A Internet appendix

LRR model anatomy

This appendix provides all derivations needed for a complete simulation of the long-run risk (LRR) model by Bansal and Yaron (2004). The macroeconomic environment of the LRR model is determined by a four-equation macro VAR:

$$\begin{aligned}
&\text{log consumption growth} && g_{t+1} = \mu_c + x_t + \sigma_t \eta_{t+1} && \text{(A-1)}\\
&\text{latent growth component} && x_{t+1} = \rho x_t + \varphi_e \sigma_t e_{t+1} && \text{(A-2)}\\
&\text{log dividend growth} && g_{d,t+1} = \mu_d + \phi x_t + \varphi_d \sigma_t u_{t+1} && \text{(A-3)}\\
&\text{latent stochastic variance} && \sigma^2_{t+1} = \sigma^2 + \nu_1(\sigma^2_t - \sigma^2) + \sigma_w w_{t+1} && \text{(A-4)}\\
&\text{with} && \eta_t, e_t, w_t, u_t \overset{i.i.d.}{\sim} \mathrm{N}(0,1). &&
\end{aligned}$$

Following Campbell and Shiller (1988), the log return to the latent aggregate wealth portfolio $r_{a,t}$ and the log return to the market portfolio $r_{m,t}$ are modeled as:

$$r_{a,t+1} = \kappa_0 + \kappa_1 z_{t+1} - z_t + g_{t+1} \tag{A-5}$$

$$r_{m,t+1} = \kappa_{0,m} + \kappa_{1,m} z_{m,t+1} - z_{m,t} + g_{d,t+1} \tag{A-6}$$

$$\text{with} \qquad z_t = A_0 + A_1 x_t + A_2 \sigma^2_t \tag{A-7}$$

$$z_{m,t} = A_{0,m} + A_{1,m} x_t + A_{2,m} \sigma^2_t, \tag{A-8}$$

where z_t denotes the latent log price-consumption ratio, and $z_{m,t}$ denotes the log price-dividend ratio and $\kappa_1 = \frac{\exp(\bar{z})}{1+\exp(\bar{z})}$ and $\kappa_0 = \ln(1 + \exp(\bar{z})) - \kappa_1 \bar{z}$. Analogous expressions hold for $\kappa_{1,m}$ and $\kappa_{0,m}$.

The Epstein and Zin (1989) utility function implies the following stochastic discount factor (SDF) that prices the assets in the economy:

$$M_{t+1} = \delta^\theta G_{t+1}^{-\frac{\theta}{\psi}} R_{a,t+1}^{-(1-\theta)}. \tag{A-9}$$

A.1 Linear approximations

To model the dependence between log returns and the log price-dividend ratio, Bansal and Yaron (2004) refer to a linear approximation suggested by Campbell and Shiller (1988). The linear relationship between the log return h_t and the log dividend-price ratio δ_t suggested by Campbell and Shiller (1988) can be derived as follows:

$$\begin{aligned} h_t &= \ln(P_{t+1} + D_t) - \ln(P_t) \\ &= \ln(P_t + D_{t-1}) + \Delta \ln(P_{t+1} + D_t) - \ln(P_t). \end{aligned}$$

We use a first-order Taylor series expansion for $\Delta \ln(P_{t+1} + D_t)$ at $P_{t+1} = P_t$ and $D_t = D_{t-1}$:

$$\begin{aligned} \Delta \ln(P_{t+1} + D_t) = \ln(P_{t+1} + D_t) - \ln(P_t + D_{t-1}) &= \ln\left(\frac{P_{t+1} + D_t}{P_t + D_{t-1}}\right) \\ &\approx \ln(1) + \frac{1}{P_t + D_{t-1}}\left[P_{t+1} + D_t - P_t - D_{t-1}\right] \\ &= \frac{P_{t+1} - P_t}{P_t + D_{t-1}} + \frac{D_t - D_{t-1}}{P_t + D_{t-1}}. \end{aligned}$$

Assuming the price is a constant fraction ρ of the price including the dividends, $P_t \approx \rho(P_t + D_{t-1})$, and hence, $D_{t-1} \approx (1-\rho)(P_t + D_{t-1})$, we can approximate:

$$\begin{aligned} \Delta \ln(P_{t+1} + D_t) &\approx \rho \frac{P_{t+1} - P_t}{P_t} + (1-\rho)\frac{D_t - D_{t-1}}{D_{t-1}} \\ &\approx \rho \Delta \ln(P_{t+1}) + (1-\rho)\Delta \ln(D_t). \end{aligned}$$

Inserting this results into the expression for h_t yields:

$$
\begin{aligned}
h_t &\approx \ln\left(P_t + D_{t-1}\right) + \rho\Delta\ln(P_{t+1}) + (1-\rho)\Delta\ln(D_t) - \ln(P_t) \\
&= \ln\left(P_t + D_{t-1}\right) + \rho\left(p_{t+1} - p_t\right) + (1-\rho)\left(d_t - d_{t+1}\right) - p_t \\
&= \ln\left(P_t + D_{t-1}\right) + \rho p_{t+1} + (1-\rho)d_t - (1-\rho)(d_{t-1} - p_t) - 2p_t \\
&= \ln\left(\frac{P_t + D_{t-1}}{P_t}\right) - (1-\rho)\delta_t + \rho p_{t+1} + (1-\rho)d_t - p_t \\
&\approx -\ln(\rho) - (1-\rho)\delta_t + \rho p_{t+1} + (1-\rho)d_t - p_t \\
&= k + \rho p_{t+1} + (1-\rho)d_t - p_t. \qquad \text{(A-10)}
\end{aligned}
$$

Note that Campbell and Shiller (1988) model a log dividend-price ratio δ_t, whereas the LRR model refers to the log price-dividend ratio $z_{m,t}$. Translating this result into the notation used by Bansal and Yaron (2004) yields:

$$
\begin{aligned}
r_{m,t} &= -\ln(\rho) - (1-\rho)(d_{t-1} - p_t) + \rho p_{t+1} + (1-\rho)d_t - p_t \\
&= -\ln(\rho) + (1-\rho)(p_t - d_{t-1}) + \rho(p_{t+1} - d_t) - (p_t - d_{t-1}) + d_t - d_{t-1} \\
&= \kappa_{0,m} + \kappa_{1,m} z_{m,t} - z_{m,t-1} + g_{d,t}, \qquad \text{(A-11)}
\end{aligned}
$$

where $\kappa_{0,m}$ and $\kappa_{1,m}$ are given by:

$$
\begin{aligned}
\kappa_{0,m} &= -\ln(\rho) + (1-\rho)z_{m,t-1} \\
\kappa_{1,m} &= \rho.
\end{aligned}
$$

We can rewrite $\kappa_{0,m}$ and $\kappa_{1,m}$ as follows:

$$\kappa_{1,m} \approx \frac{P_t}{P_t + D_{t-1}} = \frac{1}{\frac{P_t + D_{t-1}}{P_t}} = \frac{1}{1 + \frac{1}{\exp(z_{m,t})}}$$
$$= \frac{\exp(z_{m,t})}{1 + \exp(z_{m,t})}.$$

Because ρ and thus $\kappa_{1,m}$ should be a constant ratio, we use a time average to obtain a constant value:

$$\kappa_{1,m} \approx \frac{\exp(\bar{z}_m)}{1 + \exp(\bar{z}_m)}. \tag{A-12}$$

For $\kappa_{0,m}$ to be a constant, we also use a time average to obtain a constant value:

$$\kappa_{0,m} \approx -\ln(\kappa_{1,m}) + (1 - \kappa_{1,m})\bar{z}_m = \ln\left(\frac{1 + \exp(\bar{z}_m)}{\exp(\bar{z}_m}\right) + \bar{z}_m - \kappa_{1,m}\bar{z}_m$$
$$= \ln\left(1 + \exp(\bar{z}_m)\right) - \kappa_{1,m}\bar{z}_m. \tag{A-13}$$

A.2 Derivation of the coefficients A_0, A_1, and A_2

To find the expressions for the coefficients A_0, A_1, and A_2 in Equation (A-7), we use the basic asset pricing equation with the SDF from Equation (A-9):

$$\mathbb{E}_t\left[\delta^{\theta} G_{t+1}^{-\frac{\theta}{\psi}} R_{a,t+1}^{-(1-\theta)} R_{i,t+1}\right] = 1.$$

Taking the logarithm of Equation (A-9) yields:

$$m_{t+1} = \ln(M_{t+1}) = \theta \ln(\delta) - \frac{\theta}{\psi} g_{t+1} + (\theta - 1) r_{a,t+1},$$

$$\text{where} \quad g_{t+1} = \ln(G_{t+1}) \quad \text{and} \quad r_{a,t+1} = \ln(R_{a,t+1}).$$

It follows that

$$\begin{aligned} 1 &= \mathbb{E}_t\left[\exp(\ln(M_{t+1}) + r_{i,t+1})\right] \\ &= \mathbb{E}_t\left[\exp(\theta \ln(\delta) - \frac{\theta}{\psi} g_{t+1} + (\theta - 1) r_{a,t+1} + r_{i,t+1})\right]. \end{aligned} \tag{A-14}$$

The model must price any return, so the Euler equation also holds for $r_{i,t+1} = r_{a,t+1}$:

$$\begin{aligned} 1 &= \mathbb{E}_t\left[\exp\left(\theta \ln(\delta) - \frac{\theta}{\psi} g_{t+1} + \theta r_{a,t+1}\right)\right] \\ &= \exp\left(\mathbb{E}_t\left[m_{t+1} + r_{a,t+1}\right] + \frac{1}{2}\text{Var}_t\left[m_{t+1} + r_{a,t+1}\right]\right) \\ 0 &= \mathbb{E}_t\left[m_{t+1} + r_{a,t+1}\right] + \frac{1}{2}\text{Var}_t\left[m_{t+1} + r_{a,t+1}\right]. \end{aligned}$$

Inserting the linear approximation for $r_{a,t+1}$, we obtain:

$$\begin{aligned} 0 = \theta \ln \delta &- \frac{\theta}{\psi}\mathbb{E}_t(g_{t+1}) \\ &+ \theta\left[\kappa_0 + \kappa_1 A_0 + \kappa_1 A_1 \mathbb{E}_t(x_{t+1}) + \kappa_1 A_2 \mathbb{E}_t(\sigma_{t+1}^2) - A_0 - A_1 x_t - A_2 \sigma_t^2 + \mathbb{E}_t(g_{t+1})\right] \\ &+ \frac{1}{2}\left[\left(\theta - \frac{\theta}{\psi}\right)^2 \text{Var}_t(g_{t+1}) + \theta^2\left(\kappa_1^2 A_1^2 \text{Var}_t(x_{t+1}) + \kappa_1^2 A_2^2 \text{Var}_t(\sigma_{t+1}^2)\right)\right] \end{aligned}$$

because $\text{Cov}_t(g_{t+1}, x_{t+1}) = 0$, $\text{Cov}_t(g_{t+1}, \sigma_{t+1}^2) = 0$ and $\text{Cov}_t(x_{t+1}, \sigma_{t+1}^2) = 0$.

It follows that:

$$\begin{aligned}0 = \theta \ln \delta - \frac{\theta}{\psi}(\mu_c + x_t) + \theta \left[\kappa_0 + \kappa_1 A_0 + \kappa_1 A_1 \rho x_t + \kappa_1 A_2 (\sigma^2 + \nu_1(\sigma_t^2 - \sigma^2)) - A_0\right. \\ \left. - A_1 x_t - A_2 \sigma_t^2 + \mu_c + x_t\right] + \frac{1}{2}\left[\left(\theta - \frac{\theta}{\psi}\right)^2 \sigma_t^2 + \theta^2 \left(\kappa_1^2 A_1^2 \varphi_e^2 \sigma_t^2 + \kappa_1^2 A_2^2 \sigma_w^2\right)\right].\end{aligned} \tag{A-15}$$

Equation (A-15) must hold for all values of x_t, which means that all terms involving x_t must cancel out:

$$\begin{aligned}-\frac{\theta}{\psi} x_t + \theta \kappa_1 A_1 \rho x_t - \theta A_1 x_t + \theta x_t &\overset{!}{=} 0 \\ -\frac{\theta}{\psi} x_t + \theta \left[\kappa_1 A_1 \rho x_t - A_1 x_t + x_t\right] &= 0.\end{aligned} \tag{A-16}$$

Equation (A-15) also has to hold for all values of σ_t^2:

$$\begin{aligned}\theta \kappa_1 A_2 \nu_1 \sigma_t^2 - \theta A_2 \sigma_t^2 + \frac{1}{2}\left[\left(\theta - \frac{\theta}{\psi}\right)^2 \sigma_t^2 + \theta^2 A_1^2 \kappa_1^2 \varphi_e^2 \sigma_t^2\right] &\overset{!}{=} 0 \\ \left[\theta\left(\kappa_1 \nu_1 A_2 - A_2\right) + \frac{1}{2}\left(\theta - \frac{\theta}{\psi}\right)^2 + \frac{1}{2}(\theta A_1 \kappa_1 \varphi_e)^2\right] \sigma_t^2 &= 0.\end{aligned} \tag{A-17}$$

Equation (A-16) leads to the expression for the parameter A_1:

$$\begin{aligned}-\frac{\theta}{\psi} + \theta \left[\kappa_1 A_1 \rho - A_1 + 1\right] &= 0 \\ A_1 = \frac{(\frac{\theta}{\psi} - \theta)}{\theta \kappa_1 \rho - \theta} &= \frac{1 - \frac{1}{\psi}}{1 - \kappa_1 \rho}.\end{aligned} \tag{A-18}$$

Equation (A-17) leads to the expression for the parameter A_2:

$$\begin{aligned}\left[\theta\left(\kappa_1 \nu_1 - 1\right) A_2 + \frac{1}{2}\left(\theta - \frac{\theta}{\psi}\right)^2 + \frac{1}{2}(\theta A_1 \kappa_1 \varphi_e)^2\right] \sigma_t^2 &= 0 \\ -\frac{1}{2}\left[\left(\theta - \frac{\theta}{\psi}\right)^2 + (\theta A_1 \kappa_1 \varphi_e)^2\right] &= \theta(\kappa_1 \nu_1 - 1) A_2\end{aligned}$$

$$A_2 = \frac{1}{2} \frac{\left(\theta - \frac{\theta}{\psi}\right)^2 + (\theta A_1 \kappa_1 \varphi_e)^2}{\theta[1 - \kappa_1 \nu_1]}. \tag{A-19}$$

The constant can be obtained by setting the sum of all x_t and σ_t^2 terms in Equation (A-15) to zero:

$$\begin{aligned}
0 &= \theta \ln \delta - \frac{\theta}{\psi} \mu_c + \theta \left[\kappa_0 + \kappa_1 A_0 + \kappa_1 A_2 (1 - \nu_1) \sigma^2 - A_0 + \mu_c \right] + \frac{1}{2} \theta^2 \left(\kappa_1^2 A_2^2 \sigma_w^2 \right) \\
0 &= \ln \delta + \left(1 - \frac{1}{\psi} \right) \mu_c + \kappa_0 + (\kappa_1 - 1) A_0 + \kappa_1 (1 - \nu_1) \sigma^2 A_2 + \frac{1}{2} \theta (\kappa_1 A_2 \sigma_w)^2 \\
A_0 &= \frac{1}{1 - \kappa_1} \left[\ln \delta + \left(1 - \frac{1}{\psi} \right) \mu_c + \kappa_0 + \kappa_1 A_2 \sigma^2 (1 - \nu_1) + \frac{\theta}{2} (\kappa_1 A_2 \sigma_w)^2 \right]. \qquad \text{(A-20)}
\end{aligned}$$

A.3 Derivation of the coefficients $A_{0,m}$, $A_{1,m}$, and $A_{2,m}$

To find the expressions for the coefficients $A_{0,m}$, $A_{1,m}$, and $A_{2,m}$ in Equation (A-8), we use the basic asset pricing equation with the SDF from Equation (A-9) to price the return to the market portfolio. According to Equation (A-6), combined with Equations (A-1) and (A-8), $r_{m,t+1}$ is given by:

$$\begin{aligned} r_{m,t+1} = \kappa_{0,m} + \kappa_{1,m}\left[A_{0,m} + A_{1,m}x_{t+1} + A_{2,m}\sigma^2_{t+1}\right] \\ - \left[A_{0,m} + A_{1,m}x_t + A_{2,m}\sigma^2_t\right] + \mu_d + \phi x_t + \varphi_d \sigma_t u_{t+1}. \end{aligned}$$

Applying the basic pricing equation to $r_{m,t}$, we can derive the expressions for $A_{0,m}$, $A_{1,m}$, and $A_{2,m}$:

$$\begin{aligned} 1 &= \mathbb{E}_t\left[\exp(m_{t+1} + r_{m,t+1})\right] \\ 1 &= \exp\left(\mathbb{E}_t\left[m_{t+1} + r_{m,t+1}\right] + \frac{1}{2}\mathrm{Var}_t\left[m_{t+1} + r_{m,t+1}\right]\right) \\ 0 &= \theta \ln\delta - \frac{\theta}{\psi}(\mu_c + x_t) + (\theta - 1)\Big[\kappa_0 + \kappa_1 A_0 + \kappa_1 A_1 \rho x_t + \kappa_1 A_2(\sigma^2 + \nu_1(\sigma^2_t - \sigma^2)) \\ &\quad - A_0 - A_1 x_t - A_2\sigma^2_t + \mu_c + x_t\Big] + \kappa_{0,m} + \kappa_{1,m}A_{0,m} \\ &\quad + \kappa_{1,m}A_{1,m}\rho x_t + \kappa_{1,m}A_{2,m}(\sigma^2 + \nu_1(\sigma^2_t - \sigma^2)) - A_{0,m} \\ &\quad - A_{1,m}x_t - A_{2,m}\sigma^2_t + \mu_d + \phi x_t + \frac{1}{2}\mathrm{Var}_t(m_{t+1} + r_{m,t+1}). \end{aligned} \tag{A-21}$$

Derive the expression for $\mathrm{Var}_t(m_{t+1} + r_{m,t+1})$:

$$\begin{aligned} \mathrm{Var}_t(m_{t+1} + r_{m,t+1}) &= \mathrm{Var}_t\left[\theta\ln\delta - \frac{\theta}{\psi}g_{t+1} + (\theta - 1)r_{a,t+1} + r_{m,t+1}\right] \\ &= \mathrm{Var}_t\Bigg[-\frac{\theta}{\psi}g_{t+1} + (\theta - 1)\Big[\kappa_0 + \kappa_1(A_0 + A_1 x_{t+1} + A_2\sigma^2_{t+1}) - A_0 \\ &\quad - A_1 x_t - A_2\sigma^2_t + g_{t+1}\Big] + \kappa_{0,m} + \kappa_{1,m}A_{0,m} + \kappa_{1,m}A_{1,m}x_{t+1} \\ &\quad + \kappa_{1,m}A_{2,m}\sigma^2_{t+1} - A_{0,m} - A_{1,m}x_t - A_{2,m}\sigma^2_t + \mu_d + \phi x_t + \varphi_d\sigma_t u_{t+1}\Bigg] \end{aligned}$$

$$
\begin{aligned}
&= \operatorname{Var}_t\Bigg[\left(\theta - 1 - \frac{\theta}{\psi}\right) g_{t+1} + (\theta-1)\Big[\kappa_1 A_1 x_{t+1} \\
&\quad + \kappa_1 A_2 \sigma_{t+1}^2\Big] + \kappa_{1,m} A_{1,m} x_{t+1} + \kappa_{1,m} A_{2,m} \sigma_{t+1}^2 + \varphi_d \sigma_t u_{t+1}\Bigg] \\
&= \operatorname{Var}_t\Bigg[\left(\theta - 1 - \frac{\theta}{\psi}\right) g_{t+1} + \left((\theta-1)\kappa_1 A_1 + \kappa_{1,m} A_{1,m}\right) x_{t+1} \\
&\quad + \left((\theta-1)\kappa_1 A_2 + \kappa_{1,m} A_{2,m}\right) \sigma_{t+1}^2 + \varphi_d \sigma_t u_{t+1}\Bigg].
\end{aligned}
$$

Finally:

$$
\begin{aligned}
\operatorname{Var}_t(m_{t+1} + r_{m,t+1}) = &\left(\theta - 1 - \frac{\theta}{\psi}\right)^2 \sigma_t^2 + \left(\left[(\theta-1)\kappa_1 A_1 + \kappa_{1,m} A_{1,m}\right]^2 \varphi_e^2 + \varphi_d^2\right) \sigma_t^2 \\
&+ \left[(\theta-1)\kappa_1 A_2 + \kappa_{1,m} A_{2,m}\right]^2 \sigma_w^2,
\end{aligned} \tag{A-22}
$$

because $\operatorname{Cov}_t(g_{t+1}, x_{t+1}) = 0$, $\operatorname{Cov}_t(g_{t+1}, \sigma_{t+1}^2) = 0$ and $\operatorname{Cov}_t(x_{t+1}, \sigma_{t+1}^2) = 0$.

To derive the coefficient $A_{1,m}$, we insert Equation (A-22) into Equation (A-21) and collect all terms that involve x_t. They are set to zero, because the Euler equation must hold for all values of the state variables:

$$
\begin{aligned}
-\frac{\theta}{\psi} x_t + (\theta-1)\left[\kappa_1 A_1 \rho x_t - A_1 x_t + x_t\right] + \kappa_{1,m} A_{1,m} \rho x_t - A_{1,m} x_t + \phi x_t &\overset{!}{=} 0 \\
-\frac{\theta}{\psi} + (\theta-1)\left[A_1(\kappa_1 \rho - 1) + 1\right] + A_{1,m}(\kappa_{1,m}\rho - 1) + \phi &= 0 \\
-\frac{\theta}{\psi} + (\theta-1)\left[\left(\frac{1}{\psi} - 1\right) + 1\right] + A_{1,m}(\kappa_{1,m}\rho - 1) + \phi &= 0
\end{aligned}
$$

$$
A_{1,m} = \frac{-\frac{\theta}{\psi} + (\theta-1)\frac{1}{\psi} + \phi}{1 - \kappa_{1,m}\rho} = \frac{\phi - \frac{1}{\psi}}{1 - \kappa_{1,m}\rho}. \tag{A-23}
$$

To derive the coefficient $A_{2,m}$, we collect all terms involving σ_t^2 and set them to zero, because the Euler equation must hold for all values of the state variables:

$$
\begin{aligned}
&(\theta-1)(\kappa_1 A_2 \nu_1 - A_2) + \kappa_{1,m} A_{2,m} \nu_1 - A_{2,m} \\
&+ \frac{1}{2}\left[\left(\theta - 1 - \frac{\phi}{\psi}\right)^2 + (\kappa_{1,m} A_{1,m} \varphi_e - (1-\theta)\kappa_1 A_1 \varphi_e)^2 + \varphi_d^2\right] \stackrel{!}{=} 0,
\end{aligned}
$$

with $(\theta - 1 - \frac{\theta}{\psi}) = \lambda_{m,\eta}$, $(\kappa_{1,m} A_{1,m} \varphi_e) = \beta_{m,e}$, and $((1-\theta)\kappa_1 A_1 \varphi_e) = \lambda_{m,e}$:

$$
(1-\theta)(\kappa_1 \nu_1 - 1) A_2 - \frac{1}{2}\left[\lambda_{m,\eta}^2 + (\beta_{m,e} - \lambda_{m,e})^2 + \varphi_d^2\right] = A_{2,m}(\kappa_{1,m}\nu_1 - 1)
$$

$$
A_{2,m} = \frac{(1-\theta)(1-\kappa_1\nu_1)A_2 + \frac{1}{2}[\lambda_{m,\eta}^2 + (\beta_{m,e} - \lambda_{m,e})^2 + \varphi_d^2]}{(1-\kappa_{1,m}\nu_1)}. \tag{A-24}
$$

To derive $A_{0,m}$, we set the sum of all terms involving x_t and σ_t^2 in Equation (A-21) to zero:

$$
\begin{aligned}
0 &= \theta \ln\delta - \frac{\theta}{\psi}\mu_c + (\theta - 1)\left[\kappa_0 + \kappa_1 A_0 + \kappa_1 A_2 (1-\nu_1)\sigma^2 - A_0 + \mu_c\right] + \kappa_{0,m} \\
&\quad + \kappa_{1,m} A_{0,m} + \kappa_{1,m} A_{2,m} \sigma^2 (1-\nu_1) - A_{0,m} + \mu_d + \frac{1}{2}\left[(\theta-1)\kappa_1 A_2 + \kappa_{1,m} A_{2,m}\right]^2 \sigma_w^2 \\
(1-\kappa_{1,m}) A_{0,m} &= \theta \ln\delta - \frac{\theta}{\psi}\mu_c + (\theta - 1)\left[\kappa_0 + \kappa_1 A_0 + \kappa_1 A_2 (1-\nu_1)\sigma^2 - A_0 + \mu_c\right] \\
&\quad + \kappa_{0,m} + \kappa_{1,m} A_{2,m} \sigma^2 (1-\nu_1) + \mu_d + \frac{1}{2}\left[(\theta-1)\kappa_1 A_2 + \kappa_{1,m} A_{2,m}\right]^2 \sigma_w^2
\end{aligned}
$$

$$
\begin{aligned}
A_{0,m} = \frac{1}{(1-\kappa_{1,m})}\Bigg[&\theta \ln\delta - \frac{\theta}{\psi}\mu_c + (\theta - 1)\Bigg[\kappa_0 + \kappa_1 A_0 + \kappa_1 A_2 (1-\nu_1)\sigma^2 \\
&- A_0 + \mu_c\Bigg] + \kappa_{0,m} + \kappa_{1,m} A_{2,m} \sigma^2 (1-\nu_1) + \mu_d \\
&+ \frac{1}{2}\left[(\theta-1)\kappa_1 A_2 + \kappa_{1,m} A_{2,m}\right]^2 \sigma_w^2\Bigg].
\end{aligned} \tag{A-25}
$$

A.4 Representation of the risk-free rate

The formula for the risk-free rate can be derived by plugging in $r_{f,t}$ for $r_{i,t+1}$ into the basic pricing equation:

$$
\begin{aligned}
1 &= \mathbb{E}_t\left[\exp(\theta\ln(\delta) - \frac{\theta}{\psi}g_{t+1} + (\theta-1)r_{a,t+1} + r_{f,t})\right] \\
1 &= \exp\left(\mathbb{E}_t\left[\theta\ln(\delta) - \frac{\theta}{\psi}g_{t+1} + (\theta-1)r_{a,t+1} + r_{f,t}\right] + \frac{1}{2}\mathrm{Var}_t\left[-\frac{\theta}{\psi}g_{t+1} + (\theta-1)r_{a,t+1}\right]\right) \\
0 &= \theta\ln(\delta) - \frac{\theta}{\psi}\mathbb{E}_t(g_{t+1}) + (\theta-1)\mathbb{E}_t(r_{a,t+1}) + r_{f,t} + \frac{1}{2}\mathrm{Var}_t\left[\frac{\theta}{\psi}g_{t+1} + (1-\theta)r_{a,t+1}\right].
\end{aligned}
$$

The risk-free rate is thus given by:

$$
r_{f,t} = -\theta\ln(\delta) + \frac{\theta}{\psi}\mathbb{E}_t(g_{t+1}) + (1-\theta)\mathbb{E}_t(r_{a,t+1}) - \frac{1}{2}\mathrm{Var}_t(m_{t+1}).
$$

In addition, $\mathbb{E}_t(r_{a,t+1})$ can be obtained from the definition of $r_{a,t+1}$:

$$
\begin{aligned}
r_{a,t+1} &= \kappa_0 + \kappa_1 z_{t+1} - z_t + g_{t+1} \\
&= \kappa_0 + \kappa_1\left[A_0 + A_1 x_{t+1} + A_2\sigma_{t+1}^2\right] - \left[A_0 + A_1 x_t + A_2\sigma_t^2\right] + g_{t+1} \\
\mathbb{E}_t(r_{a,t+1}) &= \kappa_0 + \kappa_1\left[A_0 + A_1\rho x_t + A_2(\sigma^2 + \nu_1(\sigma_t^2 - \sigma^2))\right] - A_0 - A_1 x_t - A_2\sigma_t^2 + \mu_c + x_t.
\end{aligned}
$$

$\text{Var}_t\,(m_{t+1})$ is computed as follows:

$$\begin{aligned}
\text{Var}_t\,(m_{t+1}) &= \text{Var}_t\left[\frac{\theta}{\psi}g_{t+1} + (1-\theta)r_{a,t+1}\right] \\
&= \text{Var}_t\left[\frac{\theta}{\psi}g_{t+1} + (1-\theta)(\kappa_1 A_1 x_{t+1} + \kappa_1 A_2 \sigma_{t+1}^2 + g_{t+1})\right] \\
&= \text{Var}_t\left[\left(\frac{\theta}{\psi} + 1 - \theta\right) g_{t+1} + (1-\theta)\kappa_1 A_1 x_{t+1} + (1-\theta)\kappa_1 A_2 \sigma_{t+1}^2\right]
\end{aligned}$$

with $\text{Cov}_t(g_{t+1}, x_{t+1}) = 0$:

$$= \left(\frac{\theta}{\psi} + 1 - \theta\right)^2 \text{Var}_t(g_{t+1}) + (1-\theta)^2(\kappa_1 A_1)^2 \text{Var}_t(x_{t+1})$$

$$+ (1-\theta)^2(\kappa_1 A_2)^2 \text{Var}_t(\sigma_{t+1}^2)$$

with $(-\frac{\theta}{\psi} + \theta - 1) = \lambda_{m,\eta}$, $((1-\theta)\kappa_1 A_1 \varphi_e) = \lambda_{m,e}$, and $(1-\theta)\kappa_1 A_2 = \lambda_{m,w}$:

$$= \lambda_{m,\eta}^2 \sigma_t^2 + \lambda_{m,e}^2 \sigma_t^2 + \lambda_{m,w}^2 \sigma_w^2.$$

A.5 Risk premia

The gross risk-free rate is given by:

$$R_{f,t+1} = \frac{1}{\mathbb{E}_t(M_{t+1})}.$$

The log return on the risk-free asset is given by:

$$\begin{aligned}
\ln(R_{f,t+1}) &= -\ln\left[\mathbb{E}_t(M_{t+1})\right] \\
r_{f,t+1} &= -\ln\left[\mathbb{E}_t\left(\exp(m_{t+1})\right)\right] \\
&= -\ln\left[\exp\left(\mathbb{E}_t(m_{t+1}) + \frac{1}{2}\mathrm{Var}_t(m_{t+1})\right)\right] \\
&= -\mathbb{E}_t(m_{t+1}) - \frac{1}{2}\mathrm{Var}_t(m_{t+1}).
\end{aligned} \tag{A-26}$$

To derive the risk premium on the aggregate wealth portfolio, we use the following Euler equation to obtain the relation between $\mathbb{E}_t(m_{t+1})$ and $\mathbb{E}_t(r_{a,t+1})$:

$$\begin{aligned}
\mathbb{E}_t\left[M_{t+1}R_{a,t+1}\right] &= 1 \\
\mathbb{E}_t\left[\exp\left(m_{t+1} + r_{a,t+1}\right)\right] &= 1 \\
\exp\left(\mathbb{E}_t(m_{t+1} + r_{a,t+1}) + \frac{1}{2}\mathrm{Var}_t(m_{t+1} + r_{a,t+1})\right) &= 1 \\
\mathbb{E}_t(m_{t+1}) + \mathbb{E}_t(r_{a,t+1}) + \frac{1}{2}\mathrm{Var}_t(m_{t+1} + r_{a,t+1}) &= 0.
\end{aligned}$$

Finally:

$$\mathbb{E}_t(m_{t+1}) = -\mathbb{E}_t(r_{a,t+1}) - \frac{1}{2}\left[\mathrm{Var}_t(m_{t+1}) + \mathrm{Var}_t(r_{a,t+1}) + 2\ \mathrm{Cov}_t(m_{t+1}, r_{a,t+1})\right]. \tag{A-27}$$

Next we use Equation (A-26) and subsequently insert Equation (A-27) to determine the risk premium on the aggregate wealth portfolio:

$$\begin{aligned}
\mathbb{E}_t\left[r_{a,t+1} - r_{f,t+1}\right] &= \mathbb{E}_t\left[r_{a,t+1} + \mathbb{E}_t(m_{t+1}) + \frac{1}{2}\mathrm{Var}_t(m_{t+1})\right] \\
&= \mathbb{E}_t\Big[r_{a,t+1} - \mathbb{E}_t(r_{a,t+1}) - \frac{1}{2}\mathrm{Var}_t(m_{t+1}) - \frac{1}{2}\mathrm{Var}_t(r_{a,t+1}) \\
&\quad - \mathrm{Cov}_t(m_{t+1}, r_{a,t+1}) + \frac{1}{2}\mathrm{Var}_t(m_{t+1})\Big] \\
&= -\mathrm{Cov}_t\left[m_{t+1}, r_{a,t+1}\right] - \frac{1}{2}\mathrm{Var}_t(r_{a,t+1}) \\
&= -\mathrm{Cov}_t\left[m_{t+1} - \mathbb{E}_t(m_{t+1}), r_{a,t+1} - \mathbb{E}_t(r_{a,t+1})\right] - \frac{1}{2}\mathrm{Var}_t(r_{a,t+1}).
\end{aligned}$$

To write the risk premium in detail, we must derive the expressions $m_{t+1} - \mathbb{E}_t(m_{t+1})$ and $r_{a,t+1} - \mathbb{E}_t(r_{a,t+1})$:

$$\begin{aligned}
r_{a,t+1} &= \kappa_0 + \kappa_1 z_{t+1} - z_t + g_{t+1} \\
&= \kappa_0 + \kappa_1\left[A_0 + A_1 x_{t+1} + A_2\sigma_{t+1}^2\right] - A_0 - A_1 x_t - A_2\sigma_t^2 + g_{t+1} \\
\mathbb{E}_t(r_{a,t+1}) &= \kappa_0 + \kappa_1\left[A_0 + A_1\rho x_t + A_2(\sigma^2 + \nu_1(\sigma_t^2 - \sigma^2))\right] - A_0 - A_1 x_t - A_2\sigma_t^2 \\
&\quad + \mu_c + x_t \\
r_{a,t+1} - \mathbb{E}_t(r_{a,t+1}) &= \kappa_1 A_1\left[x_{t+1} - \rho x_t\right] + \kappa_1 A_2\left[\sigma_{t+1}^2 - \sigma^2 - \nu_1(\sigma_t^2 - \sigma^2)\right] + \left[g_{t+1} - \mu_c - x_t\right] \\
&= \kappa_1 A_1 \varphi_e \sigma_t e_{t+1} + \kappa_1 A_2 \sigma_w w_{t+1} + \sigma_t \eta_{t+1}
\end{aligned}$$

$$\begin{aligned}
m_{t+1} &= \theta\ln\delta - \frac{\theta}{\psi}g_{t+1} + (\theta - 1)r_{a,t+1} \\
\mathbb{E}_t(m_{t+1}) &= \theta\ln\delta - \frac{\theta}{\psi}\left[\mu_c + x_t\right] + (\theta - 1)\mathbb{E}_t(r_{a,t+1}) \\
m_{t+1} - \mathbb{E}_t(m_{t+1}) &= -\frac{\theta}{\psi}\left[\sigma_t\eta_{t+1}\right] + (\theta - 1)\left[\sigma_t\eta_{t+1} + \kappa_1 A_1\varphi_e\sigma_t e_{t+1} + \kappa_1 A_2\sigma_w w_{t+1}\right] \\
&= \left[\theta - 1 - \frac{\theta}{\psi}\right]\sigma_t\eta_{t+1} - (1-\theta)\kappa_1 A_1\varphi_e\sigma_t e_{t+1} - (1-\theta)\kappa_1 A_2\sigma_w w_{t+1} \\
&= \lambda_{m,\eta}\sigma_t\eta_{t+1} - \lambda_{m,e}\sigma_t e_{t+1} - \lambda_{m,w}\sigma_w w_{t+1}.
\end{aligned}$$

The risk premium on the aggregate wealth portfolio is given by:

$$\begin{aligned}
\mathbb{E}_t\left[r_{a,t+1}-r_{f,t+1}\right] =& -\mathrm{Cov}_t\left[m_{t+1}-\mathbb{E}_t(m_{t+1}), r_{a,t+1}-\mathbb{E}_t(r_{a,t+1})\right]-\frac{1}{2}\mathrm{Var}_t(r_{a,t+1}) \\
=& -\mathbb{E}_t\Big[\left(\lambda_{m,\eta}\sigma_t\eta_{t+1}-\lambda_{m,e}\sigma_t e_{t+1}-\lambda_{m,w}\sigma_w w_{t+1}\right) \\
& \quad \left(\sigma_t\eta_{t+1}+\kappa_1 A_1\varphi_e\sigma_t e_{t+1}+\kappa_1 A_2\sigma_w w_{t+1}\right)\Big] \\
& -\frac{1}{2}\left(\mathbb{E}_t\left[\sigma_t^2\eta_{t+1}^2\right]+\mathbb{E}_t\left[(\kappa_1 A_1\varphi_e)^2\sigma_t^2 e_{t+1}^2\right]+\mathbb{E}_t\left[\kappa_1^2 A_2^2\sigma_w^2 w_{t+1}^2\right]\right) \\
=& -\lambda_{m,\eta}\sigma_t^2+\lambda_{m,e}(\kappa_1 A_1\varphi_e)\sigma_t^2+\kappa_1 A_2\lambda_{m,w}\sigma_w^2 \\
& -\frac{1}{2}\left((1+(\kappa_1 A_1\varphi_e)^2)\sigma_t^2+(\kappa_1 A_2)^2\sigma_w^2\right).
\end{aligned}$$

To derive the risk premium on the market portfolio, we use the following Euler equation to obtain the relation between $\mathbb{E}_t(m_{t+1})$ and $\mathbb{E}_t(r_{m,t+1})$:

$$\begin{aligned}
\mathbb{E}_t\left[M_{t+1}R_{m,t+1}\right] &= 1 \\
\mathbb{E}_t\left[\exp\left(m_{t+1}+r_{m,t+1}\right)\right] &= 1 \\
\exp\left(\mathbb{E}_t(m_{t+1}+r_{m,t+1})+\frac{1}{2}\mathrm{Var}_t(m_{t+1}+r_{m,t+1})\right) &= 1 \\
\mathbb{E}_t(m_{t+1})+\mathbb{E}_t(r_{m,t+1})+\frac{1}{2}\mathrm{Var}_t(m_{t+1}+r_{m,t+1}) &= 0.
\end{aligned}$$

Finally:

$$\mathbb{E}_t(m_{t+1}) = -E_t(r_{m,t+1})-\frac{1}{2}\left[\mathrm{Var}_t(m_{t+1})+\mathrm{Var}_t(r_{m,t+1})+2\ \mathrm{Cov}(m_{t+1},r_{m,t+1})\right]. \tag{A-28}$$

Now we use Equations (A-26) and (A-28) to determine the risk premium:

$$\begin{aligned}
\mathbb{E}_t\left[r_{m,t+1} - r_{f,t+1}\right] &= \mathbb{E}_t\left[r_{m,t+1} + \mathbb{E}_t(m_{t+1}) + \frac{1}{2}\text{Var}_t(m_{t+1})\right] \\
&= \mathbb{E}_t\Big[r_{m,t+1} - \mathbb{E}_t(r_{m,t+1}) - \frac{1}{2}\text{Var}_t(m_{t+1}) - \frac{1}{2}\text{Var}_t(r_{m,t+1}) \\
&\quad - \text{Cov}_t(m_{t+1}, r_{m,t+1}) + \frac{1}{2}\text{Var}_t(m_{t+1})\Big] \\
&= -\text{Cov}_t\left[m_{t+1}, r_{m,t+1}\right] - \frac{1}{2}\text{Var}_t(r_{m,t+1}) \\
&= -\text{Cov}_t\left[m_{t+1} - \mathbb{E}_t(m_{t+1}), r_{m,t+1} - \mathbb{E}_t(r_{m,t+1})\right] - \frac{1}{2}\text{Var}_t(r_{m,t+1}).
\end{aligned}$$

To write the risk premium in detail, first derive the expression for $r_{m,t+1} - \mathbb{E}_t(r_{m,t+1})$:

$$\begin{aligned}
r_{m,t+1} &= \kappa_{0,m} + \kappa_{1,m} z_{m,t+1} - z_{m,t} + g_{d,t+1} \\
&= \kappa_{0,m} + \kappa_{1,m}\left[A_{0,m} + A_{1,m}x_{t+1} + A_{2,m}\sigma_{t+1}^2\right] - A_{0,m} - A_{1,m}x_t - A_{2,m}\sigma_t^2 \\
&\quad + \mu_d + \phi x_t + \varphi_d \sigma_t u_{t+1} \\
\mathbb{E}_t(r_{m,t+1}) &= \kappa_{0,m} + \kappa_{1,m}A_{0,m} + \kappa_{1,m}A_{1,m}\rho x_t + \kappa_{1,m}A_{2,m}(\sigma^2 + \nu_1(\sigma_t^2 - \sigma^2)) \\
&\quad - A_{0,m} - A_{1,m}x_t - A_{2,m}\sigma_t^2 + \mu_d + \phi x_t \\
r_{m,t+1} - \mathbb{E}_t(r_{m,t+1}) &= \kappa_{1,m}A_{1,m}\left[x_{t+1} - \rho x_t\right] + \kappa_{1,m}A_{2,m}(\sigma_{t+1}^2 - \sigma^2 - \nu_1(\sigma_t^2 - \sigma^2)) + \varphi_d\sigma_t u_{t+1} \\
&= \kappa_{1,m}A_{1,m}\varphi_e\sigma_t e_{t+1} + \kappa_{1,m}A_{2,m}\sigma_w w_{t+1} + \varphi_d\sigma_t u_{t+1}.
\end{aligned}$$

The risk premium on the market portfolio is given by:

$$
\begin{aligned}
\mathbb{E}_t\left[r_{m,t+1} - r_{f,t+1}\right] = & -\mathrm{Cov}_t\left[m_{t+1} - \mathbb{E}_t(m_{t+1}), r_{m,t+1} - \mathbb{E}_t(r_{m,t+1})\right] - \frac{1}{2}\mathrm{Var}_t(r_{m,t+1}) \\
= & -\mathbb{E}_t\Big[(\lambda_{m,\eta}\sigma_t\eta_{t+1} - \lambda_{m,e}\sigma_t e_{t+1} - \lambda_{m,w}\sigma_w w_{t+1}) \\
& \quad (\varphi_d\sigma_t u_{t+1} + \kappa_{1,m}A_{1,m}\varphi_e\sigma_t e_{t+1} + \kappa_{1,m}A_{2,m}\sigma_w w_{t+1})\Big] \\
& -\frac{1}{2}\left[\mathbb{E}_t\left(\varphi_d^2\sigma_t^2 u_{t+1}^2\right) + \mathbb{E}_t\left((\kappa_{1,m}A_{1,m}\varphi_e)^2\sigma_t^2 e_{t+1}^2\right) + \mathbb{E}_t\left(\kappa_{1,m}^2 A_{2,m}^2\sigma_w^2 w_{t+1}^2\right)\right] \\
= & \, \lambda_{m,e}\kappa_{1,m}A_{1,m}\varphi_e\sigma_t^2 + \lambda_{m,w}\kappa_{1,m}A_{2,m}\sigma_w^2 \\
& -\frac{1}{2}\left[\varphi_d^2\sigma_t^2 + (\kappa_{1,m}A_{1,m}\varphi_e)^2\sigma_t^2 + (\kappa_{1,m}A_{2,m})^2\sigma_w^2\right].
\end{aligned}
$$

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