STRONG NORMALIZABILITY AND REALIZABILITY

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This is a worked-out version of my handwritten notes of 1986. The content was explained and discussed on several occasions. In particular, both the approach and the results of this paper were reported in my talk "On Consistency and Cut Elimination" at the Logic Meeting '88 in OBERWOLFACH (Nov.6 - Nov.12, 1988), and the exhaustive abstract circulated.

0. Summary

By the appropriate formalization of the familiar impredicative cut elimination ideas we reduce strong normalizability in set theoretical language to the consistency of the appropriate comprehension axioms. In particular, this proves strong normalizability of Quine's New Foundations minus Extensionality (**NF⁻**) within Zermelo Set Theory (**Z**) extended by the existence of power-set hierarchy along $\aleph_\omega$.

1. Preliminaries

1.1. The theory **NF⁻**.

The language of **NF⁻** includes the following items:
(a) infinite list of variables (abbr.: x, y, z),
(b) one binary predicate $\in$, and one fixed predicate $\perp$,
(c) two connectives $\wedge$ and $\rightarrow$, and one quantifier $\forall$,
(d) formulae (abbr.: A, B, F, G, H), which arise from 1)–3) by familiar clauses.
Other formula-expressions are introduced in the familiar "negative" manner, i.e. e.g.

$$A \leftrightarrow B := (A \rightarrow B) \wedge (B \rightarrow A),\ \neg A := (A \rightarrow \perp),\ \exists x A := \neg \forall x \neg A,\ A \vee B := \neg(\neg A \wedge \neg B), \text{ etc..}$$

The underlying logic of **NF⁻** is the intuitionistic one. Furthermore, **NF⁻** contains all instances of the stratified (Quine's) comprehension axiom:

(**SC**) $\exists y \forall x (x \in y \leftrightarrow A)$, for each stratified formula A that does not contain y free.

1.2. Theories $\mathbf{NF^0}$ and $\mathbf{NF^1}$.

The theory $\mathbf{NF^0}$ is a conservative extension of **NF⁻** obtained by adding
(e) abstraction-terms (abbr.: ρ, τ, π) of the form $\{x \mid A\}$ where A is as in (**SC**) above.
Moreover, while substituting abstraction-terms for variables we simultaneously convert all subformulae according to the ß-reduction $\tau \in \{x \mid A\} \triangleright A[x/\tau]$. So in $\mathbf{NF^0}$ atomic formulae are expressions $\tau \in \{x \mid A\}$ where A contains atomic formulae of the rudimentary form $\rho \in y$.

The corresponding classical theory $\mathbf{NF^1}$ is specified accordingly.

1.3. Sequent calculi SEQ$\mathbf{NF^0}$ and SEQ$\mathbf{NF^1}$.

The (finite) intuitionistic sequent calculus SEQ$\mathbf{NF^0}$ standardly extends Gentzen's **LJ** (with cut) by the following symmetric abstraction rules.

(A⇒) $\dfrac{A[x/\rho], \tau \in \{x \mid A\}, \Gamma \Rightarrow H}{\rho \in \{x \mid A\}, \Gamma \Rightarrow H}$ (⇒A) $\dfrac{\Gamma \Rightarrow A[x/\rho]}{\Gamma \Rightarrow \rho \in \{x \mid A\}}$

The axioms of **SEQNF^0** are sequents of the following forms.

(F) $\perp,\Gamma \Rightarrow H$.

(I) $\rho\in\pi,\Gamma \Rightarrow \rho\in\pi$.

The corresponding classical (many-succedent) sequent calculus **SEQNF^1** based on Gentzen's **LK** is specified accordingly.

1.4. The theory **TS$^-$**.

Let **TS$^-$** be the appropriate intuitionistc variant of Specker's type theory [S] with shift isomorphisms. As compared to the language of **NF$^-$**, the language of **TS$^-$** includes:
(a) infitely many typed variables (abbr.: x^i, y^i, z^i for $i\in\mathbb{N}$),
(f) new function symbols $\sigma^k_m(-)$ for $k,m\in\mathbb{N}$ with the intended meaning $\sigma^k_m(-)$: $U^m\rightarrow U^k$.
The terms of **TS$^-$** (abbr.: ρ^i, τ^i, π^i or simply ρ, τ, π) are built up from variables by applying $\sigma^k_i(-)$ such that $\sigma^k_i(\tau^i)$ is of the type k for τ^i ranging over the terms of the type i.

Normal terms, i.e. terms in the normal form, arise by applying reductions

(R) $\sigma^k_m\circ\sigma^m_i(\tau^i)\triangleright\sigma^k_i(\tau^i)$ and $\sigma^k_k(\tau^k)\triangleright\tau^k$.

Atomic formulae of **TS$^-$** are expressions $\rho^i\in\pi^{i+1}$ for any normal terms ρ^i and π^{i+1} of the types exposed, from which arbitrary formulae are built up standardly. A formula is called solid if no σ^k_m occurs in it.

Now **TS$^-$** extends the intuitionistic predicate calculus by the following two aixoms.

(S) $x^m\in y^{m+1}\rightarrow\sigma^k_m(x^m)\in\sigma^{k+1}_{m+1}(y^{m+1})$.

(C) $\exists y^{i+1}\forall x^i(x^i\in y^{i+1}\leftrightarrow A)$, for each solid A that does not include y^{i+1} free.

1.5. The theory **TS^0**.

The correlated conservative extension **TS^0** is defined by analogy to **NF^0**, i.e. by adding in the above definition of the terms all abstraction-terms $\{x^i\,|\,A\}$ of the type i+1, where A is as in (C). Moreover, arbitrary terms may occur as parameters in the fomula A involved.

By substituting terms for variables we simultaneously convert all subformulae according to the ß-reduction $\tau^i\in\{x^i\,|\,A\}\triangleright A[x^i/\tau^i]$. Thus, in **TS^0**, normal terms are of the either form $\{x^i\,|\,A\}$ or $\sigma^k_{m+1}(\{x^m\,|\,A\})$, for $m\neq k$, such that all atomic formulae occurring in A are of the form $\rho^j\in y^{j+1}$.

1.6. The sequent calculus **SEQTS^0**.

The corresponding sequent calculus **SEQTS^0** extends **LJ** by the following symmetric rules.

$$(A\Rightarrow)\quad \frac{A[x^m/\sigma^m_k(\rho^k)],\rho^k\in\sigma^{k+1}_{m+1}(\{x^m\,|\,A\}),\Gamma \Rightarrow H}{\rho^k\in\sigma^{k+1}_{m+1}(\{x^m\,|\,A\}),\Gamma \Rightarrow H}$$

$$(\Rightarrow A)\quad \frac{\Gamma \Rightarrow A[x^m/\sigma^m_k(\rho^k)]}{\Gamma \Rightarrow \rho^k\in\sigma^{k+1}_{m+1}(\{x^m\,|\,A\})}$$

The axioms of **SEQTS^0** are sequents of the following forms.

(F) $\perp,\Gamma \Rightarrow H$.

(I) $\rho^m\in\pi^{m+1},\Gamma \Rightarrow \sigma^k_m(\rho^m)\in\sigma^{k+1}_{m+1}(\pi^{m+1})$.

1.7. The theory $\mathbf{TS^x}$.

The language of $\mathbf{TS^x}$ extends the typed terms of $\mathbf{TS^-}$ by:
(g) infinitely many untyped variables (abbr.: n, m, k) which are thought to range over $\mathbb{N}$,
(h) untyped number theoretical operations $0,+,\cdot$ as acting on $\mathbb{N}$.
(Typed terms are thought to range over the proper sets.) The corresponding untyped number-theoretical terms (abbr.: a, b, c) arise accordingly.

Formulae of $\mathbf{TS^x}$ standardly arise from the following atomic formulae:

$$a = b,\ \rho^{i} = \pi^{i} \text{ and } (a,\rho^{i}) \in \pi^{i+1}.$$

A formula is called solid if no σ^{k}_{m} occurs in it. Now $\mathbf{TS^x}$ extends the intuitionistic predicate calculus by Peano axioms (with induction) for untyped objets, and the following two aixoms.

(S) $(n,x^{m}) \in y^{m+1} \rightarrow (n,\sigma^{k}_{m}(x^{m})) \in \sigma^{k+1}_{m+1}(y^{m+1})$

(C) $\exists y^{i+1} \forall n \forall x^{i}((n,x^{i}) \in y^{i+1} \leftrightarrow A)$, for each solid A that does not contain y^{i+1} free.

2. Results

2.1. THEOREM. $\mathbf{NF^0}$ *proves* A *iff* $\Rightarrow A$ *is deducible in* SEQ$\mathbf{NF^0}$. $\mathbf{NF^0}$ *proves* $\neg A$ *iff* $A \Rightarrow \bot$ *is deducible in* SEQ$\mathbf{NF^0}$. $\mathbf{NF^1}$ *proves* A *iff* $\Rightarrow A$ *is deducible in* SEQ$\mathbf{NF^1}$. $\mathbf{NF^1}$ *proves* $\neg A$ *iff* $A \Rightarrow \bot$ *is deducible in* SEQ$\mathbf{NF^1}$.

2.2. THEOREM. $\mathbf{TS^0}$ *proves* A *iff* $\Rightarrow A$ *is deducible in* SEQ$\mathbf{TS^0}$. $\mathbf{TS^0}$ *proves* $\neg A$ *iff* $A \Rightarrow \bot$ *is deducible in* SEQ$\mathbf{TS^0}$.

Proof. These are obvious by the Gentzen's approach. □

2.3. THEOREM. *Strong normalizability in* SEQ$\mathbf{TS^0}$ *implies strong normalizability in* SEQ$\mathbf{NF^0}$. *Strong normalizability in* SEQ$\mathbf{NF^0}$ *implies strong normalizability in* SEQ$\mathbf{NF^1}$.

Proof Sketch. The first claim easily follows from the Specker's interpretation [S] of $\mathbf{NF^-}$ within $\mathbf{TS^-}$ by deleting all symbols σ^{k}_{i} from a given cut free deduction in SEQ$\mathbf{TS^0}$. The second claim follows by the $\neg\neg$-interpretation and the interpretation of $F_1,...,F_n \Rightarrow G_1,...,G_k$ in the intuitionistic form $\neg G_1,...,\neg G_k,F_1,...,F_n \Rightarrow \bot$. It should be observed that with any given chain of classical reductions is associated the corresponding chain of intuitionistic reductions. □

2.4. THEOREM. *The consistency of* $\mathbf{TS^x}$ *implies strong normalizability in* SEQ$\mathbf{TS^0}$.

The proof will be sketched in the next two sections.

2.5. THEOREM. *The consistency of* $\mathbf{Z}$ *extended by the existence of power-set hierarchy along* $\aleph_\omega$ *implies the consistency of* $\mathbf{TS^x}$.

Proof Sketch. This follows by the Jensen's approach [J]. Recall that Jensen described an interpretation of $\mathbf{TS^-}$ (in fact, with classical logic) within the initial segment of power-set hierarchy whose levels satisfy Ramsey's theorem. Since Ramsey's theorem holds in ω, this segment can be bounded by ω, and hence the theory of all finite power-set levels is strong enough. The analogous treatment of $\mathbf{TS^x}$ is possible within the initial ordinal-segnent $[0,\kappa)$ satisfying the following combinatorial property.

For any $m < \omega$, any function $F: \omega \times [\kappa]^{m} \rightarrow 2$, there is an infinite homogeneous set $H \subseteq \kappa$, i.e. such one that $(\forall n < \omega)(\forall \boldsymbol{\alpha},\boldsymbol{\beta} \in [H]^{m})(F(n,\boldsymbol{\alpha}) = F(n,\boldsymbol{\beta}))$.

This principle easily reduces to the familiar combinatorial statement

$$(\forall m < \omega)(\kappa \rightarrow (\omega)^{m}_{\omega})$$

which has a solution $\kappa = \aleph_\omega$ (assuming GCH, which is proof-theoretcally irrelevant). □

2.6. COROLLARY. *The consistency of* $\mathbf{Z}$ *extended by the existence of power-set hierarchy along* $\aleph_\omega$ *implies strong normalizability in both* SEQ$\mathbf{NF^0}$ *and* SEQ$\mathbf{NF^1}$. □

3. Proof of Theorem 2.4. Preliminary case

We prove strong normalization by the appropriate specification of the familiar approach of Girard [G] and Martin-Löf [M]. For some reasons, we prefer to argue in Kleene's terms of realizability, as familiar in the intuitionistic proof theory. The underlying idea is as follows. For the sake of brevity, we first consider pure type-theoretical formalisms $\mathbf{T^0}$, $\mathbf{T^x}$ and SEQ$\mathbf{T^0}$ obtained by deleting from $\mathbf{TS^0}$, $\mathbf{TS^x}$ and SEQ$\mathbf{TS^0}$ all shift-isomorphisms $\sigma_i^k(-)$. Thus all atomic formulae are of the form $\rho^i\in\pi^{i+1}$, or simply $\rho\in\pi$, where $\pi^{i+1}=\{x^i\,|\,A\}$.

3.1. Let D be any strongly normalizable deduction (s.n.d.) of the sequent $\Gamma,F_1,...,F_k\Rightarrow G$. D acts on s.n.d. as a k-ary function as follows. Let $D_1,...,D_k$ be s.n.d. such that every D_i is a deduction of the sequent $\Sigma_i\Rightarrow F_i$ for some Σ_i. Then let $D[D_1,...,D_k]$ denote the deduction of the correlated sequent $\Gamma,\Sigma_1,...,\Sigma_k\Rightarrow G$ that arises by successively applying cuts on $F_1,...,F_k$, whose left-hand premises are deduced by $D_1,...,D_k$, respectively. Now $D[D_1,...,D_k]$ is well-defined if it is s.n.d., which we abbreviate by $D[D_1,...,D_k]\downarrow$.

3.2. For any typed term τ of $\mathbf{T^0}$, let τ^* be some fixed term of $\mathbf{T^x}$ which has the same type and the same parameters as τ. With respect to this fixed assignment, the corresponding notion of realizability is defined as follows.

3.3. In $\mathbf{T^x}$, we define binary relation $D\Upsilon S$: "A s.n.d. D realizes a sequent S in SEQ$\mathbf{T^0}$" by transfinite recursion on $\omega\cdot\mathrm{lth}(S)+\mathrm{lth}(D_N)$, where D_N is the (uniquely determined by the Church-Rosser property) normal deduction of S to which D must eventually reduce.

3.3.1. Let $S=(F_1,...,F_k\Rightarrow G)$ where $k>0$. Set $D\Upsilon S$ iff D is a s.n.d. of the sequent $\Gamma,F_1,...,F_k\Rightarrow G$, for some Γ, such that $D\Upsilon(\Rightarrow G)$ holds, and for any s.n.d. $D_1,...,D_k$, if $D_i\Upsilon(\Rightarrow F_i)$ holds for all $0<i\leq k$, then $D[D_1,...,D_k]\downarrow$ and $D[D_1,...,D_k]\,\Upsilon(\Rightarrow G)$ holds.

3.3.2. Set $D\Upsilon(\Rightarrow G)$ iff D is a s.n.d. of the sequent $\Gamma\Rightarrow G$, for some Γ, such that one of the following holds.

3.3.2.1. $G=\perp$.

3.3.2.2. $G=(\rho\in\pi)$ and $(\ulcorner D\urcorner,\rho^*)\in\pi^*$, where $\ulcorner D\urcorner$ is the Gödelnumber of D.

3.3.2.3. $G=(A\wedge B)$ and the following holds. If D_N ends by some elimination-rule, and P_N is the immediate subdeduction in D_N of any premise with the succedent minor-predecessor G, then $P_N\Upsilon(\Rightarrow G)$ holds. Otherwise, if D_N ends by the (uniquely determined) introduction-rule, and P_N and Q_N are immediate subdeductions in D_N of premises with the succedent main-predecessors A and B, respectively, then both $P_N\Upsilon(\Rightarrow A)$ and $Q_N\Upsilon(\Rightarrow B)$ hold.

3.3.2.4. $G=(A\rightarrow B)$ and the following holds. If D_N ends by some elimination-rule, and P_N is the immediate subdeduction in D_N of any premise with the succedent minor-predecessor G, then $P_N\Upsilon(\Rightarrow G)$ holds. Otherwise, if D_N ends by the (uniquely determined) introduction-rule with the immediate subdeduction P_N, then $P_N\Upsilon(A\Rightarrow B)$ holds.

3.3.2.5. $G=(\forall x^iA)$ and the following holds. If D_N ends by some elimination-rule, and P_N is the immediate subdeduction in D_N of any premise with the succedent minor-predecessor G, then $P_N\Upsilon(\Rightarrow G)$ holds. Otherwise, if D_N ends by the (uniquely determined) introduction-rule with the immediate subdeduction P_N, then $\forall x^iP_N\Upsilon(\Rightarrow A)$ holds.

This completes the definition in $\mathbf{T^x}$ of the realizability relation $D\Upsilon S$ relative to the term-assignment $\tau\mapsto\tau^*$. It is readily seen that for any sequent S, the above definition results in the appropriate formula $n\Upsilon S$ of $\mathbf{T^x}$ whose parameters extend the ones of S by the new parameter n (as ranging over the Gödelnumbers of D).

In order to prove that all deductions in SEQ$\mathbf{T^0}$ are strongly normalizable, it will suffice to find a suitable assignment $\tau\mapsto\tau^*$ for which all deductions realize their concluding sequents. In fact, we prove the following soundness theorem.

3.4. THEOREM. *There is an assignment* $\tau\mapsto\tau^*$ *such that the following is provable in* $\mathbf{T^x}$. *For any deduction* D *of* $F_1,...,F_k\Rightarrow G$ *in* SEQ$\mathbf{T^0}$, D *realizes every subsequent of* $F_1,...,F_k\Rightarrow G$, *i.e.* $D\Upsilon(\Pi\Rightarrow G)$ *holds for any* $\Pi\subseteq F_1,...,F_k$.

Proof Sketch. Arguing in $\mathbf{T^x}$, by straightforward verification by induction on lth(D), the following condition 3.4.1 is sufficient for the required soundness of realizability.

Let ρ be any term of $\mathbf{T^0}$ of the type i, let π be any abstraction-term $\{x^i \mid A\}$ of $\mathbf{T^0}$ of the type i+1, and let C be the set of all n satisfying $(n,\rho^*)\in\{x^i \mid A\}^*$. Then

3.4.1. C is the minimal set of $n = \ulcorner D \urcorner$ such that for some Γ, D is a s.n.d. in $\mathbf{SEQT^0}$ of the sequent $\Gamma\Rightarrow\rho\in\pi$, and one of the following holds.

(a) D is the axiom $\rho\in\pi,\Sigma\Rightarrow\rho\in\pi$.

(b) D_N ends by some elimination-rule, and $\ulcorner P_N \urcorner\in C$ provided that P_N is the immediate subdeduction in D_N of any premise with the succedent minor-predecessor $\rho\in\pi$.

(c) D_N ends by the introduction-rule ($\Rightarrow$A), and $P_N \Upsilon (\Rightarrow A[x^i/\rho^*])$ holds for the (uniquely determined) immediate subdeduction P_N.

It is now readily seen from the above definition of the realizability predicate $n\Upsilon S$ that the assignment $\tau\mapsto\tau^*$ satisfying 3.4.1 is explicitly definable in $\mathbf{T^x}$ by recursion on compexity of τ by the comprehension axiom (**C**). Namely, we first set $\tau^* := \tau$ for each untyped term τ and for each variable $\tau = x^i$. Now for any term $\pi = \{x^i \mid A\}$ of $\mathbf{T^0}$, define π^* in $\mathbf{T^x}$ by (**C**) as the minimal set consisting of (n,x^i), where n codes a s.n.d. in $\mathbf{SEQT^0}$ of the sequent $\Gamma\Rightarrow\rho\in\pi$ for some Γ and ρ, such that either condition (a), (b) or (c)$^-$ holds. Here (a) and (b) are as above, whereas (c)$^-$ is obtained by replacing, in (c), $P_N \Upsilon (\Rightarrow A[x^i/\rho^*])$ by the corresponding relation $P_N \Upsilon (\Rightarrow A)$ that is defined with respect to the assignment $\tau\mapsto\tau^*$ on the parameters of A. □

3.5. COROLLARY. *The consistency of* $\mathbf{T^x}$ *implies strong normalizability in* $\mathbf{SEQT^0}$. □

4. Proof of Theorem 2.4

Let D be any s.n.d. in $\mathbf{SEQTS^0}$ of the sequent $\Gamma\Rightarrow\rho\in\pi$, where π is of the type k+1. Since by definition π (as well as ρ) is normal, it uniquely determines the corresponding pure abstract-term $\pi^\dagger = \{x^m \mid A\}$ such that $\pi = \sigma^{k+1}_{m+1}(\pi^\dagger)$. Moreover, $\pi = \pi^\dagger$ holds in the case m = k. It is readily seen that D uniquely determines the corresponding isomorphic s.n.d. $D^\dagger$ of the sequent $\Gamma\Rightarrow\sigma^m_k(\rho)\in\pi^\dagger$. Having this, we specify as follows the previous notion of realizability 3.1–3.3.

4.1. The applicability $D[D_1,...,D_k]$ on s.n.d. in $\mathbf{SEQTS^0}$ is defined analogously to 3.1.

4.2. As in 4.1, we fix some assignment $\tau\mapsto\tau^*$. For any typed term τ of $\mathbf{TS^0}$, τ^* is some term of $\mathbf{TS^x}$ which has the same type and the same parameters as τ.

4.3. Arguing in $\mathbf{TS^x}$, we define binary relation $D\Upsilon S$: "A s.n.d. D realizes a sequent S in $\mathbf{SEQTS^0}$" by transfinite recursion on $\omega\cdot$lth(S)+lth(D_N) where, as above, D_N is the normal form of D. The definition is analogous to the previous definition 3.3, except for the following modified treatment of 3.3.2.2.

4.3.2.2. $G = (\rho\in\pi)$ and $(\ulcorner D^\dagger \urcorner,\rho^*)\in\pi^*$, where $\ulcorner D^\dagger \urcorner$ is the Gödelnumber of $D^\dagger$ (see above).

For each sequent S, the resulting definition provides us with the appropriate formula $n\Upsilon S$ of $\mathbf{TS^x}$ whose parameters extend the ones of S by the new parameter n (as ranging over the Gödelnumbers of D), which formalizes the realizability relation $D\Upsilon S$ for $\mathbf{SEQTS^0}$ relative to the term-assignment $\tau\mapsto\tau^*$.

4.4. THEOREM. *There is an assignment* $\tau\mapsto\tau^*$ *such that the following is provable in* $\mathbf{TS^x}$. *For any deduction* D *of* $F_1,...,F_k\Rightarrow G$ *in* $\mathbf{SEQTS^0}$, D *realizes every subsequent of* $F_1,...,F_k\Rightarrow G$, *i.e.* $D\Upsilon(\Pi\Rightarrow G)$ *holds for any* $\Pi \subseteq F_1,...,F_k$.

Proof Sketch. As compared to the proof of 3.4, there is one new point. Namely, in order to prove that every axiom (**I**) is realizable, and in particular, in order to establish

$$\{\rho^m\in\pi^{m+1},\Gamma\Rightarrow\sigma^k_m(\rho^m)\in\sigma^{k+1}_{m+1}(\pi^{m+1})\}\Upsilon(\rho^m\in\pi^{m+1}\Rightarrow\sigma^k_m(\rho^m)\in\sigma^{k+1}_{m+1}(\pi^{m+1})),$$

we must be sure that D realizes $\Rightarrow \rho^{m}\in\pi^{m+1}$ iff it realizes $\Rightarrow \sigma^{k}_{m}(\rho^{m})\in\sigma^{k+1}_{m+1}(\pi^{m+1})$. By the clause 4.3.2.2, this yields the following assertion

(d) $$(\ulcorner D^{\dagger}\urcorner,\rho^{*})\in\pi^{*} \leftrightarrow (\ulcorner D^{\dagger}\urcorner,\sigma^{k}_{m}(\rho)^{*})\in\sigma^{k+1}_{m+1}(\pi)^{*}.$$

Now observe that (d) is derivable in $\mathbf{TS}^{\mathbf{x}}$ from the crucial axiom (S), provided that the following condition holds for every term τ of the type i

(e) $$\sigma^{k}_{i}(\tau)^{*} = \sigma^{k}_{i}(\tau^{*}).$$

(Clearly, the realizability of (I) from $\mathrm{SEQ}\mathbf{T}^{\mathbf{0}}$ is established by the identity of $\mathbf{T}^{\mathbf{x}}$.)

By this observation, the following condition 4.4.1 proves the required soundness of realizability provided that (e) is satisfied.

Let ρ be any term of $\mathbf{TS}^{\mathbf{0}}$ of the type k, let π be any term $\pi = \sigma^{k+1}_{m+1}(\pi^{\dagger})$ of $\mathbf{TS}^{\mathbf{0}}$ of the type k+1, where $\pi^{\dagger} = \{x^{m} \mid A\}$, and let C be the set of all n satisfying $(n,\rho^{*})\in\pi^{*}$. Then

4.4.1. C is the minimal set of $n = \ulcorner D^{\dagger}\urcorner$ such that for some Γ, D is a s.n.d. in $\mathrm{SEQ}\mathbf{TS}^{\mathbf{0}}$ of the sequent $\Gamma\Rightarrow\rho\in\pi$, and one of the following holds.

(a) D is the axiom (I) of the form $\sigma^{p}_{k}(\rho)\in\sigma^{p+1}_{k+1}(\pi),\Sigma\Rightarrow\rho\in\pi$.

(b) D_N ends by some elimination-rule, and $\ulcorner P_N{}^{\dagger}\urcorner\in C$ provided that P_N is the immediate subdeduction in D_N of any premise with the succedent minor-predecessor $\rho\in\pi$.

(c) D_N ends by the introduction-rule ($\Rightarrow$A), and $P_N\Upsilon(\Rightarrow A[x^{m}/\sigma^{m}_{k}(\rho^{*})])$ holds for the (uniquely determined) immediate subdeduction P_N.

The assignment $\tau\mapsto\tau^{*}$ satisfying (e) and 4.4.1 is explicitly definable in $\mathbf{TS}^{\mathbf{x}}$ by recursion on compexity of τ by the comprehension axiom (C). As in the preliminary case 3.4, we first set $\tau^{*}:=\tau$ for each untyped term τ and for each variable $\tau = x^{i}$. For any term $\tau = \sigma^{k+1}_{m+1}(\tau^{\dagger})$ where $\tau^{\dagger} = \{x^{m} \mid A\}$, m≠k, we set $\tau^{*} := \sigma^{k+1}_{m+1}((\tau^{\dagger})^{*})$. Now for any abstraction-term $\pi = \{x^{i} \mid A\}$ of $\mathbf{TS}^{\mathbf{0}}$, we define π^{*} in $\mathbf{TS}^{\mathbf{x}}$ by (C) as the minimal set consisting of (n,x^{i}), where n codes a s.n.d. in $\mathrm{SEQ}\mathbf{TS}^{\mathbf{0}}$ of the sequent $\Sigma\Rightarrow\rho\in\pi$ for some Σ and ρ, such that either condition (a), (b) or (c)$^{-}$ holds. Here (a) and (b) are as above, whereas (c)$^{-}$ is obtained by replacing, in (c), $P_N\Upsilon(\Rightarrow A[x^{m}/\sigma^{m}_{k}(\rho^{*})])$ by the corresponding relation $P_N\Upsilon(\Rightarrow A)$ that is defined with respect to the assignment $\tau\mapsto\tau^{*}$ on parameters of A.

This completes the proof of Theorem 4.4, and thereby completes the proof of Theorem 2.4. □

4.5. REMARK. In this framework, strong normalizability of a given purely comprehensive set theory reduces to the consistency of a theory that extends Peano Arithmetic by the appropriate extended comprehension axioms. That we used typed language is unimportant – it is possible to formalize our proof of strong normalizability for $\mathrm{SEQ}\mathbf{NF}^{\mathbf{0}}$ in the analogous stratified extension of $\mathbf{NF}^{\mathbf{0}}$, $\mathbf{NF}^{\mathbf{x}}$, based upon the untyped language of $(n,x)\in y$ instead of of $x\in y$. The crucial observation is that our assignment $\tau\mapsto\tau^{*}$ preserves the stratification. (The only reason for the choice of the typed theory $\mathbf{TS}^{\mathbf{x}}$ is the proof theoretical estimate given in Theorem 2.5.) On the other hand, this approach does not preserve Zermelo's comprehension axiom $\exists y\forall x(x\in y\leftrightarrow x\in\tau\wedge A(x))$, since by definition the required set π^{*} should consist of $(n,x)\in y$ for which only $(m,x)\in\tau^{*}$, for some m, but not $(n,x)\in\tau^{*}$, is required. Indeed, strong normalizability in known to fail in very rudimentary fragments of $\mathbf{Z}$.

[G]: Girard, J.Y. *Une extension de l'interpretation de Gödel a l'analyse, ...* Proc. Second Scand. Logic Symp., 63–92 (North-Holland, Amsterdam, 1971)

[J]: Jensen, R.B. *On the consistency of a slight (?) modification of Quine's New Foundations.* Synthese 19, 250–263

[M]: Martin-Löf, P. *Hauptsatz for intuitionistic simple type theory.* Logic, Methodology and Philosophy of Science IV, 279–290 (North-Holland, Amsterdam, 1973)

[S]: Specker, E. *Typical ambiguity.* Logic, Methodology and Philosophy of Science, 116–124 (Stanford Univ. Press, 1960)

L.GORDEEV: *On consistency and cut-elimination.* (OBERWOLFACH'88:Exhaustive Abstract)

I write SEQ to mean the weak rule free extension by two (symmetrical) abstraction rules

$$\frac{s\in\{x\mid\beta(x)\},\beta(s),\Gamma\vdash\varphi}{s\in\{x\mid\beta(x)\},\Gamma\vdash\varphi}\qquad\qquad\frac{\Gamma,\varphi\vdash\beta(s)}{\Gamma,\varphi\vdash s\in\{x\mid\beta(x)\}}$$

of purely "=","∈"-variant of Gentzen's intuitionistic sequent calculus. In this language, abstraction terms are built up from variables and constants $c_0,\dots,c_m$ (possibly sorted) as $\{x\mid\alpha_i(x)\}$, $i=0,\dots,n$, and are closed under substitutions, where $\alpha_1,\dots,\alpha_n$ are some fixed stratified (in Quine's sense) formulae with one distinguished parameter x, possibly including other parameters from a fixed list **z**. Moreover, I assume that all abstraction terms under consideration are in the normal form, i.e. are irreducible under the ordinary abstraction term reduction $s\in\{x\mid\beta(x)\}\triangleright\beta(s)$. Proper abstraction terms (PAT) are abstraction terms which are not variables. A subformula-occurrence φ in a PAT $a=\{x\mid\beta(x)\}$ is such an occurrence of φ in β that does not occur in any PAT occurring in β. The Minc-style elementary (cut-)reduction operator R_0 is defined by an obvious abstraction rule specification of the standard definition (cf. e.g. APAL **38**(1) p.49). Strong normalizability (SNorm) expresses that for any derivation d in SEQ, every sequence of R_0-reductions in d terminates. I describe the appropriate SEQ-translation in the terms of realizability of the familiar (for Natural Deduction, say) Girard-style strong normalizability proof in pure intuitionistic Type Theory (^{i}TT). For any type-level n, the modified proof of the corresponding ($\leq n$)-restricted statement SNorm(SEQ(^iTT_n)) easily reduces to certain simple axioms provable in $^iTT_{n+1}$ extended by the axioms formalizing elementary number theory at level 0. Furthermore, strong normalizability of the corresponding classical "many-succedent" sequent calculus SEQ(TT_n) follows (provably in the 2^{nd} order conservative extenison of HA) directly from SNorm(SEQ(^iTT_n)) by Gödel's ¬¬-interpretation together with the familiar intuitionistic interpretation of classical sequents. Using Jensen's treatment of Quine's New Foundation minus extensionality (NF-Ext), for both intuitionistic and classical sequent calculi I reduce by similar techniques the statement SNorm(SEQ(NF-Ext)) to a combinatorial principle provable well within ZF. However, a similar approach in the presence of the ω-rule for abstraction terms would certainly assume (at least) the truth of the Domain Completion Axiom (DCA) of the form $(\forall y)\mathrm{VPAT}(y)$, where VPAT(y) (i.e. in words: "y *is the value of some* PAT") is an abbreviation of $(\bigvee_{i\leq m})(y=c_i)\vee(\exists z)(\bigvee_{i\leq n})(\forall x)(x\in y\Leftrightarrow\alpha_i(x,z))$. Let SEP be the corresponding separation axioms $(\bigwedge_{i\leq n})(\exists y)(\forall x)(x\in y\Leftrightarrow\alpha_i(x))$. I ask under which assumptions SEP+DCA is consistent.

1. I conjecture that the following condition (C) implies the consistency of SEP+DCA.

(C): "*Let* a *be* PAT, $(Qx)\varphi(x)$ *be any subformula-occurrence in* a ($Q=\forall,\exists$). *Let* a′, a″ *be obtained by substituting* $(Qx)(\varphi(x)\wedge\mathrm{VPAT}(x))$, *resp.* $(Qx)(\varphi(x)\vee\neg\mathrm{VPAT}(x))$ *for* $(Qx)\varphi(x)$ *in* a. *Then* a′, a″ *are both* PAT *modulo provability in classical Predicate Calculus.*"

Fact. In the particular case of 1^{st} order unsorted purely "∈"-language without constants, this logical conjecture implies Con(NF).

1.1 Let me explain the latter underlined expression. First of all, NF-Ext appears very natural and remarkably simple in formalization (as compared to TT), while still satisfying strong normalizability - contrary to more familiar formal set theoretical systems (s.a. Zermelo's Z) based on nonstratified separation. Secondly, the axiom of extensionality, although having (so far) no plausible explanation in purely logical terms, classically proves the axiom of infinity (Specker) and thus dramatically increases both expressive and proof theoretical strength (Jensen) of NF-Ext.

Now turning to the above conjecture (let me call it C1), I say that a given list $\langle c_0,\dots,c_m;\alpha_1,\dots,\alpha_n\rangle$ of constants and stratified formulae with parameters x,z is Herbrand-consistent (H-consistent) iff SEP+DCA is consistent (in the ordinary sense of being contradiction free). The notion of H-consistency seems natural and important because by asserting only SEP we wish to "construct" the universe consisting of PAT c_i, $\{x\mid\alpha_j(x)\}$ - this will resemble the familiar number theoretical formalism where objects are thought to be generated from 0 by the successor operation $\lambda x.S(x)$ so that eventually we get $(\forall x)(x=0\vee(\exists z)(x=S(z)))$. It seems something must go wrong if the separation axioms SEP suddenly become incompatible with DCA. Let me now briefly illustrate the notion of H-(in)consistency.

[see ⟹]

For arbitrary constants $c_0,\ldots,c_m$, any list of quantifier free formulae is H-consistent, and obviously satisfies (C). Moreover, so is any complete list of formulae, i.e. such that every stratified separation term is PAT. That complete lists do exist follows from the finite axiomatizability of NF-Ext. Loosely speaking, $\langle c_0,\ldots,c_m;\alpha_0,\ldots,\alpha_n\rangle$ certainly is H-consistent if it includes "reversable" formulae, i.e. if something like $(\forall x)(x\in y\Leftrightarrow x\in\bigcup\{y\})$ can be expressed in the terms of $(\alpha_0,\ldots,\alpha_n)$-separations. On the other hand, the list $\langle c;(\forall Y)\neg(x\in Y)\rangle$ is H-inconsistent already in 2^{nd} order logic. For take $A:=\{x\,|\,(\forall Y)\neg(x\in Y)\}$, then DCA expresses $(\forall Y)(\forall x)(x\in Y\Leftrightarrow x\in A)$, hence $c\in A \Leftrightarrow c\in\{x\,|\,(\forall Y)\neg(x\in Y)\} \Leftrightarrow (\forall Y)\neg(c\in Y) \Leftrightarrow_{(DCA)} \neg(c\in A)$. Clearly, $\langle\emptyset;(\forall y)\neg(x\in y)\rangle$ is H-inconsistent also in 1^{st} logic via $a\in a \Leftrightarrow a\in\{x\,|\,(\forall y)\neg(x\in y)\} \Leftrightarrow (\forall y)\neg(a\in y) \Leftrightarrow_{(DCA)} \neg(a\in a)$ for $a:=\{x\,|\,(\forall y)\neg(x\in y)\}$. Note that these counter-examples do not satisfy (C). For otherwise (in the latter case, say) in 1^{st} order classical logic we would derive $(\forall y)\neg(x\in y) \Leftrightarrow (\forall y)(\neg(x\in y)\wedge(\forall u)(u\in y\Leftrightarrow(\forall v)\neg(u\in v))) \Leftrightarrow (\forall y)(\neg(x\in y)\vee\neg(\forall u)(u\in y\Leftrightarrow(\forall v)\neg(u\in v)))$. However, the first equivalence is easily refutable. For example, take the two-element structure $\{0,1\}$ with the only relation $1\in 1$, then $(\forall y)\neg(0\in y)$ holds but $1\in 1\Leftrightarrow(\forall v)\neg(1\in v)$ obviously fails.

Thus the question concerning nontrivial sufficient conditions to H-consistency appears quite natural from the ordinary "algebraic-logical" viewpoint. The counter-example above may suggest that sometimes H-inconsistency fails simply because that list of PAT is too small, i.e. is not closed under certain "logically obvious" operations (e.g. the negation). In this context observe that the condition (C) is minimal w.r.t. the provability in classical Predicate Calculus, since the subformula $\varphi(x)$ is equivalent to both $\varphi(x)\wedge\mathrm{VPAT}(x))$ and $\varphi(x)\vee\neg\mathrm{VPAT}(x))$ under the desired assumption DCA. On the other hand, the real structure of H-consistent theories is far more complicated, and certainly it cannot be explaind in the (C)-terms. For example, one can extend H-inconsistent list $\langle\emptyset;(\forall y)\neg(x\in y)\rangle$ (see above) by adding the negation $(\exists y)(x\in y)$. The extended list $\langle\emptyset;(\forall y)\neg(x\in y),(\exists y)(x\in y)\rangle$ is H-consistent since the corresponding theory SEP+DCA has the two-element model $\{0,1\}$ with $0\in 1\in 1$, but it does not satisfy the condition (C).

Finally, observe that the conjecture (C1), i.e. (C)$\Rightarrow$Con(SEP+DCA), is "logically neutral" in the sense that its intuitionistic variant (^{i}C1): (C)$\Rightarrow$Con(^{i}(SEP+DCA)) is of the same proof theoretical strength. For in the intuitionistic formalism, Gödel's $\neg\neg$-interpretation yields classical model of SEP+DCA provided that each formula α_i does not include $\vee,\exists$. The latter is easily achievable since the condition (C) is invariant w.r.t. classically valid transformations of subformulae in α_i. In other words, from the intuitionistic viewpoint it suffices to apply (C) to negative formulae α_i for which intuitionistic and classical provability are essentially equivalent. Thus (^{i}C1) implies the consistency of NF (with classical logic and Extensionality) just as well as (C1) (see above). This refers to a more complicated situation concerning the latter theory, because it is unknown whether ^{i}NF is proof theoretically as strong as NF, or even whether the infinity axiom is provable in ^{i}NF. For the same reasons, the conjecture (C1) (if it should be true) would present the theory of (possibly) infinite sets in somewhat "logical" way, as compared to the familiar set theories which need a special axiom postulating the existence of an infinite set (otherwise, these set theories would proof theoretically collapse into the theory of numbers).

2. My <u>second conjecture</u> (C2) <u>is (C)$\Rightarrow$SNorm(SEQ$^\omega$(^{i}SEP))</u> i.e. in words: "*The intuitionistic sequent calculus corresponding to* SEP, *extended by two symmetrical ω-rules for abstraction terms instead of Gentzen's finite $\forall$-introduction and $\exists$-elimination, respectively, is strongly normalizable*". As usual in Proof Theory, already the corresponding primitive recursive conjecture (PRC2) seems to be sufficiently strong, where in (PRC2) I consider only primitive resursive infinite derivations and primitive recursive sequences of reductions. Note that by standard arguments, (PRC2) implies (C)$\Rightarrow$Con(^{i}(SEP+DCA+PATIND)), and hence by Gödel's $\neg\neg$-interpretation, (PRC2) also implies (C)$\Rightarrow$Con(SEP+DCA+PATIND). Here PATIND stands for the corresponding PAT-induction axiom expressing that for any formula $\varphi(x)$, $(\forall x)\varphi(x)$ holds provided that φ holds for each constant c_i and is hereditarily closed under each operation $\{x\,|\,\alpha_i(x)\}$. By the previous arguments, (C2) implies (C)$\Rightarrow$SNorm(SEQ$^\omega$(SEP)). Moreover, by similar arguments which show, loosely speaking, that strong normalizability is closed under syntactical interpretations, (C2) seems to imply (C)$\Rightarrow$SNorm(NF) for both intuitionistic and classical variants of NF with Extensionality w.r.t. the "naturally" defined notion of reduction in the presence of the corresponding (symmetrical) extensionality rules.

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