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[c : A] \qquad [b : A \supset B]}{Ap(b, c) : B} \qquad \dfrac{[c : A] \qquad [a : A \supset (B \supset C)]}{Ap(a, c) : B \supset C}}{Ap(Ap(a, c), Ap(b, c)) : C}}{\lambda c.Ap(Ap(a, c), Ap(b, c)) : A \supset C}}{\lambda bc.Ap(Ap(a, c), Ap(b, c)) : (A \supset B) \supset (A \supset C)}}{\lambda abc.Ap(Ap(a, c), Ap(b, c)) : (A \supset (B \supset C)) \supset ((A \supset B) \supset (A \supset C))}$$

Martin-Löf's *Intuitionistic Theory of Types*

One of the more interesting development of the "propositions-as-types" framework was Martin-Löf's *Intuitionistic Theory of Types*, **ITT** for short, published in the early 1970s. In fact, he rated that framework as the best way to accommodate the Brouwerian principle according to which logic comes *after* mathematics, logical theorems being nothing but mathematical theorems in extreme generality. In fact, in **ITT** logical operations on propositions are interpreted as certain mathematical operations on sets: in particular, $\forall$ is interpreted as Cartesian product Π, and $\exists$ as coproduct (disjoint union) Σ.

"Proposition-type" vs "Judgement-category"

- The first form of judgement, "A tp", or "A prop", is explained by stating that in order to comprehend a type (a proposition) it is necessary to know the formation rules for its *canonical* objects (proofs), *and* under what conditions two canonical elements of A are equal.
- In order to be able to assert that set A is not void, it is not necessary to produce a canonical object of type A, it is sufficient to provide a *method* to evaluate any other object (proof) which is given in a non-canonical form.
- In this way, we know *what a type (a proposition) is*, but it is impossible to speak of the type of all the types, writing something like "$tp : tp$".

"Proposition-type" vs "Judgement-category" II

- The notion of type is open: we have not exhausted the possibilities of defining new types. On the contrary, we can always add new types, provided we specify what the canonical elements are and what canonical identical elements of the types in question are.
- This fact is mirrored in the requirement that the collection of all types is not a type, but a *category*, and "*A* tp" is not a proposition but a judgement.
- This is the way in which the previously stressed second form of "circularity" is faced.

Dependent Types

A fundamental ingredient of **ITT**, peculiarly fitting to our argument, is the notion of *dependent* types, that is to say, types which may themselves depend on terms (of a certain type). The rules for the operator of Disjoint Union, Σ, are especially relevant for our point: the *formation* rule states that by assuming "A tp" and $\dfrac{x : A}{B(x) : tp}$, we can introduce the type $(\Sigma x : A)B(x)$, whose canonical object are the pairs $< a, b >$ with $a : A$ and $b : B(a)$ (this is the ($I\Sigma$-rule)).

Dependent Types II

The most remarkable rules are the elimination ($E\Sigma$) and equality ($U\Sigma$) rules:
if $c : (\Sigma x : A)B(x)$, and $p(c) = fst(c)$ and $q(c) = snd(c)$, we get the rules:

$$\frac{c : (\Sigma x : A)B(x) \qquad \dfrac{[x : A, y : B(x)]}{x : A}}{p(c) : A}$$

and

$$\frac{c : (\Sigma x : A)B(x) \qquad \dfrac{[x : A, y : B(x)]}{y : B[x := p(c)]}}{q(c) : B(p(c))}$$

Dependent Types III

We have to note that the second rule involves a term for *objects*, $p(c)$, introduced by the first rule, and which is built by means of a term for *proofs*, c. The propositions-as-types perspective allows us to introduce types explicitly depending on terms. It becomes possible that proofs occur in propositions which, therefore, will not speak only of individuals but will also express properties of proofs.

Dependent Types IV

From the operator Σ it is possible to extract the rules for $\exists$ in a form which is considerably stronger than usual. Saving the proof-objects, we can have elimination rules able to provide both the first and the second projection (as an existential quantifier reasonable from a constructive point of view is expected to do):

$$\frac{c : (\exists x : A)B(x)}{p(c) : A} \qquad \frac{c : (\exists x : A)B(x)}{q(c) : B(p(c))}$$

And Martin-Löf stresses that in the conclusion of the second rule we have the introduction of a term for objects, $p(c)$, which is built by means of a term for proofs, c, occurring in the premise. Moreover, we can suppress the proof-object $q(c)$ in the conclusion, getting the judgement $B(p(c)) \quad \mathit{true}$, but not in the premise, being happy with the judgement $(\exists x : A)B(x) \quad \mathit{true}$, since the conclusion *depends* on the proof-object c.

It is interesting to consider the paper Howard, W. [1980] with regard to the logical tools exploited. From a general point of view, in fact, arguments have an evident close link with (Intuitionistic) natural deduction of the 1930s and the more or less contemporary simply-typed λ-calculus. This statement, however, deserves some qualification. Actually, in the § 1. of his paper Howard calls "sequent calculus" his calculus $P(\supset)$:

(1.1) All sequents of the form $A \rightarrow A$

$$(1.2)\ \frac{\Gamma, A \rightarrow B}{\Gamma \rightarrow A \supset B}$$

$$(1.3)\ \frac{\Gamma \rightarrow A \qquad \Delta \rightarrow A \rightarrow B}{\Gamma, \Delta \rightarrow B}$$

(1.4) Thinning, permutation and contraction rules.

The calculus $P(\supset)$, however, doesn't appropriately fit Howard's statement. His rules (1.1)-(1.4), in fact, are nothing but natural deduction rules in *sequent formulation*, consisting of rules managing introduction and elimination of $\supset$, plus the assumption rule and some (weak) structural rules: thinning, permutation and contraction in the left part of the sequent.

I think that it could be pertinent to our argument to note that at least until Prawitz's monograph *Natural Deduction* (1965), and also afterward, acquaintance with Gentzen's work passed mainly through the first (published) work on the consistency of arithmetic of 1936, where exactly a natural deduction calculus in sequent formulation is exploited, enriched by the structural rules adopted by Howard himself in the system $P(\supset)$.
And also pertinent is to recall that in *Note added 1979* Howard emphasizes that his previous *Addition of* $\neg$ *and* $\wedge$ *to* $P(\supset)$ would have gained a better setting if framed within D. Prawitz's theory of Gentzen's system of natural deduction; a step pointed to Howard by P. Martin-Löf.

This fact is confirmed by the rules of term formation that he details in section 2. of the paper. These rules, by exploiting the propositions-as-types framework, can be framed in the following way: given the availability of variables of any type: $X : A, Y : B, \ldots$ (this is the rule (2.1), and is aimed at matching the assumption rule), we are given the rules of λ-abstraction and application (matching, respectively, introduction and elimination of $\supset$), which can be expressed as follows:

$$\lambda\text{-abstraction}\ \frac{\Gamma, X : A \rightarrow F : B}{\Gamma \rightarrow (\lambda X : A.F) : A \supset B}\ (2.2)$$

$$\text{Application}\ \frac{\Gamma \rightarrow G : A \supset B \qquad \Gamma \rightarrow H : A}{\Gamma \rightarrow (GH) : B}\ (2.3)$$

The general point of view changes in section 5., where Howard specifies that in the framework of the system $P(\supset)$, the cut rule (previously not mentioned) corresponds to supplying the simply-typed λ-calculus with a new term operator $G[X := F]$ which *substitutes* F for the free variable X in G, with X and F being of the same type, and which is ruled by the following type inference:

$$\frac{\Gamma \to F : A \qquad \Gamma, X : A \to G : B}{\Gamma \to G[X := F] : B}$$

This is a rule of Explicit Substitution (ES, let's say) that corresponds to the explicit substitution of terms into terms in the λ-calculus, and whose typing rule is closely related to the cut rule.

Operating on terms, in fact, the rule (ES) translates the meaning of the cut rule –a proper rule *of* the sequent calculus– in an operation *on* derivations in natural deduction (working at the meta-level, so to speak). A new derivation is obtained by composing two derivations of the natural deduction calculus which share as, respectively, assertion and assumption, a same formula A and removing either reference to A, as given by the conclusions:

$$\begin{array}{c} \Gamma \\ \vdots \\ A \end{array} \quad \text{combined with} \quad \begin{array}{c} \Delta, A \\ \vdots \\ B \end{array} \quad \text{becomes} \quad \begin{array}{c} \Delta \quad \begin{array}{c} \Gamma \\ \vdots \\ A \end{array} \\ \vdots \\ B. \end{array}$$

Considering things from the "λ-terms" point of view, the rule (ES) can be obtained by combining the rule of λ-abstraction with the rule of application:

$$\dfrac{\dfrac{\Gamma, X : A \to G : B}{\Gamma \to (\lambda X : A.G : B) : A \supset B} \qquad \Gamma \to F : A}{\Gamma \to ((\lambda X : A.G : B)F) = G[X := F] : B}$$

stating in this way that some derivation (the derivation, coded by the term F, of the proposition A, within the context Γ) is to be substituted for an assumption (of the proposition A labeled by the variable X within the context Γ) in another derivation (the derivation, coded by the term G, of the proposition B).

At this stage, Howard judges –not without reason– that adding (ES) to the system $P(\supset)$ of § 1. is not sufficient to guarantee that the cut elimination procedure is able to match the normalization procedure. Thus, he decides to follow Curry's suggestion and replaces the (previously reminded) application rule of λ-calculus with the following rule:

$$\frac{\Gamma \to X : A_1 \supset (A_2 \supset \ldots (A_n \supset B) \ldots) \qquad \ldots \Gamma_i \to F_i : A_i \ldots}{\Gamma, \Gamma_1, \ldots, \Gamma_n \to XF_1 \ldots F_n : B} \; i \leq n$$

Then, Howard searches for an accommodation of this rule within the system $P(\supset)$. To this aim he modifies rule 1.3 –actually a *natural deduction* rule of $\supset$-elimination– into the following *sequent* rule of left $\supset$-introduction:

$$\frac{\ldots \Gamma_i \to A_i \ldots}{\Gamma_1, \ldots, \Gamma_n, (A_1 \supset (A_2 \supset \ldots (A_n \supset B) \ldots)) \to B} \; i \leq n$$

In order to detect this rule as a rule of left $\supset$-introduction, one has to consider that it can be obtained by n applications of the usual rule of left $\supset$-introduction of the sequent calculus with the right premise being an initial sequent $B, \Delta \to B$:

$$\cfrac{\Gamma_1 \to A_1 \qquad \cfrac{\cfrac{\Gamma_{n-1} \to A_{n-1} \qquad \cfrac{\Gamma_n \to A_n \qquad B, \Delta \to B}{\Gamma_n, A_n \supset B, \Delta \to B}}{\Gamma_{n-1}, \Gamma_n, A_{n-1} \supset (A_n \supset B), \Delta \to B}}{\vdots \\ \Gamma_2, \ldots, \Gamma_n, (A_2 \supset \ldots (A_n \supset B) \ldots), \Delta \to B}}{\Gamma_1, \ldots, \Gamma_n, (A_1 \supset (A_2 \supset \ldots (A_n \supset B) \ldots)), \Delta \to B}$$

It could be pertinent to note that sequent calculus *replaces* the rule ($\supset E$) of natural deduction calculus with ($\supset L$), a rule that can be seen as a generalization of cut. In fact, if in ($\supset L$) we let A be equal to B we get

$$\frac{\Gamma \rightarrow \Theta, A \qquad A, \Delta \rightarrow \Lambda}{A \supset A, \Gamma, \Delta \rightarrow, \Theta, \Lambda} \supset L$$

from which cut follows by disregarding in the antecedent of the lower sequent the tautological formula $A \supset A$:

$$\frac{\Gamma \rightarrow \Theta, A \qquad A, \Delta \rightarrow \Lambda}{\Gamma, \Delta \rightarrow \Theta, \Lambda.} \text{Cut}$$

The restriction imposed on the left ⊃-introduction rule is aimed to guarantee that the rule occurs in the form that is the image (under the translation from natural deduction to sequent calculus) of the ⊃-elimination rule (For a general discussion of the question, see von Plato, J. [2011], Tesconi, L. [2010] and [2011].)

$$\dfrac{\begin{array}{c}\vdots\\ \Gamma \rightarrow \mathbf{A} \supset \mathbf{B}\end{array} \qquad \dfrac{\begin{array}{c}\vdots\\ \Delta \rightarrow \mathbf{A}\end{array} \qquad \begin{array}{c}\vdots\\ \Sigma, B \rightarrow B\end{array}}{\Sigma, \Delta, A \supset B \rightarrow B}}{\Gamma, \Sigma, \Delta \rightarrow \mathbf{B}}\ \text{Cut}$$

This is the framework which allows us to establish a one-to-one correspondence between the sequent calculus (modified according to Curry's suggestion) and the natural deduction calculus, and which in turn engenders the correspondence between cut-free and canonical proofs (corresponding, in turn, to λ-terms in canonical form), and also the correspondence between cut-elimination procedure and the elimination of maximal formulas, where the last one corresponds, for λ-terms, to the reduction procedure to β-normal form.

I think that this short excursus on Howard's paper allows us to answer the initial question, certainly with some amount of *liberal* interpretation, by saying that natural deduction calculus is (also for Howard) the *natural* candidate to occupy the logic pole of the correspondence. We have in fact seen that Howard starts by exploiting a natural deduction calculus in sequent formulation; the latter is then enriched with the cut rule in the form of a substitution rule. In order to get the correspondence between this calculus and the simply typed λ-calculus, Howard modifies the first one with the addition of a special left $\supset$-introduction rule, and the second with an application rule suggested by Curry. The sequent calculus obtained at last in this way is such that it can be put in a one-to-one correspondence with the natural deduction calculus.

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THANKS

FOR YOUR ATTENTION