To appear in revised form in: M. van Atten, P. Boldini, M. Bourdeau, G. Heinzmann (eds.), One Hundred Years of Intuitionism (1907-2007), Basel: Birkhäuser 2008.

Lorenzen's operative justification of intuitionistic logic

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Introduction

With his *Introduction to Operative Logic and Mathematics*[1], which first appeared in 1955, Paul Lorenzen became an exponent of an approach to the foundations of logic and mathematics, which is both formalistic and intuitionistic in spirit. *Formalistic* because its basis is the purely syntactical handling of symbols — or "figures", as Lorenzen preferred to say —, and *intuitionistic* because the insight into the validity of admissibility statements justifies the laws of logic. It is also intuitionistic with respect to its result, as Heyting's formalism of intuitionistic logic is legitimatised this way. Along with taking formal calculi as its basis, the notion of an inductive definition becomes fundamental. Together with a theory of abstraction and the idea of transfinitely iterating inductive definitions, Lorenzen devised a novel foundation for mathematics, many aspects of which still deserve attention. When he wrote his *Operative Logic*, neither a full-fledged theory of inductive definitions nor a proof-theoretic semantics for logical constants was available. A decade later, Lorenzen's inversion principle was used and extended by Prawitz (1965) in his theory of natural deduction, and in the 1970s, the idea of inversion was used for a logical semantics in terms of proofs by Dummett, Martin-Löf, Prawitz and others. Another aspect which makes Lorenzen's theory interesting from a modern point of view, is that in his protologic he anticipated certain views of rule-based reasoning and free equality which much later became central to the theory of resolution and logic programming. Lorenzen's inversion principle in its general form — that is, not in its restricted application in logic — is closely related to principles of definitional reflection in logic programming (Schroeder-Heister 2007). The idea that logical introduction rules are but a special case of rules defining (atomic) propositions was used in a different form in Martin-Löf's (1971) theory of iterated inductive definitions. Thus there are various interesting points from which we might take a closer look at *Operative Logic*.

Unfortunately, Lorenzen had already lost interest in the subject when issues such as proof-theoretic semantics and resolution-based reasoning became more popular in logic. Within the narrower realm of logic, he had already given up the operative approach in favour of dialogical logic by the end of the 1950s, perhaps motivated by discussions

[1] "Einführung in die operative Logik und Mathematik", henceforth quoted as "OL". A major earlier paper propagating the operative approach is Lorenzen (1950). For a biography of Lorenzen see Thiel(1996), for a bibliography of his work see [Lorenzen] (1996).

with Tarski in 1957-58 at the Institute for Advanced Study at Princeton (Lorenz 2001, p. 257). From the early 1960s on, he broadened his approach from the philosophy of logic and mathematics to geometry, the philosophy of science, and ethics, partly in collaboration with Wilhelm Kamlah, after having taken the philosophical chair in Erlangen in 1962. He became the head of a philosophical school in Germany which, as it favoured a normative and anti-empiricist foundation of science, was opposed to the analytic philosophy of science represented in Germany by Wolfgang Stegmüller, who was strongly influenced by Carnap, and for whom the rational reconstruction of science was the primary goal. Stegmüller once even criticized the Erlangen school for developing a "metascience of science fiction" (Stegmüller 1973, p. 26). The fact that Lorenzen found philosophical allies even within hermeneutics and Habermas' critical theory of society further contributed to the split in German philosophy of science. By the end of the 1970s, when the proof-theoretic foundation of logic had become a topic within the philosophy of language, Lorenzen had begun to make completely different issues his main agenda, including the philosophy of politics. Correspondingly, he did not take notice of the rising interest in rule-based theories and proof-theoretic approaches within the realm of computer science.

In this paper we concentrate on the logical aspects of the *Operative Logic*, i.e., on Lorenzen's attempt to base intuitionistic logic on admissibility principles, leaving aside his general theory of rule-based reasoning and his foundation of mathematics. In particular, we compare his approach to the theories of Dummett and Prawitz, who consider introduction rules as the defining properties of logical constants and derive valid logical laws by using ideas closely related to Lorenzen's inversion principle. In the first section, we discuss those aspects of Lorenzen's protologic, i.e. of his general theory of calculi and admissible rules which are most relevant to the foundation of deductive logic. In the second section, we reconstruct his theory of iterated implications and meta-calculi, and extract from his remarks a validity concept for sentences alias higher-level rules. We show that Lorenzen's theory of meta-calculi can be given a rendering that makes it a sensible theory of implication, and of logical constants in general. In the third section we compare (our reconstruction of) Lorenzen's approach with proof-theoretic semantics in the tradition of Dummett and Prawitz. Although, as shown in the appendix, Lorenzen's and Prawitz's validity concepts can be translated into one another, Lorenzen's theory has certain deficiencies when looked at from an epistemological point of view. Unlike Dummett and Prawitz, Lorenzen does not pay sufficient attention to the manifestation aspect of meaning, which for constants such as conjunction and disjunction is embodied in their introduction rules, but is absent in the crucial case of implication, for which in Lorenzen's framework no structural introduction rule is defined.

1 Basic protological concepts

The first part of the *Operative Logic* is entitled "Protologic" as it is conceptually prior to, and more general than logic. It is a theory of formal systems (calculi) and develops general principles for establishing the admissibility of inference rules in such systems. Logic in the narrower sense is a particular application of admissibility theory dealing with logical constants and with the iteration of admissibility.

1.1 Calculi

Lorenzen starts with elementary calculi (OL, §1) which permit to generate words (strings of signs) over an arbitrary (finite) alphabet. The elements of the alphabet are called *atoms*, the words are called *sentences* ("Aussagen"). A calculus K is specified by giving certain initial formulas ("Anfänge") A and rules $A_1, \ldots, A_n \to A$, where an initial formula is the limiting case of a rule (for $n = 0$). In the following, we also write "$\to A$" for an initial formula. We also speak of *primitive* inference rules and (in the premiss-free case) of *axioms*, if we want to make clear that they are the rules on which inferences in K are based.[2] Formulas are words composed of atoms and variables. The variables occurring in formulas are either *eigenvariables* (OL, p. 16) or *object variables* (OL, pp. 26f.). Eigenvariables of K can be substituted only with sentences derived in K itself, whereas object variables are to be substituted with sentences of a different calculus. Obviously, if a rule contains eigenvariables, the substitution instances of a rule are defined simultaneously with the derivations in K. For example, if

$$K_1 \left\{ \begin{array}{rl} \to | & (R_1) \\ x \to x| & (R_2) \end{array} \right.$$

is a calculus with the eigenvariable x, then in the derivation

$$\dfrac{\dfrac{\dfrac{}{|}R_1}{||}R_2}{|||}R_2$$

the second line is obtained from the first one by using the substitution instance $| \to ||$ of R_2, which is only defined after the first line has been derived (and thus $|$ becomes substitutable for x). Similarly, the step from the second to the third line is based on the substitution instance $|| \to |||$ of R_3 which relies on that $||$ has already been derived and is thus substitutible for x. Lorenzen needs this approach based on eigenvariables as for him the specification of any domain proceeds via calculi and rules, which means that one has always to begin with a calculus for which no external substitution range for object variables is given. When object variables standing for sentences derivable in a calculus K_0 are used in K, it is presupposed, of course, that each atom of K_0 is an

[2]Lorenzen avoids the term "axiom" because of traditional connotations of something being "evident", see OL, p. 7.

atom of K. As a matter of terminology, we call a formula or rule *closed*, if it contains no variable, and *open* otherwise.

It is obvious that this approach is closely related to the formalist ideas of Post (1921) and Curry (1952). As formulas are just strings of atoms and variables, Lorenzen starts with an arbitrary word structure rather than the functor-argument structure common in logic. This makes his approach particularly general. The most appropriate more modern point of view would be to look at his calculi as inductive definitions (see Aczel 1977). Concerning the foundations of mathematics, Lorenzen's *Operative Logic* can be viewed as a programme to embed mathematics into a theory of inductive definitions.

1.2 Admissibility

The main concept on which logic is based is *admissibility* (OL, §2). This notion, which was coined by Lorenzen, has entered logical textbooks as a standard proof-theoretic concept. A rule R is called *admissible* in a calculus K, if its addition to the primitive rules of K — resulting in an extended calculus $K + R$ — does not enlarge the set of derivable sentences. If $\vdash_K A$ denotes the derivability of A in K, then R is admissible in K if

$$\vdash_{K+R} A \text{ implies } \vdash_K A \tag{1}$$

for every sentence A. However, as for Lorenzen admissibility is the central concept on which the notion of implication is based, he cannot give the implication (1) as its definition. Rather, admissibility is given an operative meaning by reference to the notion of an *elimination procedure* (OL, §3). R is admissible in K, if every application of R *can be eliminated* from every derivation in $K+R$. The implicational relation between existential statements expressed in (1) is reduced to the insight that a certain procedure reduces any given derivation in $K + R$ in such a way that the resulting derivation does no longer use R. According to Lorenzen, this is the sort of insight (evidence) on which constructive logic and mathematics is based. It goes beyond the insight that something is derivable in K, but is still something which has a "definite" meaning. It is the admission of this sort of evidence which makes Lorenzen an intuitionist rather than a formalist. In its various variants, intuitionism has been based on the notion of a procedure or construction. The interpretation of implication in the BHK-semantics of the logical signs is based on it, as are more formalized notions like realizability. In proposing the admissibility of rules as based on elimination procedures, Lorenzen adds a new perspective to this concept.

Admissibility is to be distinguished from the derivability from assumptions. Lorenzen puts strong emphasis on this (obvious) fact, as this makes admissibility theory an enterprise, which is not merely formalistic in the sense of verifying formal derivations.

For example, given the calculus

$$K_2 \begin{cases} \quad\quad\quad \rightarrow\ n<n| \\ m<n\ \rightarrow\ m<n| \end{cases}$$

with m, n being object variables for lists of strokes (i.e., sentences derivable in K_1), then the rule $m < n \to m \mid < n \mid$ is admissible but not derivable (Lorenzen 1959, p. 170). Here, for sentences $A_1, \ldots, A_n, A$, the derivability of a rule $A_1, \ldots, A_n \to A$ in a calculus K means that $A_1, \ldots, A_n \vdash_K A$, i.e., A is derivable from $A_1, \ldots, A_n$ as assumptions. An open rule is derivable, if all its closed substitution instances are derivable. For example, $||||<|||$ is not derivable in K_2 from the assumption $|||<||$. The standard proof-theoretic example of a rule admissible but not derivable is that of the cut rule in the first-order sequent calculus.

Lorenzen establishes five basic protological principles for the generation of admissibility statements. Due to lack of space, we only sketch the three of them which are immediately relevant to logic. The two further principles — the *induction principle* and the *equality principle* —, though highly relevant, have their main application in the foundation of mathematics. For rules $R_1, \ldots, R_n, R$ we follow Lorenzen in using the notation $R_1, \ldots, R_n \vdash_K R$ to express that R can be eliminated from derivations in K by using $R_1, \ldots, R_n$ as additional rules of inference. In particular, this means that R is admissible, if $R_1, \ldots, R_n$ are admissible.[3]

1.2.1 The deduction principle

Though the admissibility of a rule does not imply its derivability, the converse, which is called the *deduction principle* (OL, p. 26), is true. If for every closed substitution instance $A'_1, \ldots, A'_n \to A'$ of a rule $A_1, \ldots, A_n \to A$ we have $A'_1, \ldots, A'_n \vdash_K A'$, then $A_1, \ldots, A_n \to A$ is an admissible rule of K. As an elimination procedure, we just have to replace every application of $A_1, \ldots, A_n \to A$ by the derivation of A' from $A'_1, \ldots, A'_n$ for the appropriate instance $A'_1, \ldots, A'_n \to A'$ of $A_1, \ldots, A_n \to A$.

It should be noted that the deduction principle is not a formal introduction rule for an implication statement. Lorenzen does not consider Gentzen-style natural deduction rules like

$$\dfrac{\begin{matrix}[A] \\ \vdots \\ B\end{matrix}}{A \to B}$$

where an assumption can be *discharged*. Rather, establishing the admissibility of the rule $A_1, \ldots, A_n \to A$ is a metalinguistic procedure which involves a step of reflection. It

[3]Lorenzen also characterizes $R_1, \ldots, R_n \vdash_K R$ as the admissibility of R in the calculus K extended with $R_1, \ldots, R_n$ as additional primitive rules (OL, pp. 24, 40). This is, however, a non-monotonic notion, and not what Lorenzen intends.

can be turned into a formal assertion of a sentence only in a so-called "meta-calculus" (see Section 2.1).

1.2.2 The inversion principle

The *inversion principle* (OL, pp. 30f.)is the most relevant principle for logic. It governs the relationship between introduction and elimination rules for logical constants (in the *Operative Logic* only used for conjunction, disjunction and existential quantification, see Section 2.3). The term "inversion principle" was later adopted by Prawitz (1965) in his theory of natural deduction (see Section 3). For simplicity, we just consider the propositional case without object variables. Suppose the calculus K is given by the following rules, where for each i $(1 \leq i \leq n)$, Γ_i stands for a list of sentences:

$$K \left\{ \begin{array}{c} \Gamma_1 \to A \\ \vdots \\ \Gamma_n \to A \end{array} \right.$$

Then the following holds for any sentence C:

$$\text{(IP)} \quad \Gamma_1 \to C, \ \ldots \ , \Gamma_n \to C \ \vdash_K \ A \to C$$

The validity of this principle is easily seen: An application of $A \to C$ leads from A to C, where A can only be derived in K according to one of the 'introduction rules' $\Gamma_i \to A$ for A. So we have the following situation:

$$\cfrac{\cfrac{\Gamma_i}{A}\,\Gamma_i \to A}{C}\,A \to C$$

Obviously, this an be replaced with

$$\frac{\Gamma_i}{C}\,\Gamma_i \to C$$

thus eliminating the application of $A \to C$. Since $\Gamma_i \to C$ is available for every i, this elimination procedure works in any case no matter from which Γ_i the sentence A was introduced in the first place.

Reading the calculus K as an inductive definition, this principle can be regarded as an interpretation of the extremal clause sometimes used to finish an inductive definition ("nothing else is A"): Everything that follows from each defining condition of A, follows from A itself. Note, however, that this principle is not laid down as a primitive rule interpreting the inductive definition, but is proved by means of an elimination procedure. This elimination procedure, in eliminating an application of $A \to C$, removes A as a 'maximum sentence' which is first introduced by means of $\Gamma_i \to A$ and then eliminated by means of $A \to C$. In the case of $n = 1$, this principle amounts to the 'direct' inversion of a rule. Suppose

$$A_1, \ldots, A_n \to A$$

is the only primitive rule of K having A as its head. Then from

$$A_1, \dots, A_n \to C \vdash_K A \to C$$

we obtain the admissibility of the rules:

$$\begin{array}{l} A \to A_1 \\ \quad\vdots \\ A \to A_n \end{array}$$

by choosing C in turn to be $A_1, \dots, A_n$ and using the fact that $A_1, \dots, A_n \to A_i$ is trivially admissible.

The inversion principle is not of much use when restricted to sentences. Even for the treatment of propositional logic, variables for sentences are necessary. Unfortunately, Lorenzen's formulation for the case with object variables is mistaken, and his own (in the second edition of OL) and others' (e.g., Lorenz 1980) attempts at repairing it failed as well. Hermes (1959) gives a correct version which suffers, however, from certain deficiencies of the general framework, which are due to the fact that unification theory and resolution-based reasoning was not known yet. Schroeder-Heister (2007) presents an examination of these approaches and a comparison with more modern principles, in particular with the prinpiple of *definitional reflection* developed by Hallnäs and Schroeder-Heister (1990/91) in the context of logic programming and later used as general definitional device (Schroeder-Heister (1993, 2008a), Hallnäs (1991, 2006).

1.2.3 The underivability principle

The *underivability principle* (OL, pp. 36f.) says that, if the sentence A is not derivable in K, then any rule $A \to B$ is admissible in K. The reason for this result is simply that, if the premiss A of $A \to B$ is underivable in K, then $A \to B$ cannot be applied to a derivable sentence of K, so $A \to B$ can be added to the primitive rules of K without generating new derivable sentences. Lorenzen does not consider the underivability principle as a limiting case of the inversion principle, for the case in which there is no defining rule for A available in K, but considers it a principle of its own. Obviously, the underivability principle can be used as the basis of the intuitionistic absurdity rule and the intuitionistic concept of negation, although it is not entirely clear whether this is intended by Lorenzen (see Section 2.4). In any case, this principle is needed at other places in his protologic, e.g., within his theory of inequality (OL, pp. 89f.)

2 Lorenzen's foundation of intuitionistic logic

An investigation of Lorenzen's foundation of mathematics would now study his theory of abstraction in connection with his theory of inductive definitions, in particular his idea of transfinitely iterating inductive definitions through the handling of language levels ("Sprachschichten"). Here, we discuss his justification of deductive logic based on the admissibility principles mentioned, in particular on the inversion principle, and relate it to proof-theoretic semantics in the sense of Dummett and Prawitz. For simplicity, we focus on the propositional case. Universal quantification essentially behaves like implication, and existential quantification like conjunction or disjunction.

Lorenzen divides his discussion into that of consequence logic (OL, §6), which contains his treatment of implication (and universal quantification), of conjunction and disjunction (and existential quantification) (OL, §7) and of negation (OL, §8). The separation of the treatment of implication from that of conjunction and disjunction indicates that there is a fundamental split between two sorts of connectives. He even speaks of implication (and universal quantification and negation) as "improper" logical constants as opposed to the "proper" ones of conjunction and disjunction (and existential quantification) (OL, p. 172).

As a matter of terminology, we speak of "implication" throughout when $\to$ is used as a sentence-forming operator (rather than a rule arrow), whereas Lorenzen in the *Operative Logic* does not use any specific terminology, and in his later writings propagates the term "subjunction" (and "adjunction" for "disjunction"). Furthermore, we use a tree-like notation of derivations in contradistinction to Lorenzen's linear one. Finally, we only consider closed rules, i.e., rules without variables, and identify open rules with the sets of their instances, thus admitting that the set of primitive rules of a calculus is infinite. When we speak of the finite specification of calculi, this is to be understood in the sense of a specification by means of finitely many rule *schemata*.

2.1 Consequence logic and the theory of meta-calculi

Lorenzen's theory of implication is based on the idea that an implicational sentence $A \to B$ expresses the admissibility of the rule $A \to B$, so the assertion of an implication is justified if this implication, when read as a rule, is admissible. In this sense an implication expresses a meta-statement about a calculus. This has a clear meaning as long as there is no iteration of the implication sign. In order to cope with iterated implications, Lorenzen develops the idea of finitely iterated meta-calculi. In the following, we try to built a coherent theory based on his scarce remarks on the specification of meta-calculi. We take the freedom to deviate from his notation and terminology, and to extend it whenever this seems appropriate to us.

From the beginning, we have to consider the possibility that calculi are extended with inference rules for additional constants, and that such extensions are available at any meta-level. We speak of a *definitional extension* and of constants *defined* in this extension, as it is the intention of such an extension to fix their meaning. In practice, these constants will be conjunction and disjunction [and existential quantification] (see Section 2.3). A definitional extension consists of formation rules $\mathbb{F}$ and primitive inference rules $\mathbb{D}$, where $\mathbb{F}^n$ and $\mathbb{D}^n$ are the formation rules and primitive inference rules, respectively, for level n. We assume that at each level n, the formation rules $\mathbb{F}^n$ determine how to construct a level-n-sentence from atomic level-n-sentences, where in $\mathbb{F}^n$ these atomic level-n-setences are not specified, but are just represented by variables. Starting from a basic calculus K and a definitional extension with formation rules $\mathbb{F}$, we define level-n-sentences and level-n-rules, and the sentences and rules of each meta-calculus M^nK as follows.

Level-n-sentences and level-n-rules:

(i) *Every sentence of K is a level-0-rule.*

(ii) *Every level-n-rule is an atomic level-n-sentence.*

(iii) *Every atomic level-n-sentence is a level-n-sentence.*

(iv) *Every expression generated from atomic level-n-sentences by means of the formation rules $\mathbb{F}^n$ is a level-n-sentence.*

(v) *If $A_1, \ldots, A_n, A$ are level-n-sentences, then $A_1, \ldots, A_n \rightarrow A$ is a level-$(n+1)$-rule.*

Sentences and rules of M^nK:

(i) *Every level-$(n+1)$-rule is a rule of M^nK.*

(ii) *Every level-n-sentence is a sentence of M^nK.*

We write MK for M^1K. However, in our terminology, M^0K is not just K, but the definitional extension of K (in practice: K extended with conjunction and disjunction [and existential quantification]). Although, syntactically, level-n-rules and level-n-sentences are the same, they are associated with different systems. As it transforms level-n-sentences into a level-n-sentence, a level-$(n+1)$-rule is a rule of M^nK, whereas a level-$(n+1)$-sentence is a sentence of $M^{n+1}K$. So the basic idea is that every rule over sentences of M^nK is at the same time an atomic sentence of $M^{n+1}K$. To avoid parentheses, Lorenzen uses dots to indicate precedence. For example,

$$\rightarrow A;\ B_1, B_2 \rightarrow C\ \dot{\rightarrow}\ E \rightarrow F$$

is a level-2-rule (and thus a rule of MK), and at the same time an atomic level-2-sentence (and thus a sentence of M^2K).

$$A \rightarrow B_1 \wedge B_2\ \dot{\rightarrow}\ \rightarrow C\ \ddot{;}\ \rightarrow E_1\ \dot{\wedge}\ D_1, D_2 \rightarrow E_2\ \dot{\rightarrow}\ F \rightarrow G\ \ddot{\rightarrow}\ \rightarrow H\ \dot{\rightarrow}\ I \rightarrow J$$

is a level-3-rule (and thus a rule of M^2K), and at the same time an atomic level-3-sentence (and thus a sentence of M^3K), where it is supposed that conjunctions are available (at least) at levels 0 and 1. Lorenzen also uses a notation with unequal levels such as $A \dot{\rightarrow} B \rightarrow C$. This is just shorthand for $\rightarrow A \dot{\rightarrow} B \rightarrow C$, and analogously for similar notations.

It remains to specify the primitive inference rules of M^nK with respect to a definitional extension $\mathbb{D}$. The primitive inference rules $\mathbb{D}^n$ available at level n govern the expressions formed by means of $\mathbb{F}^n$ and are called *definitional rules.* These rules transfer level-n-sentences into level-n-sentences, i.e. they have the form of level-$(n+1)$-rules. In practice $\mathbb{D}^n$ will contain the introduction rules for conjunction and disjunction [and existential quantification] (see Section 2.3). As the rules of $\mathbb{D}^n$ are considered definitional, they are expected to be conservative, i.e., they must not extend the class of primitive sentences, which are derivable in M^nK without using $\mathbb{D}^n$.[4] In practice this requirement is trivially fulfilled, as the introduction rules considered only generate non-atomic level-$(n+1)$-sentences, i.e., sentences containing the constant introduced.

Concerning atomic sentences,, the primitive inference rules of M^0K are just the primitive rules of K. For $M^{n+1}K$, Lorenzen extends the usual notion of a finitely specified calculus in favour of a system of which it is only required that every atomic sentence derivable in $M^{n+1}K$, when read as a level-$(n+1)$-rule, be admissible in M^nK (OL, p. 42). This means that we may add an admissible rule of M^0K (i.e., K extended with rules $\mathbb{D}^0$ for defined constants) as an axiom to MK, an admissible rule of MK as an axiom to M^2K etc. At first glance this looks like a set-theoretic closure notion of admissibility, at least for the implicational fragment without definitional extension, in which all sentences are atomic and thus have the form of rules: The rules admissible in K form the class of axioms of MK; the rules which do not properly extend this class, form the class of axioms of M^2K; and so on. However, even at higher levels, and even in the purely implicational case, Lorenzen wants to keep the idea that admissibility is tied to an operative elimination procedure. Therefore, $M^{n+1}K$ is to contain not only axioms but also proper inference rules (in addition to the definitional rules of $\mathbb{D}^{n+1}$), namely certain rules of which it has been shown beforehand that they lead from admissible level-n-rules to admissible level-n-rules. Once we have shown this for a particular rule (by means of an elimination procedure!), we may, so to speak, store this result as a primitive inference rule and establish further sentences of $M^{n+1}K$ by a formal derivation. For example, the admissibility of

(I) $A_1, \ldots, A_k \rightarrow A_i \quad (1 \leq i \leq k)$

is obvious for any calculus K, so all rules of this form can be taken as axioms of MK and therefore of any $M^{n+1}K$. Furthermore, since for Γ as $A_1, \ldots, A_k$,

$$(\Gamma \rightarrow B_1), (\Gamma \rightarrow B_m), (B_1, \ldots, B_m \rightarrow A) \vdash_K \Gamma \rightarrow A$$

holds for any K, we may use the rule

(II) $\Gamma \rightarrow B_1; \ \ldots \ ; \Gamma \rightarrow B_m; B_1, \ldots, B_m \rightarrow A \ \dot{\rightarrow} \ \Gamma \rightarrow A$

as a primitive inference rule of MK and therefore of any $M^{n+1}K$. Rules like (I) and (II), which are available for any calculus K, are called *universally admissible* (OL,

[4]Instead of the now common term "conservative", Lorenzen uses the term "relatively admissible".

pp. 42–44). Now we can, for example, establish die admissibility of the level-2-rule

$$A \to B_1; B_1, B_2 \to B \ \dot{\to}\ A, B_2 \to B$$

by formally deriving $A, B_2 \to B$ from $A \to B_1$ and $B_1, B_2 \to B$ in MK using (I) and (II) (OL, p. 43).

Hence, even if the meta-calculi are not fully formalized by specifying finite sets of axioms and rules, certain formal rules are available even in the implicational fragment, which allow for formal derivations. Meta-calculi are *open systems* in the sense that inference principles once justified may be used as formal rules.

So far, as formal rules of the meta-calculi, we have discussed definitional rules $\mathbb{D}^n$ and universally admissible inference principles like (I) or (II). Further rules result from the admissibility principles, especially from the the inversion principle. For example, if, for sentences A, B and C, K contains the rules

$$\begin{cases} A \to C \\ B \to C \end{cases}$$

then the inversion principle says that

$$A \to D, B \to D \vdash_K C \to D$$

which justifies to use the rule

$$A \to D; B \to D \ \dot{\to}\ C \to D$$

as a primitive rule of MK (and at the same time as an axiom of M^2K). In general, each admissibility principle discussed in Section 1.2 generates formal inferences in appropriate meta-calculi.

Viewed in this way, in a meta-calculus a variety of formal rules is available. Therefore, even though it is an open system, the protological admissibility principles are applicable to a meta-calculus directly. For example, the *deduction principle* holds without restriction: If in M^nK we can formally prove A from $A_1, \ldots, A_k$ (for level-n-rules alias M^nK-sentences $A_1, \ldots, A_k, A$), we have established the sentence $A_1, \ldots, A_k \to A$ as an axiom of $M^{n+1}K$. Even the *inversion principle*, though in general dependent on a fixed set of rules and therefore non-monotonic, can be used at the meta-level in certain cases, namely if we know that, in the open system, there cannot be further introduction rules for a sentence C beyond those of a certain form. This can be achieved by definition, e.g., when C is a conjunction or disjunction, so that the elimination inferences inverting the introduction rules for conjunction and disjunction are available at any level.

Furthermore, as a universal principle to generate formal level-$(n+1)$-rules from admissible level-n-rules, we may use that $\to A_1, \ldots, \to A_k \ \dot{\to} \to A$ is a derivable rule of $M^{n+1}K$ if $A_1, \ldots, A_k \to A$ is admissible in M^nK. This results from

$$\frac{\to A_1, \ldots, \to A_k \quad A_1, \ldots, A_k \to A}{\to A} \; ,$$

which is an application of the universally admissible rule (II), by removing the level-1-axiom (= admissible rule of K) $A_1, \ldots, A_k \to A$.[5] We call this the *principle of lifting*. In particular, since $\to A$ is admissible in $M^n K$ iff A is derivable in $M^n K$, $\to A$ is derivable in $M^{n+1}K$ iff A is derivable in $M^n K$. When we use the downwards direction of this law, we speak of *downlifting*.

Thus Lorenzen's much criticized doctrine of meta-calculi is perfectly sensible when a meta-calculus $M^{n+1}K$ is viewed as an open system, which not only collects admissible level-n-rules as axioms, but in which general admissibility principles can be used as formal rules of inference. Thus derivations in the open system $M^n K$ with respect to a definitional extension $\mathbb{D}$ proceed by the following primitive inference rules.

Primitive rules of $M^n K$:

(i) Every rule admissible in $M^n K$ can be chosen as an axiom of $M^{n+1}K$.

(ii) Every rule that leads from rules admissible in $M^n K$ to rules admissible in $M^n K$ can be chosen as a primitive inference rule of $M^{n+1}K$.

(iii) Every rule of $\mathbb{D}^n$ is a primitive rule of $M^n K$.

If clause (iii) is lacking, i.e., in the purely implicational case, then in every $M^{n+1}K$, all admissible rules of $M^{n+1}K$ may be considered to be derivable. However, as we have an open system, it is not intended that these inference rules are available as primitive rules. As primitive rules one would rather choose those rules, of which it has already been demonstrated that they lead from admissible rules to admissible rules of $M^n K$. Derivability and admissibility are distinct anyway at the ground level (i.e., in K), and also when a definitional extension is present. For example, when the definitional extension contains introduction rules for conjunction, the elimination rules for conjunction are available as primitive rules only at the next higher level (see Section 2.3).

Besides (I) and (II), Lorenzen considers the universally admissible rules of importation and exportation to be fundamental (OL, p. 46).

$$\text{(III)}\quad A_1; \ldots; A_m \;\dot{\to}\; A_{m+1}, \ldots, A_n \;\to\; A \quad \ddot{\to} \quad A_1; \ldots; A_{m-1} \;\dot{\to}\; A_m, \ldots, A_n \;\to\; A$$
$$A_1; \ldots; A_{m-1} \;\dot{\to}\; A_m, \ldots, A_n \;\to\; A \quad \ddot{\to} \quad A_1; \ldots; A_m \;\dot{\to}\; A_{m+1}, \ldots, A_n \;\to\; A$$

Based on our discussion, the following notions of a valid sentence and a valid consequence can be destilled out of Lorenzen's theory (Lorenzen himself does not use a concept of validity):

[5] This follows by using (II). Lorenzen only remarks that this rule is admissible in $M^{n+1}K$ (Lorenzen 1955, 44).

Lorenzen's concept of validity:

(i) *A level-n-sentence A is valid with respect to K and $\mathbb{D}$, if it is derivable in M^nK.*

(ii) *A level-n-sentence A is a valid consequence of level-n-sentences $A_1, \ldots, A_n$ with respect to K and $\mathbb{D}$, if $A_1, \ldots, A_n \rightarrow A$ is a valid level-$(n+1)$-sentence, i.e. a sentence derivable in $M^{n+1}K$ and therefore a rule admissible in M^nK.*

(iii) *A level-$(n+1)$-sentence is valid with respect to $\mathbb{D}$, if for every K it is valid with respect to K and $\mathbb{D}$, i.e., if A is universally admissible with respect to $\mathbb{D}$ as a level-$(n+1)$-rule.*

(iv) *A level-$(n+1)$-sentence is logically valid, if it is valid with respect to $\mathbb{D}$, where $\mathbb{D}$ contains exactly the introduction rules for the logical operators (see Section 2.3).*

(v) *A level-$(n+1)$-sentence is a valid sentence of positive implicational logic, if it is valid with respect to $\mathbb{D} = \emptyset$.*

This is not a formalistic definition of validity, as it is not reducible to derivability in a fully formalized system, but relates to open meta-calculi. It is not an exclusively semantical definition either, as it is based on fully formalized derivations on the ground level and rule-based definitional extensions at all levels. This position in between a formalistic way of thinking and an exclusively semantical approach puts Lorenzen's theory into a close relationship to proof-theoretic semantics in the Dummett-Prawitz-style (see Section 3).

Lorenzen finally arrives at the positive implicational calculus governed by the rules I–III, which is the intuitionistically accepted part of implicational logic. However, he cannot prove that there are no valid sentences beyond those derivable in positive implicational logic. He proves some sort of closure property of the system which is, however, different from semantic completeness[6]. So Lorenzen is faced with a completeness conjecture he cannot prove, quite similar to the one later made by Prawitz (1973, p. 246). It should also be noted that Lorenzen does not fully distinguish between implication as a connective and consequence as a relation between sentences. Consequence is an admissibility statement which at the same time can be asserted as a formal sentence. This is due to the fact that by means of a reflection step, any admissibility statement can be formally expressed at the next higher level as a sentence. This reflection step is not an inference step within a single system, which, as we shall argue, is a fundamental weakness of Lorenzen's approach (see Section 3.2).

2.2 Digression on Lorenzen and natural deduction

Since according to Lorenzen, establishing an implication always means a jump to a higher level, we are not allowed to pass *from* A to B in K whenever $A \rightarrow B$ is admissible. For example, if K is extended with $A, B \rightarrow A \wedge B$ as an introduction rule for

[6]Although Lorenzen claims that this property is near to completeness (OL, p. 49)

conjunction, the elimination $A \wedge B \rightarrow A$ is admissible in this extension; however, as a primitive inference rule, we are only entitled to use it at the meta-level. This means that

$$\frac{A \wedge B}{A}$$

would not be a valid step at the ground level. However,

$$\frac{\rightarrow A \wedge B}{\rightarrow A}$$

would be a correct step at the meta-level due to lifting. This shows that there are derivable rules of the form $\rightarrow A_1; \ldots; \rightarrow A_k \dot{\rightarrow} \rightarrow A$ at the meta-level, whose object-level counterparts $A_1, \ldots, A_k \rightarrow A$ are only admissible. The meta-level complements the object level by making admissible rules available (in 'lifted' shape) as formal derivation rules.

The fact that for every rule $A_1, \ldots, A_k \rightarrow A$ derivable (and therefore admissible) in K the rule $\rightarrow A_1; \ldots; \rightarrow A_k \dot{\rightarrow} \rightarrow A$ can be assumed as a primitive rule of MK, shows that any derivation $\mathcal{D}$ of A from $A_1, \ldots, A_k$ in K can be represented as a derivation of $\rightarrow A$ from $\rightarrow A_1, \ldots, \rightarrow A_k$ in MK. This means that even reasoning in natural deduction can be formally reconstructed within Lorenzen's framework, even though originally no formal inference rules are available that discharge assumptions. The introduction rule for implication can be made understandable in the meta-calculus by means of

$$\frac{\begin{array}{c}[\rightarrow A]\\ \vdots \\ \rightarrow B\end{array}}{A \rightarrow B} \tag{2}$$

Since $A \rightarrow B$ is admissible in K whenever $\rightarrow B$ is derivable from $\rightarrow A$ in MK, we can safely add (2) to MK and create a natural-deduction-style $\rightarrow$-introduction rule.

Using (2), and denoting by $\stackrel{m}{\rightarrow} A$ the m-fold iteration of the arrow, we can even associate with any natural-deduction derivation of A from $A_1, \ldots, A_k$ a derivation of $\stackrel{m}{\rightarrow} A$ from $\stackrel{m_1}{\rightarrow} A_1, \ldots, \stackrel{m_k}{\rightarrow} A_k$ in $M^n K$ for certain $m_1, \ldots, m_k, m, n$, where the iterations of the arrow not only equalize the (possibly different) levels of $A_1, \ldots, A_k, A$, but where the numbers $m_1, \ldots, m_k, m$ also depend on the levels of the sentences internally reached within the natural-deduction derivation.[7]

Therefore, it would be wrong to claim that Lorenzen's way of reasoning is entirely foreign to the idea of natural deduction. The basic difference is that in Lorenzen $A \rightarrow B$

[7]In fact, we might consider a system which switches between the different levels. In that case the arrow would have to be treated as a kind of modal operator. We would always be allowed to pass from $\rightarrow A$ to A, but from A to $\rightarrow A$ only if all assumptions are prefixed with $\rightarrow$.

means that the derivability of B follows from that of A (i.e., $\to B$ from $\to A$), not that B itself follows from A itself. This means that the semantical content of $A \to B$ is properly kept. Even according to Dummett and Prawitz, the semantical meaning of an open derivation of B from A consists in that a (valid) closed derivation of B can be generated given a (valid) closed derivation of A.

2.3 Conjunction and disjunction

Lorenzen's treatment of conjunction and disjunction (and existential quantification) proceeds by adding introduction rules for these connectives to a calculus K and showing that the corresponding elimination inferences are admissible. Assuming that $\wedge$ and $\vee$ do not occur as atome of K, then given the additional rules

$$\begin{cases} A, B \to A \wedge B \\ \quad A \to A \vee B \\ \quad B \to A \vee B \end{cases}$$

the rules

$$A \wedge B \to A$$
$$A \wedge B \to B$$

and

$$A \to C; B \to C \ \dot{\to} \ A \vee B \to C$$

are admissible. They are both straightforward applications of the inversion principle. For $\vee$ this is obvious, for $\wedge$ this becomes obvious if we infer $\wedge$-elimination from

$$A, B \to C \ \dot{\to} \ A \wedge B \to C$$

using the universally admissible rules $A, B \to A$ and $A, B \to B$, which are instances of (I). Combining these rules with the laws of consequence logic, the system of positive intuitionistic propositional logic is obtained.

It should be noted that these rules are available at any level. For example, if A, B, C, D are sentences of K, we can infer $A \to B \,\dot{\wedge}\, C \to D$ from $A \to B$ and $C \to D$ in MK, etc. This means in particular, as the conjunction sign is a fresh symbol at every level, that by means of inversion we obtain the elimination rule at this level. This is an example of an application of the inversion principle within a open (i.e., not fully formalized) system.

Besides conjunction and disjunction rules intrinsic to every level, there are lifted translations of the introduction and elimination rules of the lower levels available. In MK, from $\to A$ and $\to B$ we can infer both $\to A \,\dot{\wedge} \to B$ (by conjunction introduction in MK) and $\to A \wedge B$ (by conjunction introduction in K, lifted to MK). Conversely, from $\to A \wedge B$ we can infer both $\to A$ and $\to B$ (by lifting $\wedge$-elimination, which is admissible in M^0K, to MK) and thus $\to A \,\dot{\wedge} \to B$. From $\to A \,\dot{\wedge} \to B$ we do not obtain $\to A \wedge B$, as $\dot{\wedge}$-elimination in only admissible in MK but not derivable.

Lorenzen was the first to formulate an inversion principle in sufficient detail and to apply it to infer elimination rules from introduction rules. This is one of the lasting achievements of his *Operative Logic*.

2.4 Negation

As one would expect, negation is introduced via absurdity. Define a *falsum* sentence $\bot$[8] as a sentence such that $\bot \to A$ is admissible for every sentence A of K. Then the negation $\neg A$ of A is defined as $A \to \bot$. This is unambiguous as for any two *falsum* sentences $\bot$, $\bot'$, the rules $\bot \to \bot'$ and $\bot' \to \bot$ are admissible. Together with the laws of implication and the admissibility of *ex falso quodlibet*, we obtain intuitionistic propositional logic in the usual way.

Since a *falsum* sentence is by definition a sentence for which *ex falso quodlibet* holds, the underivability principle does not formally enter the theory of negation. In fact, Lorenzen presupposes that a *falsum* sentence is always available in the calculus considered, or that otherwise we could find a surrogate *ad hoc*[9]. However, if we have to construct a *falsum* sentence on the basis of the given calculus K, we would enlarge K by a fresh constant $\bot$ without an introduction rule. When applied to the empty set of introduction rules for $\bot$, the underivability principle establishes $\bot$ as a *falsum* sentence. By presupposing a *falsum* sentence to be given in the underlying calculus K, Lorenzen actually argues in favour of minimal logic with *ex falso quodlibet* being a property of K and not of its extension with logical constants.[10] This is even more surprizing as Lorenzen uses the underivability principle anyway, namely in his theory of equality (OL, pp. 89f.). As an explanation we can only speculate that for Lorenzen a symbol which does not occur in any rule does not have an operative meaning.

2.5 Classical logic

To interpret classical logic within his framework, Lorenzen uses an adaptation of Gödel's double-negation translation devised by E. Wette (OL, pp. 80–84). He shows that, when starting from a basic calculus K_0, all whose sentences A are *stable* in that the double-negation law $\neg\neg A \to A$ is valid, i.e., $A \to \bot \dot{\to} \bot \ddot{\to} A$ is admissible in M^2K, then the following holds. Suppose K is an extension of K_0 such that the heads of all primitive rules of K, which are not at the same time primitive rules of K_0, do not belong to K_0. (This trivially implies that K is a conservative extension of K_0.) Then

[8] Lorenzen uses the notation $\bigwedge$ (OL, 1st ed.) or $\curlywedge$ (OL, 2nd ed.), but does not give it a name.

[9] By the latter he means the conjunction of the propositions under consideration (see OL, pp. 63, 75).

[10] The careful reading of §8 of OL supports this view. There Lorenzen speaks of the "complete" reduction of negation to implication (OL, p. 75). On the other hand, in §5 he speaks of the underivability principle as the justification of *ex falso quodlibet* (OL, p. 37). So there is some tension between different views.

every sentence of K_0 derived in K by means of classical principles (e.g., by assuming the double-negation law $\neg\neg B \rightarrow B$ or the *tertium non datur* $B \vee \neg B$ for arbitrary sentences B of K) is already derivable in K_0, i.e., the classical extension of K is a conservative extension of K_0.

3 Lorenzen's approach and Dummett-Prawitz-style proof-theoretic semantics

Lorenzen's idea to justify certain inference rules by admissibility proofs, and in particular his inversion principle as a central tool, is closely related to Dummett's and Prawitz's approaches towards proof-theoretic semantics. Technically, these approaches are based on proof-theoretic ideas and results originally developed by Prawitz (1965) in the context of natural deduction and later extended and used by Tait (1967), Martin-Löf (1971), Girard (1971) and Prawitz (1971) in the context of logic and type theories, in particular in proofs of strong normalization. Philosophically, they received much underpinning within Dummett's (1991)[11] 'verificationist' theory of meaning. They can be traced back to Gentzen's remark that the introduction inferences in natural deduction may be viewed as definitions, and the elimination inferences as a sort of consequences thereof (Gentzen, 1934/35, p. 189). Prawitz (1965) used Lorenzen's term "inversion principle" to describe the relationship between introduction and elimination rules in general, not only for the cases of conjunction and disjunction [and existential quantification], in which Lorenzen used the inversion principle. In the following we compare Prawitz's notion of validity with Lorenzen's notions of admissibility and derivability in meta-calculi..[12] We do not include a discussion of Martin-Löf's approach, which is in the same spirit, since due to his distinction between proofs as acts and proofs as objects he adds an additional layer of understanding which is beyond the scope of our paper.[13]

3.1 Prawitz's definition of valdity

The following we give a simplified version of Prawitz's definition of validity for positive propositional logic (i.e., logic based on implication, conjunction and disjunction). Prawitz defined the validity of derivations with respect to atomic systems. We may

[11]Dummett (1991) is a convenient reference as it contains the essence of Dummett's logical semantics in monograph form. Dummett's basic papers on the philosophical basis of intuitionistic logic date back to the 1970s (see Schroeder-Heister 2005, which contains an extensive bibliography).

[12]For a detailed discussion of Prawitz's notion of validity and the notion of proof-theoretic semantics in general see Schroeder-Heister (2006) and Kahle & Schroeder-Heister (2006). As original papers by Prawitz, see, e.g. Prawitz (1973, 1974, 1978, 1985, 2006).

[13]It might be mentioned that Per-Martin Löf's definition of *computability* is closely related to Prawitz's definition discussed below. However, although it was conceived earlier, Prawitz's definition is not merely an adaptation of it. For a discussion of this issue see Schroeder-Heister (2006).

identify such an atomic system with a calculus in Lorenzen's sense, although we are aware that Prawitz from the very beginning uses a term-formula structure rather than an arbitrary word structure. This difference is not really essential when it comes to logic and logical constants. Here and in the following we speak of "derivations" throughout, even though these 'derivations' are intended to carry a meaning and to be epistemologically significant. The term "proof" will only be used in the context of metalinguistic considerations. As the concept of a *valid derivation* presupposes a neutral idea of *derivation* which includes invalid ones, by a derivation we understand a tree-like structure composed of arbitrary inference steps which, in the context of Prawitz-validity, may discharge assumptions. The introduction rules for the logical connectives are considered to be valid by definition, i.e., introduction rules are "self-justifying" in Dummett's terminology (Dummett 1991, p. 251). Furthermore, it is assumed that certain reduction procedures are available which reduce derivations to other derivations of the same end formula (from the same or less open assumptions). In a more elaborated form these procedures form a parameter of the definition of validity. Here we assume for simplicity that they are implicitly fixed. Typically, the standard reductions used in normalization proofs are assumed to be given, but other reductions can be thought of as well. A derivation using an introduction rule in the last step is called *canonical*.

Prawitz's definition of validity

(i) Every closed derivation in the atomic system is valid.

(ii) A closed canonical derivation is valid, if its immediate subderivations are valid.

(iii) A closed noncanonical derivation is valid, if it reduces to a valid canonical derivation.

(iv) An open derivation $\begin{array}{c} A_1 \ldots A_n \\ \mathcal{D} \\ A \end{array}$ *is valid, if for every list of closed valid derivations* $\begin{array}{c} \mathcal{D}_i \\ A_i \end{array}$ $(1 \leq i \leq n)$*, the derivation* $\begin{array}{c} \mathcal{D}_1 \quad\; \mathcal{D}_n \\ A_1 \ldots A_n \\ \mathcal{D} \\ A \end{array}$ *is valid.*

The following three basic features of this definition deserve to be pointed out.

(i) The definition applies to derivations, not just to rules. The validity of a rule $\dfrac{A_1 \;\ldots\; A_n}{A}$ results as a limiting case, if that rule is considered a one-step derivation with the open assumptions $A_1, \ldots, A_n$ and the end formula A. This rule is valid, if for all valid closed derivations $\begin{array}{c} \mathcal{D}_i \\ A_i \end{array}$ $(1 \leq i \leq n)$, the closed derivation

$\dfrac{\begin{matrix}\mathcal{D}_1 & & \mathcal{D}_n\\ A_1 & \ldots & A_n\end{matrix}}{A}$ is valid. So the validity of a rule is explained in terms of the validity of derivations, rather than the other way round, according to which a derivation would be valid if it consists of applications of valid rules. In that whole derivations and not rules are the primary focus, Prawitz's definition of validity is *global* rather than *local*.

(ii) Closed derivations are semantically prior to open ones. A valid open derivation of A from $A_1, \ldots, A_n$ is semantically interpreted as something that yields a valid closed derivation of A when supplemented with valid closed derivations of $A_1, \ldots, A_n$. This means that open assumptions in natural deduction derivations are interpreted as placeholders for closed derivations. If we call closed derivations *categorical* and open derivations *hypothetical*, we may speak of the priority of the categorical over the hypothetical.

(iii) The definition of validity rests on the distinction between canonical and non-canonical derivations. Canonical derivations are valid if they end with an introduction step applied to valid premiss derivations, whereas non-canonical derivations are valid, if they *reduce* to valid canonical derivations. For this *canonical vs. non-canonical* distinction it is crucial that a natural-deduction model of derivation is chosen, as only there a genuine introduction rule for implication is available.

Comparing Prawitz's approach with Lorenzen's, there are both similarities and differences.

Ad (i) Lorenzen's approach is definitely no global. The validity predicate assigned to him in Section 2.1 means the validity of a rule in the sense of its admissibility — or the derivability of this rule read as a sentence in a meta-calculus — rather than the validity of a derivation as a whole. When we described meta-calculi as open systems, we nevertheless understood them as something in which to proceed from axioms according to certain rules. So validity as derivability in a meta-calculus ultimately requires the justification of certain inference rules. However, this difference is not so big as it might appear at first glance. If we consider a rule a one-step derivation, then the Prawitz-validity of this derivation means essentially the same as admissibility: We have to demonstrate that it does not extend the class of valid closed derivations, i.e. that it is admissible as a rule with respect to this class.[14] Conversely, we could extend Lorenzen's definition of the admissibility of a rule to that of the admissibility of a derivation

[14]This is made clear in Moriconi & Tesconi (2008). See also Schroeder-Heister (2008b).

by considering any derivation $\begin{matrix} A_1, \ldots, A_n \\ \mathcal{D} \\ A \end{matrix}$ of A from $A_1, \ldots, A_n$ to be admissible, if the rule $A_1, \ldots, A_n \rightarrow A$ is admissible. In practice, when applying Prawitz's definition of validity, the relevant reduction cases are the standard elimination inferences, which are one-step derivations, and most of the more complicated derivations would, even in Prawitz conception, be handled as composed of single steps which are justified as separately. That a Prawitz-valid derivation establishes a Lorenzen-valid sentence or rule, and vice versa, is shown in the appendix.

Ad (ii) In Lorenzen, too, closed derivations are primary as compared to open derivations. That $A \rightarrow B$ is admissible in K means that any application of $A \rightarrow B$ can be eliminated from a closed derivation in $K + \{A \rightarrow B\}$. Otherwise the crucial distinction between admissible and derivable inference rules would break down. In this respect both Lorenzen and Prawitz unequivocally belong to the intuitionistic tradition of defining a construction in the categorical sense first and then define a hypothetical construction by the transmission of the categorical concept. This is, incidentally, also the position of classical semantics, in which the categorical concept of truth (in a structure) is defined first, and the hypothetical concept of consequence is defined by reference to the transmission of truth. An alternative position, which takes the concept of consequence primary, would go beyond both Lorenzen's and Prawitz's conception of validity, and also beyond the classical view of truth as the basis of consequence. Such ideas are developed in Hallnäs (1991, 2006), Schroeder-Heister (1993, 2004, 2008a) and Schroeder-Heister & Contu (2005).

Ad (iii) As the *canonical vs. non-canonical* distinction constitutes the fundamental difference between the two approaches, this point is dealt with in a separate section.

3.2 Implication and the *canonical vs. non-canonical* distinction

The distinction between canonical and non-canonical derivations, which figures prominently in Dummett-Prawitz-style semantics, has an analogue in Lorenzen's conception only in the cases of conjunction and disjunction [and existential quantification], but not in the case of implication [and negation and universal quantification]. For conjunction and disjunction proper introduction rules are given, and corresponding elimination inferences are established by applying the inversion principle. The split between conjunction/disjunction and implication is for Lorenzen one between proper and improper connectives and constitutes a difference between proper calculi and improper metacalculi. For Lorenzen, establishing an implication $A \rightarrow B$ means either a reflection step resulting from establishing the admissibility of the rule $A \rightarrow B$, or by inferring $A \rightarrow B$

in the meta-calculus using inference principles available there (such as the universally admissible rules like (I) – (III), or the inversion principle IP), which are themselves based on such reflection steps. Semantically, Prawitz's definition of validity is in the same spirit, as the validity of an open derivation is defined in a way corresponding to an admissibility statement. However, for Prawitz-validity it is crucial that for implication there is a *formal* canonical introduction step available in analogy to conjunction and disjunction, namely the formal step from an open derivation $\begin{array}{c}A\\ \mathcal{D}\\ B\end{array}$ to the assertion of $A \rightarrow B$. This is a *syntactical* step within an extension of the basic system, which is possible in a natural-deduction style framework permitting the discharging of assumptions. The syntactic application of the $\rightarrow$-introduction rule is different from any semantic validity (or admissibility) claim. From the standpoint of Prawitz's conception, Lorenzen's introduction of implication as an assertion in the meta-calculus passes directly to the semantical level. From the standpoint of Lorenzen's conception, Prawitz adds an intermediate syntactical step when defining the validity of a derivation of an implication.

There are at least three points Prawitz can claim in his favour.

(i) *Uniformity.* Implication is treated on par with conjunction and disjunction. Correspondingly, the inversion principle can be applied to generate the elimination inferences for all logical constants, whereas in Lorenzen it applies only to conjunction and disjunction [and existential quantification], which are the distinguished constants with explicit introduction rules.

Commentary. Uniformity is not a value in itself. Even according to Prawitz, the distinguished character of implication remains present in the way open derivations are handled. The natural-deduction-style introduction rule for $\rightarrow$ forces him to combine the validity of open derivations with that of closed derivations in a joint inductive definition, and to deal with the introduction rules at the same level as the rules justified as valid. Lorenzen's idea to interpret implication by a jump to the meta-calculus separates these levels. For him the rule $A \wedge B \rightarrow A$, as a rule justified by the inversion principle, would reside as an axiom only in the meta-calculus unlike the rule $A, B \rightarrow A{\wedge}B$ which 'defines' conjunction and is therefore already available in the basic definitional extension of K.

(ii) *Direct vs. indirect derivations.* A basic feature of Dummett-Prawitz-style proof-theoretic semantics is the distinction between direct and indirect closed derivations. A direct derivation arrives at its conclusion by means of a 'self-justifying' inference step and is thus called canonical, whereas an indirect derivation is one that reduces (= can be transferred) to such a form and is called non-canonical. The distinction between canonical and non-canonical derivations corresponds to the one between direct and indirect knowledge or evidence. This crucial epistemological distinction is lacking in Lorenzen in the case of implication.

Commentary. For implication a distinction between direct and indirect derivations can even be drawn within Lorenzen's conception in a natural way. An implication sentence $A \to B$ can be asserted (i) as an axiom of MK based on a demonstration of the admissibility in K of the rule $A \to B$, or (ii) by means of a formal derivation in MK using axioms and rules already shown to be valid. (i) is a direct derivation of $A \to B$ as it relies on a demonstration of admissibility, whereas (ii) is an indirect derivation as it relies on other admissibility demonstrations carried out beforehand, e.g., on proofs that certain rules are universally admissible. An indirect derivation according to (ii) has the feature that it reduces to a direct one according to (i) by spelling out the admissibility proofs relied on, thus arriving at an admissibility proof of $A \to B$. So the postulate that there be a *direct vs. indirect* distinction does not speak unequivocally in favour of Prawitz.

(iii) *Behaviourally significant knowledge of meaning.* Proof-theoretic semantics in the Dummett-Prawitz tradition is an epistemological approach to semantics, according to which we must distinguish between mere knowledge of meaning and inferences correctly drawn on the basis of this knowledge. This knowledge of meaning must manifest itself in observable behaviour. This manifestation is the correct application of introduction inferences. It presupposes that introduction inferences are solely structurally and thus syntactically specified, and therefore decidable patterns. This condition is not met in Lorenzen, as even the 'direct' establishment of an implication such as $A \wedge B \to A$ is not a syntactical step but rests on the insight into a (elimination) procedure.

Commentary. This is the decisive argument against Lorenzen's treatment of implication. As Lorenzen requires even of a most direct derivation of an implication sentence that an admissibility proof be carried out, there is no behaviourally significant way of showing that a reasoner knows the meaning of implication. The proper epistemological approach demands that a certain part of the semantically correct usage be behaviourally significant. Syntactically specified introduction rules satisfy exactly this requirement. In a Gentzen-style introduction rule for implication the conclusion $A \to B$ captures a structural feature of the deduction, namely that there is a derivation of B from A, still independent of whether this hypothetical derivation itself is valid, quite analogously to the fact that, e.g., the introduction rule for a conjunction $A \wedge B$ captures the structural feature that there is a pair of derivations, one of A and one of B, still independent of the validity of the derivations of A and B themselves. Even though in Dummett-Prawitz-semantics the validity of a hypothetical derivation is defined by reference to a transformation procedure, i.e., in a way closely related to admissibility, the canonical derivation of the corresponding implication is based on a separate syntactical step from a hypothetical derivation to a derivation of an implication. For example, though the validity of the one-step derivation

$$\frac{A \wedge B}{A}$$

rests on a transformation, establishing the implication $A \wedge B \to A$ rests on an additional canonical introduction step:

$$\dfrac{\dfrac{[A \wedge B]^{(1)}}{A}}{A \to B}\,(1)$$

Dummett-Prawitz-semantics is based on a subtle analysis of the epistemological content of a derivation which goes beyond the standard intuitionistic way embodied in BHK-semantics and in Lorenzen-style semantics. Like BHK-semantics, Lorenzen's semantics of implication is an attempt to give an epistemological semantics of implication without paying sufficient respect to the way knowledge of meaning manifests itself. Although Lorenzen puts much emphasis on the operative handling of syntactic figures, the manifestation-free intuitionistic epistemology is not overcome by Lorenzen.

3.3 Lorenzen's way out: Dialogue games

Lorenzen's turn away from operative logic to dialogical logic and game-theoretical semantics can be seen as a way out of these deficiencies. As dialogue semantics distinguishes between matters of games and matters of strategies, we can assign the knowledge of meaning to the game level and the validity of consequence statements to the strategy level. Someone who masters the syntactically specified game rules for the logical conntctives would be considered to know their meaning, whereas the availability of a strategy, which needs the handling of a constructive procedure (thus resembling admissibility), would be responsible for the logical facts which turn out to hold. There are some indications in Lorenzen's texts supporting this view.[15]

If this is correct, then both Dummett-Prawitz-style proof-theoretic semantics and Lorenzen-style dialogical semantics would be ways out of the shortcomings of the operative approach (and of the intuitionistic BKH-approach as well). Both Dummett-Prawitz semantics and dialogical semantics would be epistemological approaches doing justice to the requirement of a behavioural manifestation of meaning.

[15]See Lorenzen (1961) and Lorenz (2001).

4 Appendix: Prawitz-validity vs. Lorenzen-validity

First we define a translation which assigns to each sentence and rule of positive propositional logic a level-n-sentence in Lorenzen's sense, which essentially means equalizing unequal implication levels. Let $\overset{m}{\to}$ denote the m-fold iteration of the arrow, and let $\ell(A) := n$, if A is a Lorenzen-style level-n-sentence (i.e., $\ell(A)$ is the level of A).

Translation of sentences of positive propositional logic into Lorenzen-style sentences:

$A' := A$, *if* A *is atomic.*
$(A \,\square\, B)' \;:=\; \overset{m}{\to}\, A' \,\square\; \overset{n}{\to}\, B'$,
    where $m = max(0, \ell(B') - \ell(A')), n = max(0, \ell(A') - \ell(B'))$, *if* $\square$ *is* $\wedge$ *or* $\vee$.
$(A_1, \ldots, A_k \to A_{k+1})' \;:=\; \overset{m_1}{\to}\, A'_1, \ldots, \overset{m_k}{\to}\, A'_k \;\;\to\;\; \overset{m_{k+1}}{\to}\, A'_{k+1}$
    where $m_i = max(0, max(\ell(A'_1), \ldots, \ell(A'_{k+1})) - \ell(A'_i))$

Conversely, to each level-n-sentence in Lorenzen's sense we assign a sentence of positive propositional logic, which essentially means replacing the comma by conjunction and removing rules without premisses:

Translation of Lorenzen-style sentences into sentences of positive propositional logic:

$A^* := A$, *if* A *is atomic.*

$(A \,\square\, B)^* := A^* \,\square\, B^*$, *if* $\square$ *is* $\wedge$ *or* $\vee$.
$(A_1, \ldots, A_k \to A)^* := A_1^* \wedge \;\ldots\; \wedge A_k^* \;\to\; A^*$
$(\to A)^* := A^*$

We assume that the notions of validity are relativized to a fixed atomic calculus K and that the definitional extension $\mathbb{D}$ considered for Lorenzen-validity contains the introduction rules for conjunction and disjunction at every level, and nothing else.
Then we can show the following:

Theorem. For all sentences of positive propositional logic $A_1, \ldots, A_k, A$ *the following holds:*

(i) There is an open Prawitz-valid derivation $\begin{array}{c} A_1, \ldots, A_k \\ \mathcal{D} \\ A \end{array}$ *iff* $(A_1, \ldots, A_k \to A)'$ *is Lorenzen-valid.*

(ii) There is a closed Prawitz-valid derivation $\begin{array}{c} \mathcal{D} \\ A \end{array}$ *iff* A' *is Lorenzen-valid.*

Proof. We first remark that the second assertion can be viewed as a special case of the first one for empty $k = 0$, since $\to A$ is a Lorenzen-valid level-$(n+1)$-sentence [i.e., a rule admissible in $M^n K$] iff A is a Lorenzen-valid level-n-sentence [i.e., a sentence derivable in $M^n K$]. We prove (i) simultaneously by induction on the complexity of

$A_1, \ldots, A_k, A$. For a closed derivation of an atomic sentence, which is a derivation in K, nothing is to show.

Suppose A is a conjunction $A_1 \wedge A_2$, and a closed Prawitz-valid derivation of A is given. Then according to the definition of Prawitz-validity, this derivation reduces to one of A using $\wedge$-introduction in the last step. We apply the induction hypothesis to its premiss derivations and obtain Lorenzen-style derivations of A_1' and A_2' in appropriate meta-calculi. If they are of unequal levels, we use lifting to obtain derivations of $\stackrel{m_1}{\to} A_1'$ and $\stackrel{m_2}{\to} A_2'$ in the same meta-calculus, from which by $\wedge$-introduction in this meta-calculus we obtain A'. Conversely, if $A_1' \wedge A_2'$ is Lorenzen-valid, then there is a Lorenzen-style derivation in the appropriate meta-calculus using the $\wedge$-introduction rule in the last step. From its premiss derivations we obtain derivations of A_1' and A_2' (using downlifting if necessary). Application of the induction hypothesis gives us closed Prawitz-valid derivations of A_1 and A_2, and thus a closed Prawitz-valid derivation of A by means of $\wedge$-introduction. — If A is a disjunction, we argue analogously.

Suppose A is an implication $A_1 \to A_2$, and a closed derivation of A is given. Then this derivation is — or reduces to — a derivation using $\to$-introduction in the last step, with a Prawitz-valid derivation $\begin{array}{c} A_1 \\ \mathcal{D} \\ A_2 \end{array}$ as its premiss. By induction hypothesis we know that $(A_1 \to A_2)'$ is Lorenzen-valid. Conversely, suppose that $(A_1 \to A_2)'$ is a Lorenzen-valid level-$(n+1)$-sentence, i.e., that it is derivable in $M^{n+1}K$. We know that $(A_1 \to A_2)'$ has the form $\stackrel{m_1}{\to} A_1' \to \stackrel{m_2}{\to} A_2'$ for appropriate m_1 and m_2, i.e., that it is an atomic level-$(n+1)$-sentence, i.e., that it has the form of a level-$(n+1)$-rule. Therefore, its derivability in $M^{n+1}K$ implies that it is (as a rule) admissible in M^nK, i.e., we can transform every derivation (in M^nK) of $\stackrel{m_1}{\to} A_1'$ into one of $\stackrel{m_2}{\to} A_2'$. Since by induction hypothesis we know that, if there is a closed valid Prawitz-derivation $\begin{array}{c} \mathcal{D}' \\ A_1 \end{array}$, then A_1' is Lorenzen-valid, i.e. derivable in the appropriate metacalculus, we also know (by lifting) that, if there is a closed valid Prawitz-derivation $\begin{array}{c} \mathcal{D}' \\ A_1 \end{array}$, $\stackrel{m_1}{\to} A_1'$ is derivable in M^nK. Transforming this derivation into one of $\stackrel{m_2}{\to} A_2'$, and observing that this yields a derivation of A_2' in the appropriate meta-calculus (using downlifting, if necessary), by induction hypothesis we obtain a closed Prawitz-valid derivation of A_2. This yields a closed Prawitz-valid derivation of the form

$$\dfrac{\dfrac{[A_1]}{A_2}}{A_1 \to A_2}$$

where the one-step derivation from A_1 to A_2 is justified by the procedure just described.

Now suppose there is an open Prawitz-valid derivation $\begin{array}{c} A_1, \ldots, A_k \\ \mathcal{D} \\ A_{k+1} \end{array}$. To show that $(A_1, \ldots, A_k \to A_{k+1})'$ is Lorenzen-valid, we suppose that $(A_1, \ldots, A_k \to A_{k+1})'$ is a level-$(n+1)$-sentence and that derivations in M^nK of $\stackrel{m_1}{\to} A_1', \ldots, \stackrel{m_k}{\to} A_k'$ are given. By downlifting (if necessary), we know that there are derivations of $A'_1, \ldots, A'_k$ in appropriate meta-calculi, which means that $A'_1, \ldots, A'_k$ are Lorenzen-valid. By induction hypothesis there are closed Prawitz-valid derivations $\genfrac{}{}{0pt}{}{\mathcal{D}_1}{A_1}, \ldots, \genfrac{}{}{0pt}{}{\mathcal{D}_k}{A_k}$. Combining them with the open Prawitz-valid derivation which is given, yields a closed Prawitz-valid derivation of A_{k+1}, from which by induction hypothesis we conclude that A'_{k+1} is Lorenzen-valid. By applying lifting (if necessary), we obtain that $\overset{m_{k+1}}{\to} A'_{k+1}$ is Lorenzen-valid, i.e., derivable in M^nK. Conversely, suppose that $(A_1, \ldots, A_k \to A_{k+1})'$ is Lorenzen-valid. Then we know that, for some n, $(A_1, \ldots, A_k \to A_{k+1})'$ is an an atomic level-$(n+1)$-sentence of the form $\overset{m_1}{\to} A'_1, \ldots, \overset{m_k}{\to} A'_k \ \to \ \overset{m_{k+1}}{\to} A'_{k+1}$. Since it is derivable in $M^{n+1}K$, it is (as a rule) admissible in M^nK, i.e., we can transform every list of derivations in M^nK of the level-n-sentences $\overset{m_1}{\to} A'_1, \ldots, \overset{m_k}{\to} A'_k$ into a derivation in M^nK of $\overset{m_{k+1}}{\to} A'_{k+1}$. By applying the induction hypothesis (and lifting, if necessary), we obtain a procedure which transforms every list of closed Prawitz-valid derivations $\genfrac{}{}{0pt}{}{\mathcal{D}_1}{A_1}, \ldots, \genfrac{}{}{0pt}{}{\mathcal{D}_k}{A_k}$ into a closed Prawitz-valid derivation of A_{k+1}. Thus the open one-step derivation $\dfrac{A_1 \ \ldots \ A_k}{A_{k+1}}$ is Prawitz-valid.

Theorem. A level-n-sentence A is Lorenzen-valid iff for the sentence A^* there is a closed Prawitz-valid derivation.

Proof. This follows from the previous result by observing that for any level-n-sentence A, we have that $(A^*)'$ is the same as A modulo lifting, i.e., modulo replacing subsentences B with a lifted form $\overset{k}{\to} B$.

References

Aczel, P. (1977). An introduction to inductive definitions. In: J. Barwise (ed.), *Handbook of Mathematical Logic*, North Holland: Amsterdam, 739–782.

Curry, Haskell B. (1952). *Outlines of a Formalist Philosophy of Mathematics*, North-Holland: Amsterdam.

Dummett, M. (1991). *The Logical Basis of Metaphysics*, London: Duckworth.

Gentzen, G. (1935). Untersuchungen über das logische Schließen. *Mathematische Zeitschrift*, **39** (1934/35), 176–210, 405–431, English translation in: M.E. Szabo (ed.), *The Collected Papers of Gerhard Gentzen*, Amsterdam: North Holland 1969, 68–131.

Girard, Jean-Yves. (1971). Une extension de l'interprétation de Gödel à l'analyse, et son application à l'élimination des coupures dans l'analyse et la théorie des types. In: J.E. Fenstad (ed.), *Proceedings of the 2nd Scandinavian Logic Symposium (Oslo 1970)*, Amsterdam: North Holland, 63–92.

Hallnäs, L. (1991). Partial Inductive Definitions. *Theoretical Computer Science* **87**, 115–142.

Hallnäs, L. (2006). On the proof-theoretic foundation of general definition theory. In: R. Kahle & P. Schroeder-Heister (eds.), *Proof-Theoretic Semantics*. Special issue of *Synthese*, **148**(3), 589–602.

Hallnäs, L. & Schroeder-Heister, P. (1990/91) A proof-theoretic approach to logic programming. I. Clauses as rules. *Journal of Logic and Computation*, **1** (1990), 261–283; II. Programs as definitions, *ibid.* **1** (1991), 635–660.

Hermes, H. (1959). Zum Inversionsprinzip der operativen Logik. In: A. Heyting (ed.), *Constructivity in Mathematics: Proceedings of the Colloquium Held at Amsterdam, 1957*, Amsterdam: North-Holland, 62–68.

Kahle, R. & Schroeder-Heister, P. (eds.). 2006. *Proof-Theoretic Semantics*. Special issue of *Synthese*, **148**(3), 503–756. Introduction: Ibid., 503–506.

Lorenz, K. (1980). Logik, operative. Entry in: J. Ritter & K. Gründer (eds.), *Historisches Wörterbuch der Philosophie*, Vol. 5, Basel: Schwabe, 444–452.

Lorenz, K. (2001). Basic objectives of dialogue logic in historical perspective. *Synthese*, **127**, 255–263.

Lorenzen,P. (1950). Konstruktive Begründung der Mathematik. *Mathematische Zeitschrift*, **53**, 162–201.

Lorenzen, P. (1955). *Einführung in die operative Logik und Mathematik*. Springer: Berlin 1955, 2nd ed. Berlin 1969. (Referred to as "OL").

Lorenzen, P. (1959). Über die Begriffe "Beweis" und "Definition". In: A. Heyting (ed.), *Constructivity in Mathematics: Proceedings of the Colloquium Held at Amsterdam, 1957*, Amsterdam: North-Holland, 169–177.

[Lorenzen, P.] (1996). Bibliographie der Schriften von Paul Lorenzen. *Journal for General Philosophy of Science*, **27**, 187–202.

Lorenzen, P. (1961). Ein dialogisches Konstruktivitätskriterium. In: *Infinitistic Methods: Proceedings of the Symposium on Foundations of Mathematics (Warsaw 1959)*, Oxford: Pergamon Press and Warszawa: Państwowe Wydawnictwo Naukowe, 193–200.

Martin-Löf, Per. (1971). Hauptsatz for the intuitionistic theory of iterated inductive definitions. In: J.E. Fenstad (Hrsg.), *Proceedings of the 2nd Scandinavian Logic Symposium (Oslo 1970)*, Amsterdam: North Holland, 179–216.

Moriconi, E. & Tesconi, L. (2008). On inversion principles. *History and Philosophy of Logic* (in press).

Post, E. L. (1921). Introduction to a general theory of elementary propositions. *American Journal of Mathematics*, **43**, 163–185.

Prawitz, D. (1965). *Natural Deduction: A Proof-Theoretical Study*, Stockholm: Almqvist & Wiksell. Reprinted Mineola, N.Y.: Dover Publ., 2006.

Prawitz, D. (1971). Ideas and results in proof theory. In: J.E. Fenstad (Hrsg.), *Proceedings of the 2nd Scandinavian Logic Symposium (Oslo 1970)*, Amsterdam: North Holland, 235–308.

Prawitz, D. (1973). Towards a Foundation of a General Proof Theory, in: P. Suppes et al. (eds.), Logic, Methodology, and Philosophy of Science IV, Amsterdam, 225–250.

Prawitz, D. (1974). On the idea of a general proof theory, *Synthese* **27**, 63–77.

Prawitz, D. (1978). Proofs and the meaning and completeness of the logical constants. In: J. Hintikka et al. (eds.), *Essays on Mathematical and Philosophical Logic*, Dordrecht: Reidel, 25–40 (revised German translation in *Conceptus* **16** (1982)).

Prawitz, D. (1985). Remarks on some approaches to the concept of logical consequence, *Synthese* **62**, 152–171.

Prawitz, D. (2006). Meaning approached via proofs. In: R. Kahle & P. Schroeder-Heister (eds.), *Proof-Theoretic Semantics.* Special issue of *Synthese*, **148**(3), 507–524.

Schroeder-Heister, P. (1993). Rules of definitional reflection. In: *8th Annual IEEE Symposium on Logic in Computer Science (Montreal 1993).* IEEE Computer Society Press, Los Alamitos, 222–232.

Schroeder-Heister, P. (2004). On the notion of *assumption* in logical systems. In: R. Bluhm & C. Nimtz (eds.), *Selected Papers Contributed to the Sections of GAP5, Fifth International Congress of the Society for Analytical Philosophy, Bielefeld, 22-26 September 2003*, Paderborn: mentis, 27–48.

Schroeder-Heister, P. (2005). Dummett, Michael. Entry in: J. Mittelstraß (ed.), *Enzyklopädie und Wissenschaftstheorie*, Vol. 2, Stuttgart: Metzler, 258–262.

Schroeder-Heister, P. (2006). Validity concepts in proof-theoretic semantics. In: R. Kahle & P. Schroeder-Heister (eds.), *Proof-Theoretic Semantics*. Special issue of *Synthese*, **148**(3) (2006), 525–571.

Schroeder-Heister, P. (2007). Generalized definitional reflection and the inversion principle. *Logica Universalis*, **1**, 355–376.

Schroeder-Heister, P. (2008a). Proof-theoretic vs. model-theoretic consequence. In: *Logica 2007 Yearbook*. Prague: Czech Academy of Sciences (in print).

Schroeder-Heister, P. (2008b). Lorenzens operative Logik und moderne beweistheoretische Semantik. In: J. Mittelstraß (ed.), *Der Konstruktivismus in der Philosophie im Ausgang von Wilhelm Kamlah und Paul Lorenzen* (Beiträge zum Symposium in Konstanz, September 2005). Paderborn: mentis (in print).

Schroeder-Heister, P. & Contu, P. (2005). Folgerung. In: W. Spohn, P. Schroeder-Heister & E.J. Olsson (eds.), *Logik in der Philosophie*, Heidelberg: Synchron Wissenschaftsverlag, 247–276.

Stegmüller, W. (1973). Probleme und Resultate der Wissenschaftstheorie und Analytischen Philosophie, Band IV: Personelle und Statistische Wahrscheinlichkeit, 1. Halbband: Personelle Wahrscheinlichkeit und Rationale Entscheidung, Berlin: Springer.

Tait, William W. 1967. Intensional interpretations of functionals of finite type I. *Journal of Symbolic Logic* **32**. 198–212.

Thiel, C. (1996). Paul Lorenzen (1915-1994). *Journal for General Philosophy of Science*, **27**, 1–13.