Constructive semantics, admissibility of rules and the validity of Peirce's law

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Abstract

In his approach to proof-theoretic semantics, Sandqvist claims to provide a justification of classical logic without using the principle of bivalence. Following ideas by Prawitz, his semantics relies on the idea that logical systems extend atomic systems, so-called 'bases', with respect to which the validity of logically complex formulas is defined. We relate this approach to admissibility-based semantics and show that the latter significantly differs from the former. We also relate it to semantics based on the notion of construction, in which case the results obtained are essentially the same as Sandqvist's. We argue that the form of rules admitted in atomic bases determines which logical rules are validated, as is the fact of whether bases are conceived as information states, which can be monotonely extended, or as non-extensible inductive definitions. This shows that the format of atomic bases is a highly relevant issue in proof-theoretic semantics.

Keywords: Proof-theoretic semantics, constructive semantics, classical logic, intuitionistic logic, double negation law, Peirce's law, Peirce's rule, admissibility.

1 Introduction

In Dummett-Prawitz-style proof-theoretic semantics, the meaning of a proposition is given in terms of what conditions must be fulfilled in order to assert the proposition. If the condition to assert a proposition is the possession of a proof, then a constructive semantics requires a description of what are proofs of basic propositions and of what are proofs of logically complex propositions. The description is usually given as an inductive definition. Among others, Dummett [3] and Prawitz [12–14] are references for this approach (for an overview see [17, 18]). Recently, Sandqvist [16] proposed a semantics for logically complex propositions, which is closely related to constructive semantics, and which takes the form of an inductive definition too. It is of special interest as it claims to be a justification of classical logic without making use of the principle of bivalence.

We focus on the logical constant of implication. The implicational fragment $\{\supset\}$ of natural deduction for minimal logic *NM* (and also for intuitionistic logic *NI*) is given by the following introduction

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and elimination rules:

$$\dfrac{\begin{matrix}[\varphi]\\ \vdots \\ \psi\end{matrix}}{\varphi\supset\psi}\,(\supset\mathrm{I}) \qquad\qquad \dfrac{\varphi \quad \varphi\supset\psi}{\psi}\,(\supset\mathrm{E})$$

These rules also hold in the fragment $\{\supset\}$ of natural deduction for classical logic *NK*, although they do not suffice to obtain all classical theorems in this fragment. Classical principles such as Peirce's law $((\varphi\supset\psi)\supset\varphi)\supset\varphi$ are provable when one of the following two versions of *Peirce's rule* is added:

$$\dfrac{\begin{matrix}[\varphi\supset\psi]\\ \vdots \\ \varphi\end{matrix}}{\varphi} \qquad\qquad \dfrac{(\varphi\supset\psi)\supset\varphi}{\varphi}$$

Each version can be derived from the other by using $(\supset\mathrm{I})$ and $(\supset\mathrm{E})$. Let *NP* be *NM* plus (one version of) Peirce's rule. The addition of *ex falso quodlibet*

$$\dfrac{\bot}{\varphi}$$

to *NM* gives *NI*, and the addition of it to *NP* gives *NK*. Peirce's rule is not justifiable constructively, since its addition to *NI* allows to prove *tertium non datur*, which is normally rejected as being non-constructive (see Heyting [5, p. 103f.]).

2 Sandqvist's semantics for classical logic

Sandqvist [16] has proposed a semantic justification of classical logic by validating the rule of double negation elimination for the fragment $\{\supset,\bot,\forall\}$. Once double negation elimination is established, the other logical constants can be defined, and the justification becomes a justification for classical logic without using the principle of bivalence. Sandqvist considers bases B which are sets of basic sequents, i.e. relations between finite sets of basic sentences and basic sentences, where basic sentences are formulas containing neither logical constants nor free variables. In the following, basic sentences are also called atomic sentences, atomic formulas or atoms. We are not considering quantifiers, as they are not relevant to the arguments put forward here. This means that we can consider atoms to be propositional variables, and that we need not terminologically distinguish between sentences and formulas. The language of all bases is the same, i.e. the set of all atoms. A basis B can thus be viewed as a set of atomic inferences, i.e. of production rules

$$\frac{p_1 \quad \dots \quad p_n}{p_{n+1}} \tag{0}$$

where the p_i are atoms. Such rules are called *basic rules*. In the following, letters p,q, with and without indices, stand for atoms, φ,ψ,χ for formulas and Γ for finite sets of formulas (Sandqvist considers only finite sets of formulas, though infinite sets could be considered as well). For a given basis B, Sandqvist then defines inductively a relation $\Vdash_B$, expressing valid inferability, between finite sets Γ of sentences and sentences as follows (*ibid.*, p. 213; his Definition 2.1, we omit the clause for $\forall$):

Definition 2.1

$$\Vdash_B p \iff \textit{Every set of atoms closed under } B \textit{ contains } p \quad (1)$$

$$\textit{For non-empty } \Gamma\textit{: } \Gamma \Vdash_B \varphi \iff \Vdash_C \varphi \textit{ for every } C \supseteq B \textit{such that } \Vdash_C \psi \textit{ for every } \psi \in \Gamma \quad (2)$$

$$\Vdash_B \varphi \supset \psi \iff \varphi \Vdash_B \psi \quad (3)$$

$$\Vdash_B \bot \iff \Vdash_B p \textit{ for every atom } p \quad (4)$$

An inference

$$\frac{\varphi_1 \quad \dots \quad \varphi_n}{\varphi}$$

is valid for B if $\varphi_1,\dots,\varphi_n \Vdash_B \varphi$. It is considered logically valid if it is valid for every B. If $\Gamma \Vdash_B \varphi$ holds for every B, then we also speak of logical consequence, denoted by $\Gamma \Vdash \varphi$. A justification of the inference of double negation elimination consists thus in showing that this inference is a logical consequence, stated by Sandqvist in his main lemma (*ibid.*, p. 216, his Lemma 4):

Lemma 2.2
$(\varphi \supset \bot) \supset \bot \Vdash_B \varphi$ for any formula φ and basis B.[1]

This lemma is proved in three steps. First Sandqvist uses as another lemma (*ibid.*, p. 216, his Lemma 3) that the usual introduction and elimination rules for the fragment $\{\supset, \bot, \forall\}$ of minimal logic hold for the inferential relation $\Vdash_B$. Then he argues that, given this fact, it is sufficient to prove the result for atomic φ, as 'non-basic applications of double-negation elimination can be reduced to basic ones in the manner, *mutatis mutandis*, of Prawitz [11, pp. 39–40]' (*ibid.*, p. 216). Finally, which is the main part of the proof, he demonstrates that Lemma 2.2 holds for atomic φ.

As minimal logic together with the double negation law constitutes classical logic, this yields Sandqvist's main theorem (*ibid.*, p. 214, his Theorem 2):

Theorem 2.3
If φ is a classical consequence of Γ, then $\Gamma \Vdash \varphi$

which is seen as a justification of classical logic (*ibid.*, p. 214):

> Thus, whatever your attitude towards particular inferences among atoms, in so far as your use of logical compounds is governed by the semantics we have formulated, you have no choice but to accept all classically valid sentences and inferences.

Technically speaking, Sandqvist proves the semantic correctness of classical logic with respect to his semantics. This is highly significant, as Sandqvist's semantics is closely related to constructive semantics often proposed for intuitionistic logic (see Section 4).

2.1 Remark on intuitionistic versus classical disjunction

It should be emphasized that Theorem 2.3 applies without restriction only to the fragment of classical logic Sandqvist is considering, i.e. the fragment based on $\{\supset, \bot, \forall\}$. If we include disjunction (or analogously existential quantification), then the validity of double negation elimination can no longer

[1] More precisely, Sandqvist uses a consequence relation referring to substitutions for free variables, which is not significant in the context of propositional logic considered here.

be reduced to the atomic case, provided disjunction is given its intuitionistic interpretation according to the following semantical rule:

$$\Vdash_B \varphi \vee \psi \Longleftrightarrow \Vdash_B \varphi \text{ or } \Vdash_B \psi \tag{5}$$

In particular, Lemma 2.2 does not necessarily hold, if φ has the form $\chi \vee \neg\chi$ (where $\neg\chi$ stands for $\chi \supset \bot$). Thus Sandqvist has not given an example of a semantically valid law which is not derivable in intuitionistic logic, if disjunction is understood intuitionistically, since of a valid law we must expect that all its substitution instances, including those containing disjunction, are valid as well. Of course, if we understand disjunction $\varphi \vee \psi$ not in its intuitionistic sense (5) but in its classical sense by means of one of its equivalents, for example $\neg\varphi \supset \psi$, then Lemma 2.2 provides such an example. Sandqvist is fully aware of this fact and mentions this point explicitly (*ibid.*, p. 215).

This restriction concerning intuitionistic disjunction affects the significance of Sandqvist's result only marginally. The fact that Sandqvist's semantics validates the laws of classical logic for intuitionistically understood implication is a crucial point which is against basic intuitions of intuitionism, in particular, as he does not presuppose that atomic formulas behave classically, but instead proves that this is the case. Taking this into account we can interpret Sandqvist's result as follows: if disjunction is understood classically, then the intuitionistic semantics proposed in Definition 2.1 renders all classical laws valid. This is definitely not something an intuitionist or constructivist would accept and a very remarkable conceptual result. If one does not want to understand this result as a constructive justification of classical logic (this is how Sandqvist reads it), then it should be interpreted as exhibiting certain deficiencies of the underlying semantics (see Section 5 below).

3 Admissibility versus validity

In order to put Sandqvist's semantics into a wider perspective, we relate his notion of validity to the more common notion of admissibility. We show that Peirce's rule is universally admissible, i.e. admissible in every basis B, and point out the difference to validity in Sandqvist's sense.

3.1 Admissible rules and validity

A rule of the form

$$\frac{\varphi_1 \quad \ldots \quad \varphi_n}{\psi}$$

is *admissible* when it is guaranteed that there is a formal proof of the conclusion if there are formal proofs of the premisses. It is *derivable* if there is a derivation having as open assumptions no more than the premisses $\varphi_1, \ldots, \varphi_n$ of the rule and as endformula its conclusion ψ.

We will show that Peirce's rule is admissible in *NM*. An admissible but non-derivable rule can only be added to a given system of natural deduction if its application is restricted to premisses which do not depend on assumptions. Otherwise, Peirce's law would be provable in *NM* by implication introduction ($\supset$I):

$$\dfrac{\dfrac{[(\varphi \supset \psi) \supset \varphi]}{\varphi}\,(\text{Peirce's rule})}{((\varphi \supset \psi) \supset \varphi) \supset \varphi}\,(\supset\text{I})$$

It can easily be seen that for implications between atoms universal validity in Sandqvist's sense and universal admissibility coincide: Let $\vdash^0_B$ denote the derivability relation given by the basic rules of a basis B. Then Sandqvist's clause (1) can be equivalently reformulated as follows:

$$\Vdash_B p \Longleftrightarrow \vdash^0_B p$$

Let $\Gamma = \{p_1,\ldots,p_n\}$ be a set of atoms and q an atom. For the empty basis, Sandqvist's clause (2) implies

$$\Gamma \Vdash q \Longleftrightarrow \textit{For all bases } B\textit{: If } \vdash^0_B p_1,\ldots,\vdash^0_B p_n\textit{, then } \vdash^0_B q.$$

This means that $\Gamma \Vdash q$ holds if and only if the basic rule

$$\frac{p_1 \quad \ldots \quad p_n}{q}$$

is admissible in all bases B, a fact that we call *universal admissibility*. That is, for atoms and the empty basis the notion of validity as expressed by $\Vdash$ is equivalent to the notion of universal admissibility. Therefore, for the empty basis and for atoms p and q, we have

$$\Vdash p \supset q \Longleftrightarrow \textit{For all bases } B\textit{: If } \vdash^0_B p\textit{, then } \vdash^0_B q.$$

In other words, $p \supset q$ is universally valid if and only if the basic rule $\frac{p}{q}$ is universally admissible. Interestingly, admissibility is also a sufficient condition for the validity of some other sentences more complex than atomic implications $p \supset q$. We show this for the atomic case of Peirce's rule considered below. Let us denote by $\vdash_B \varphi$ that φ is derivable in *NM* extended with the rules of the basis B. We first show a completeness theorem for the derivability of atoms from assumptions which are either atoms or implications between atoms.

Theorem 3.1 (Atomic completeness)
Suppose $\varphi_1,\ldots,\varphi_n \Vdash_B p$, where each φ_i $(1 \le i \le n)$ is either an atom p_i or an implication $p_i \supset q_i$ between atoms. Then $\varphi_1,\ldots,\varphi_n \vdash_B p$.

Proof. We know that p is derivable in every extension C of B, for which $\Vdash_C \varphi_i$ holds for every i $(1 \le i \le n)$. Now consider the extension C' of B, which is obtained from B in the following way: For every i we add p_i as an axiom, if φ_i is an atom p_i, and we add

$$\frac{p_i}{q_i}$$

as a basic rule, if φ_i is an implication $p_i \supset q_i$. Then $\Vdash_{C'} \varphi_i$ for every i $(1 \le i \le n)$. Therefore p is derivable in C'. If we now replace every application of a rule

$$\frac{p_i}{q_i}$$

by an application of $(\supset$E)

$$\frac{p_i \quad p_i \supset q_i}{q_i}$$

we obtain a derivation of p from $\varphi_1,\ldots,\varphi_n$ over B. ∎

3.2 Admissibility of Peirce's rule

Instead of reconsidering the rule of double negation elimination, we concentrate on Peirce's rule

$$\frac{(\varphi \supset \psi) \supset \varphi}{\varphi} \tag{6}$$

and on Peirce's law $((\varphi \supset \psi) \supset \varphi) \supset \varphi$. This has the advantage that only the clauses (1), (2) and (3) of Definition 2.1 are relevant, whereas the justification of double negation elimination depends in addition on clause (4) for $\bot$. Thus any possible reservations one might have about clause (4) do not interfere with the following discussion. As a justification of Peirce's rule amounts to a justification of classical logic just as a justification of double negation elimination does, using Peirce's rule to this effect is not an undue choice (as Sandqvist points out himself; *ibid.*, p. 215).

In the context of natural deduction, concentrating on Peirce's rule and on Peirce's law means that only the fragment $\{\supset\}$ needs to be taken into account. Peirce's law is not derivable in the fragment $\{\supset\}$ of *NM*. Consequently, Peirce's rule is not derivable either.

Although Peirce's rule for atoms is not derivable in the fragment $\{\supset\}$ of *NM*, it can be shown to be admissible for any basis of production rules.[2]

Theorem 3.2 (Admissibility of Peirce's rule for atoms)
For all atoms p and q, and any basis B, if there is a closed derivation (i.e. a proof) of $(p \supset q) \supset p$ in the fragment $\{\supset\}$ of *NM* over B, then there is a closed derivation of p in this fragment.

Proof. (1) A closed derivation π of $(p \supset q) \supset p$ in the fragment $\{\supset\}$ of *NM* over any basis B can be transformed into a closed derivation π_1 in normal form, according to theorem 2 in Prawitz [11, chapter III, p. 40][3].
(2) A closed normal derivation of a non-atomic formula must use an introduction rule in the last step:

$$\pi_1: \quad \frac{\begin{array}{c}[p \supset q] \\ \vdots \\ p\end{array}}{(p \supset q) \supset p}\ (\supset \mathrm{I})$$

In case no assumption $p \supset q$ is discharged at the application of $(\supset\mathrm{I})$, we already have a closed derivation of p. Otherwise we assume that at least one occurrence of $p \supset q$ is discharged at the application of $(\supset\mathrm{I})$.
(3) The subderivation

$$\pi_2: \quad \begin{array}{c} p \supset q \\ \vdots \\ p \end{array}$$

of p in π_1 is also in normal form.

[2]This has been pointed out similarly for double negation elimination by Prawitz (unpublished notes of 2008 on Sandqvist's results).

[3]This theorem holds likewise for extensions by basic rules.

(4) The last rule application in π_2 is not an application of ($\supset$I). Thus π_2 has the form:

$$\pi_2:\quad \cfrac{\pi_3\left\{\begin{array}{c}\vdots\\ p\end{array}\right. \qquad p\supset q}{\left.\begin{array}{c} q\\ \vdots\\ p\end{array}\right\}\text{basic rules}}\ (\supset\text{E})$$

(5) Consider the subderivation π_3 having p as conclusion and let $n := 3$.
(6) If π_n is open, then the open assumption must be $p\supset q$. If π_n is closed, it is a proof of p. Thus π_n has the form:

$$\pi_n:\quad \cfrac{\pi_{n+1}\left\{\begin{array}{c}\vdots\\ p\end{array}\right. \qquad p\supset q}{\left.\begin{array}{c} q\\ \vdots\\ p\end{array}\right\}\text{basic rules}}\ (\supset\text{E})$$

(7) We increase n by 1 and go back to the beginning of (6). The procedure is repeated until we arrive at a closed derivation π_{3+m} of p, for $m \geq 1$, which must eventually be the case, since every derivation is a finite tree. ∎

For atoms, the universal validity of Peirce's rule follows from Theorems 3.1 and 3.2.

Theorem 3.3 (Validity of Peirce's rule for atoms)
For any atoms p and q, and any basis B, Peirce's rule is valid in B, i.e. $(p\supset q)\supset p \Vdash_B p$.

Proof. Suppose that for an extension C of B it holds that $\Vdash_C (p\supset q)\supset p$. By clause (3) we have $p\supset q \Vdash_C p$. From Theorem 3.1 we know that, if $p\supset q \Vdash_C p$, then $p\supset q \vdash_C p$, and therefore $\vdash_C (p\supset q)\supset p$. By Theorem 3.2 we have $\vdash_C p$. By clause (1) we have $\Vdash_C p$. Hence $(p\supset q)\supset p \Vdash_B p$. ∎

This means that we can obtain part of what is said in Lemma 2.2 and Theorem 2.3 by considerations concerning admissibility. In fact, the ideas used in the proofs of Theorems 3.1 and 3.2 are very similar to Sandqvist's proof of Lemma 2.2.

3.3 The difference between admissibility and validity

In spite of the fact that the results expressed in Theorems 2.3 and 3.2 overlap, admissibility and validity do not coincide. Even if we only consider implicational logic, then, unlike validity, the admissibility of Peirce's rule (6) cannot be lifted from the case, where φ and ψ are atomic, to the case, in which they are arbitrary formulas. In the case of validity, and thus in the proof of Lemma 2.2, one can use the fact that the standard introduction and elimination rules for implication preserve validity. However, this is not the case for admissibility. Here implication introduction ($\supset$I) does not necessarily preserve admissibility. If the step from φ to ψ is admissible with respect to a basis B (i.e. $\vdash_B \varphi$ implies $\vdash_B \psi$), then it is not necessarily the case that $\varphi\supset\psi$ is derivable in B (i.e. $\vdash_B \varphi\supset\psi$), if we do not have the restrictions of Theorem 3.1. For the derivability of $\varphi\supset\psi$ we would need a proof of ψ from φ rather than just a procedure that transforms a proof of φ into one of ψ.

That it is actually impossible to obtain a result corresponding to Theorem 2.3, but based on the notion of admissibility rather than validity, follows from [10]. There Mints shows that every implicational formula which is admissible for all its substitution instances is derivable in minimal logic. This means that a semantics based on the concept of admissibility justifies minimal rather than classical logic.[4]

The fundamental difference between validity and admissibility is that validity is iterated when it comes to nested implications and is understood as derivability only in the case of atoms. Roughly speaking, the validity of the rule

$$\frac{p_1 \supset p_2}{p_3 \supset p_4}$$

means that p_4 is derivable whenever p_3 is derivable, provided that p_2 is derivable whenever p_1 is derivable. The admissibility of this rule means that $p_3 \supset p_4$ is derivable whenever $p_1 \supset p_2$ is derivable, which is a conceptually different matter. In fact, already Lorenzen [7], who coined the concept of admissibility, uses an iteration of the admissibility concept in his notion of metacalculi, when it comes to the interpretation of implication. He thus framed his idea of iterated admissibility corresponding to notions of constructive validity.

4 Constructive semantics and incompleteness of minimal logic

Sandqvist's semantics resembles other forms of constructive semantics as put forward in the BHK-tradition. Prawitz [12, p. 278] gives a semantic clause for implication that has the same structure as Sandqvist's clauses (2) and (3) when combined. He defines *constructions of sentences* over a Post system **S** as basis by the following induction:

(i) k is a construction of an atomic sentence φ over **S** if and only if k is a derivation of φ in **S**.

(ii) k is a construction of a sentence $\varphi \supset \psi$ over **S** if and only if k is a constructive object of the type of $\varphi \supset \psi$ and for each extension $\mathbf{S}'$ of **S** and for each construction k' of φ over $\mathbf{S}'$, $k(k')$ is a construction of ψ over $\mathbf{S}'$.

As Post systems are given as sets of production rules, the bases used in Prawitz's semantics are the same as in Sandqvist's semantics. The noticeable difference to Sandqvist's semantics is that Prawitz's semantic clause (ii) for implication contains the additional requirement of having a *construction k* transforming any proof of φ into a proof of ψ in order to establish the validity of $\varphi \supset \psi$. The fact that a derivation in a Post system is considered to be a construction for an atom is a reason to consider a rule belonging to the basis as a construction; this is implicit in clause (i). As basic rules are one-step derivations, basic rules are constructions too.

Adding the requirement of having such a construction k to Sandqvist's clauses (1) and (2) yields the following clauses:

$$\Vdash_B p \text{ by a construction } k \iff \vdash^0_B p \text{ by a derivation } k \text{ in } B \tag{1'}$$

If $\Gamma = \{\psi_1, \ldots, \psi_n\}$ is non-empty: $\Gamma \Vdash_B \varphi$ by a construction k

$$\iff \Vdash_C \varphi \text{ by construction } k \text{ applied to constructions } k_1, \ldots, k_n \text{ for every } C \supseteq B \text{ such that } \Vdash_C \psi_i \text{ by construction } k_i \tag{2'}$$

[4]Note that Mints's result cannot be used to claim that the rule ($\supset$I) is preserving admissibility and thus can be used in a justification of Peirce's rule. Mints's result of the completeness of implicational logic with respect to admissibility does not imply that ($\supset$I) preserves admissibility for every basis B, but only that universal admissibility is preserved, i.e. that, whenever $\vdash_B \varphi$ implies $\vdash_B \psi$ for every B, then ψ is derivable from φ in minimal logic.

We prove that for the thus modified clauses Peirce's law is valid. This shows that Sandqvist's results and our comments on it have a more general range of application, pertaining to all constructive semantics that can be given this or a similar form.

THEOREM 4.1
For atomic p and q, $(p\supset q)\supset p \Vdash_B p$ holds constructively, for all bases B.

PROOF. Suppose that k' establishes $\Vdash_B (p\supset q)\supset p$, where B is a basis of production rules. We show that there is a construction k such that $k(k')$ establishes $\Vdash_B p$.

We consider the extension $B' = B \cup \left\{\frac{p}{q}\right\}$. Thus $\Vdash_{B'} p\supset q$. In other words, the basic rule $\frac{p}{q}$ is a construction k'' transforming proofs of p into a proof of q. By monotonicity (B' extends B) k' also establishes $\Vdash_{B'} (p\supset q)\supset p$. Therefore $\Vdash_{B'} p$ by construction $k'(k'')$. As p is atomic, we have $\vdash^0_{B'} p$ according to clause (1′) (this corresponds to Prawitz's clause (i)), i.e. the construction $k'(k'')$ is a derivation of p in basis B'.

There are two cases: First, if the derivation $k'(k'')$ does not use the basic rule $\frac{p}{q}$, then the derivation is already a derivation of p in B. Second, if the derivation $k'(k'')$ uses it, then there is a topmost occurrence of this rule, and the subderivation of its premiss is already a derivation of p in B. In both cases $\vdash^0_B p$ by a derivation k^* and, consequently, $\Vdash_B p$ by construction k^* (where k^* is $k(k')$) ∎

As the rules ($\supset$I) and ($\supset$E) can be validated by the semantics, Peirce's law $((\varphi\supset\psi)\supset\varphi)\supset\varphi$ is valid for arbitrary implicational formulas φ and ψ. Peirce's law is not valid for formulas containing constructive disjunctions (see Section 2.1).

5 Some critical remarks

Sandqvist's results can be read as a refutation of the constructivist claim that proof-theoretic or construction-theoretic semantics of a certain kind justifies intuitionistic or at least minimal logic as the adequate system of reasoning. Going even further, they say that classical logic might be more appropriate if we are prepared to understand disjunction (and existential quantification) in a classical sense (see Section 2.1). However, we would like to point to some presuppositions made by Sandqvist in his notion of validity (which pertain to related constructivist notions), whose modification leads to a different concept of validity which then does not necessarily lead to classical logic.

5.1 The notion of an atomic basis

Both Sandqvist's and constructivist approaches such as Prawitz's [12] consider an atomic basis to be given by a set of production rules of the form (0). However, one might consider atomic bases which have rules of the form

$$\frac{\begin{matrix}[\Gamma_1] \\ \vdots \\ p_1\end{matrix} \quad \cdots \quad \begin{matrix}[\Gamma_n] \\ \vdots \\ p_n\end{matrix}}{p_{n+1}} \tag{7}$$

where the Γ_i are (possibly empty) sets of atoms that can be discharged. Production rules are then the limiting case of all Γ_i being empty. Both Lemma 2.2 and Theorem 3.2 hold for the fragments $\{\supset\}$ resp. $\{\supset,\bot\}$ of *NM* over bases of production rules and do not generalize to bases with rules of the form (7). For example, if B consisted solely of the rule

$$\begin{array}{c}[p]\\ \vdots\\ \dfrac{q}{p}\end{array} \tag{8}$$

then there would be a closed derivation proving $(p \supset q) \supset p$ in the implicational fragment $\{\supset\}$ of NM over B, without p being derivable in B, so that the validity or admissibility of Peirce's rule could not be shown. The rationale for using atomic discharging rules of the form (7) would be that discharging rules are present in the logical system, implication introduction ($\supset$I) being an example. If this is a sensible type of rule, why should one not allow for this form of rules at the atomic level as well? This is at least an issue that needs further clarification. Sandqvist's argument in favour of classical logic discussed here is not applicable to such a revised notion of an atomic basis. However, we do not make the positive claim that the constructive semantics obtained by permitting atomic bases with discharge of assumptions is appropriate for minimal or intuitionistic logic.

It should be mentioned that the notion of validity needs to be revised if atomic discharging rules are admitted. For Γ_0 only containing atoms, we would in (1) define $\Gamma_0 \Vdash_B p$ to hold if there is a derivation of p from Γ_0 in B. In clause (2), for Γ_0 containing only atoms and Γ only non-atoms, we would define $\Gamma, \Gamma_0 \Vdash_B \varphi$ to hold, if $\Gamma_0, \Gamma_0' \Vdash_C \varphi$ holds for every set of atoms Γ_0' and every $C \supseteq B$ such that $\Gamma_0' \Vdash_C \psi$ for every $\psi \in \Gamma$.

5.2 Extensions of atomic systems and the definitional view

Clause (2) quantifies over all extensions C of the basis B in order to define hypothetical validity. This means that the definition closely resembles the definition of validity in a special Kripke model. Considering extensions is crucial for the proofs reported or given in this article. In the proofs of Lemma 2.2 and Theorem 3.2 one constructs an extension of a given basis by adjoining a certain rule (see, e.g. the proof of Theorem 3.1). The rationale behind the usage of extensions is normally that bases, similar to worlds or reference points in Kripke frames, represent some state of information. This information may increase, and consequents of hypothetical judgements are evaluated with respect to a possible increase in information which validates the antecedents. Considering extensions guarantees monotonicity: It can easily be proved that any consequence valid in B continues to be valid in every extension of B (see [17]).

However, this is not the only possible attitude towards bases one might take. Another option would be to consider the atomic rules in a basis to be definitions. Suppose in our basis we have rules which govern a predicate P rather than just propositional variables (to which we have here confined ourselves to simplify our exposition). These rules can be viewed as an inductive definition of P (see Aczel [1]). We do not want that this definition is extended with additional clauses, as this would change the definition. For example, if the natural numbers are defined by the rules saying that (i) 0 is a natural number and (ii) Nx is a natural number provided x is a natural number, then these clauses exhaust the definition of natural number. Adding clauses would change the definition of natural number to the definition of a different predicate. Giving a definition of a predicate is different from describing the information one has about it. With respect to a definition we do not want to have monotonicity. In the formulation of inductive definitions this is sometimes expressed in the form of an extremal clause: 'Nothing else defines P.' Formally, this idea can be captured in a framework with introduction and elimination rules for atomic predicates, as in the theory of iterated inductive definitions (see [8]) or in the theory of definitional reflection (see [4], [18]). This means that a basis is no longer just a system of rules for atoms, but a more complex reasoning system.[5]

6 Conclusion

There are various options in the definition of proof-theoretic validity, some of which would block Sandqvist's argument in favour of classical logic. There are other options not discussed here, for example the requirement that the extensions of bases must be consistent in some sense.[6] We do not want to argue for any of these options, but point to the fact that so far no convincing canonical proof-theoretic notion of validity has been proposed which exactly distinguishes intuitionistic or minimal logic, something which advocates of these logics such as Lorenzen, Dummett or Prawitz might have expected or have actually conjectured. Additional results (submitted for publication), which unlike Sandqvist include constructive disjunction, cast further doubts on the appropriateness of notions of proof-theoretic validity for this purpose.[7] In any case, it is absolutely crucial for proof-theoretic semantics how the notion of an atomic basis is defined. This is not so surprising after all, as it corresponds to the notion of a structure in model-theoretic semantics, in which formulas are evaluated. A different (and more negative) conclusion that might perhaps be drawn would be that in intuitionistic and minimal logic, the introduction and elimination rules for logical constants together with the inversion principle governing them (see [2]) should speak for themselves, without there being a further semantical justification in form of an external notion of validity (see [9, p. 25]).

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[5]Prawitz, in his proof-theoretic definitions of validity, has given up considering extensions of bases from 1973 on (which he still considered in his constructive semantics of 1971 sketched in Section 4). In recent personal communication, he has confirmed that the definitional view of rules for atomic predicates was one of the reasons for his change, though he has never formally specified the formalism for bases that would result from this view and that would have to be an ingredient of the theory of bases.—Lorenzen's [7] admissibility semantics of implication does not consider extensions, without adopting a definitional view. Interestingly, Lorenzen shows that in his semantics, Peirce's law can be justified if one admits classical means of reasoning in the metalanguage (*ibid.*, p. 52).

[6]This seems to be the presupposition underlying Litland's [6] approach.

[7]More precisely, it can be shown that Mints's rule $\dfrac{(\varphi\supset\psi)\supset(\varphi\vee\chi)}{((\varphi\supset\psi)\supset\varphi)\vee((\varphi\supset\psi)\supset\chi)}$ is valid, while being non-derivable in *NI*.

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