Chapter 15
Harmony in Proof-Theoretic Semantics: A Reductive Analysis

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Abstract We distinguish between the *foundational analysis* of logical constants, which treats all connectives in a single general framework, and the *reductive analysis*, which studies general connectives in terms of the standard operators. With every list of introduction or elimination rules proposed for an n-ary connective c, we associate a certain formula of second-order intuitionistic propositional logic. The formula corresponding to given introduction rules expresses the *introduction meaning*, the formula corresponding to given elimination rules the *elimination meaning* of c. We say that introduction and elimination rules for c are in *harmony* with each other when introduction meaning and elimination meaning match. Introduction or elimination rules are called *flat*, if they can discharge only formulas, but not rules as assumptions. We can show that not every connective with flat introduction rules has harmonious flat elimination rules, and conversely, that not every connective with flat elimination rules has harmonious flat introduction rules. If a harmonious characterisation of a connective is given, it can be explicitly defined in terms of the standard operators for implication, conjunction, disjunction, falsum and (propositional) universal quantification, namely by its introduction meaning or (equivalently) by its elimination meaning. It is argued that the reductive analysis of logical constants implicitly underlies Prawitz's (1979) proposal for a general schema for introduction and elimination rules.

Dag Prawitz's (1979) article was one of the main starting points of my Dr. phil. thesis, whose external examiner Dag became in 1981. His work on proofs and meaning has been a great source of inspiration to me ever since. I am extremely happy to be able to contribute the present paper to this volume dedicated to his work. It may be read as an extended commentary which further pursues the approach Dag initiated with his article.—This work was carried out within the French-German ANR-DFG project "Hypothetical Reasoning: Its Proof-Theoretic Analysis" (HYPOTHESES) (DFG Schr 275/16-2). I should like to thank Luca Tranchini for very helpful discussions on the topic of this paper, and Thomas Piecha and Dag Prawitz for many helpful remarks. I am also grateful to an anonymous reviewer of the *Review of Symbolic Logic* for comments and suggestions on the paper by Olkhovikov and the author (2014a) that have been useful for the revision of the current paper.

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15.1 Introduction: Reductive Proof-Theoretic Semantics

The proof-theoretic semantics of logical constants is predominantly concerned with the meaning of the standard connectives, which in intuitionistic propositional logic are implication ($\to$), conjunction ($\wedge$), disjunction ($\vee$) and absurdity ($\bot$) (see Schroeder-Heister 2012a). Even if we confine ourselves to intuitionistic logic, and here to the propositional case, this is a severe limitation. It is natural to ask how one should deal with arbitrary n-ary propositional connectives. For example, in a natural deduction framework, one should discuss what introduction and elimination rules for such connectives look like, and what it means that these rules are in harmony with each other, a requirement standardly made in proof-theoretic semantics. These questions will be the subject of this paper. The seminal paper on this topic within the framework of natural deduction is Prawitz (1979).[1]

Unlike Prawitz, we shall not try to formulate a general schema for elimination rules given certain introduction rules. We do not attempt the reverse procedure either—starting from eliminations and trying to formulate a general schema for introductions. We shall rather propose general schemas both for introduction and for elimination rules, and then formulate a criterion that tells when such rules are in harmony with each other. This criterion will not be based on the syntactic form of these rules but on their content. Certain elimination rules will be harmonious with given introduction rules not because they have a specific form which is developed from that of the introduction rules, or vice versa. We shall instead associate with each set of introduction rules for a connective c the introduction meaning c^I of c according to these rules, which describes the content of the introduction rules. Likewise, with each set of elimination rules for c we shall associate the elimination meaning c^E of c according to these rules, which describes the content of the elimination rules. We call the introduction and elimination rules for c harmonious when c^I and c^E are equivalent.

This presupposes that we have a language at our disposal in which we can express introduction and elimination meanings c^I and c^E, and a deductive system in which we can establish their equivalence. As such a language and system we use intuitionistic propositional logic **PL**, sometimes including universal propositional quantifiers, i.e., second-order intuitionistic propositional logic **PL2**. The introduction and elimination meanings c^I and c^E are formulas of **PL** (and **PL2**, respectively). This means that we take the logic of the standard intuitionistic operators for granted and explain the meaning of arbitrary n-ary connectives with respect to them. Therefore we call this approach a *reductive* analysis of logical constants.

[1] For the framework of the sequent calculus, the seminal paper is von Kutschera (1968), most results of which can be carried over to the natural-deduction framework.

Being reductive, our approach differs from *foundational* approaches, according to which the introduction and elimination rules for arbitrary n-ary connectives as well as those for the standard operators are accommodated in a single basic framework. Such foundational approaches, which have been proposed, for example, by von Kutschera (1968) and Schroeder-Heister (1984), carry a certain conceptual and notational overhead with them, which is not needed for all purposes. Many results can be established within the reductive framework which nonetheless pertain to the foundational frameworks. The results on flattening mentioned below are of this kind. A foundational approach corresponding to the reductive one of this paper is carried out in Schroeder-Heister (2014a).

Based on a schema S that describes the general form of introductions, and another schema S' that describes the general form of eliminations, each for an n-ary constant c, and using the introduction and elimination meanings c^I and c^E of c with respect to these schemas, we can ask questions such as the following:

1. Given certain introduction rules for c satisfying the schema S, and certain elimination rules for c satisfying the schema S', are these introduction and elimination rules in harmony with each other? If they are not in harmony with each other, when do they guarantee at least conservativeness? When do they guarantee the uniqueness of c?
2. Given certain introduction rules for c satisfying the schema S, are there elimination rules for c satisfying the schema S', such that these introduction and elimination rules are in harmony with each other?
3. Conversely, given certain elimination rules for c satisfying the schema S', are there introduction rules for c satisfying the schema S, such that these intruduction and elimination rules are in harmony with each other?

Answers to these questions are facilitated by the great technical advantages of our reductive approach. As we can express the strengh of introduction and elimination rules directly in terms of propositional formulas without referring to the rules themselves, we gain access to the apparatus of standard (second-order) propositional logic with all its well-established methods. We can use such methods to prove results about the possible forms of formulas intuitionistically equivalent to c^I or to c^E. This enables us in particular to establish negative results about the shape of harmonious introduction and elimination rules, i.e., results telling us that rules of certain restricted forms are *not* appropriate as introduction or elimination rules. This is important for the discussion of so-called "general" elimination rules in the sense of Dyckhoff, Tennant, Lopez-Escobar and von Plato (see Schroeder-Heister 2014b). More generally, these results constrain the structure of introduction and elimination rules in the foundational framework with rules of higher levels (Schroeder-Heister 1984).[2] This is due to the fact that entities in the foundational framework such as higher-level rules have an implicational counterpart in standard propositional logic, so that negative results about the latter carry over to the former. Our central result here are the non-flattening theorems, according to which there are not always elimination rules in

[2] And, correspondingly, those of von Kutschera's (1968) framework.

harmony with given introduction rules, or introduction rules in harmony with given elimination rules, which are *flat* in the sense that only formulas (and not 'higher' entities such as rules) can be discharged as assumptions.[3]

Traditionally, in investigations of n-ary connectives, proof-theoretic semantics has been concerned with the question: Given introduction rules of a certain form, how do we have to frame elimination rules such that they are in harmony with the introductions? This question is certainly related to our questions above (in particular to the second one), but covers only part of the problem. The formulation of harmonious elimination rules which are obtained as unique syntactic functions of introduction rules[4] gives us a certain principle of harmony. However, it cannot tell us anything about the relation between introductions and eliminations which are not of the form considered. A connective such as Prior's (1960) *tonk* would simply be ill-defined, as its given elimination rules are stronger than the harmonious elimination rule based on its given introduction rule. In our reductive framework, which is completely symmetric with respect to introduction and elimination rules (as it starts with independent general schemas for both of them), *tonk* has a well defined introduction meaning $tonk^I$ as well as a well-defined elimination meaning $tonk^E$ whose mutual relationship we can investigate (see below Sect. 15.5 and Table 15.3).

In Sects. 15.2–15.5 we give independent schemas for introduction and elimination rules for n-ary propositional operators c and define and characterize introduction and elimination meanings c^I and c^E with respect to them. We define harmony in terms of c^I and c^E and relate it to Belnap's (1962) criteria of conservativeness and uniqueness. We then report some positive and negative results concerning the form of harmonious elimination and introduction rules starting from introduction or elimination rules, respectively. The negative results mainly concern the fact of whether rules can be flattened in a certain way.

When introduction and elimination meanings c^I and c^E of c coincide, then c can be defined by either c^I or c^E. This relates our result to the investigation of the functional completeness of logical connectives. Here, in the intuitionistic framework, "functional completeness" means the expressive completeness in analogy to truth-functional completeness in classical logic.[5] We shall deal with this topic in Sect. 15.6. In Sect. 15.7 we relate our approach to Prawitz's (1979) paper. We argue that his approach is best regarded as a reductive rather than a foundational approach, even if it was not intended as such.

The final Sect. 15.8 sketches some problems a more foundational approach faces in comparison to the reductive approach advocated here. It is claimed that modus

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[3] The term 'flattening' has been coined by Read (see Read 2014, this volume).

[4] Already Gentzen spoke of eliminations as "functions" of introductions in the frequently quoted passage of Gentzen (1934/35, p. 189) that Prawitz (1965) first drew attention to.

[5] It is convenient to use the term "functional completeness" to distinguish this matter from semantic completeness, which is an entirely different issue. The term "functional" is definitely not perfect, but evokes the right associations. One might think of rules as transforming proofs into proofs, and therefore of "proof functions", in the intuitionistic case.

ponens and the two projections as elimination rules for implication and conjunction, respectively, are fundamental and cannot be superseded by more general rules.

15.2 Introduction and Elimination Rules

We consider introduction and elimination rules for n-ary connectives c in a natural deduction framework. As the general form of an introduction rule for c we propose the following:

$$(c\mathrm{I}) \quad \frac{\begin{matrix}[\Gamma_1] & & [\Gamma_\ell] \\ s_1 & \ldots & s_\ell\end{matrix}}{c(p_1, \ldots, p_n)} \quad , \tag{15.1}$$

where $s_1, \ldots, s_\ell$ are propositional variables and the Γ_i are (possibly empty) lists of propositional variables, which can be discharged at the application of (cI). As a limiting case we allow for $\ell = 0$ (which covers the case of the truth constant $\top$). All propositional variables occurring in the rule must be among $p_1, \ldots, p_n$. Schema (15.1) corresponds to the schema proposed in Prawitz (1979).

Evidently, the introduction rules for the standard intuitionistic connectives $\wedge, \vee, \rightarrow$

$$\frac{p_1 \quad p_2}{p_1 \wedge p_2} \qquad \frac{p_1}{p_1 \vee p_2} \qquad \frac{p_2}{p_1 \vee p_2} \qquad \frac{\begin{matrix}[p_1] \\ p_2\end{matrix}}{p_1 \rightarrow p_2}$$

fall under this schema, with ℓ being 2 in the case of conjunction and 1 in the case of disjunction and implication, and with the Γ_i being empty in the case of conjunction and disjunction, and Γ_1 consisting just of p_1 in the case of implication.

Our schema for introduction rules is quite restricted. We do not, for example, allow for any connective occurring above the inference line. This means that we cannot, for example, characterize negation by an introduction rule referring to absurdity in its premiss

$$(\neg\, \mathrm{I}) \quad \frac{\begin{matrix}[p_1] \\ \bot\end{matrix}}{\neg p_1} \quad .$$

However, for the point we want to make our schema is sufficient. It is easy to extend it to the case where connectives are introduced in a certain order, where an 'earlier' connective can be used to define a 'later' one.[6] The fact that we do not consider 'extra

[6] Prawitz (1979) works in a more general context, allowing for dependencies between connectives. More involved is the problem of self-referential operators, the premisses of whose introduction rules may contain the operator being introduced. We do not discuss this problem here (see Schroeder-Heister 2012b). Tranchini (2014) has pointed out that our reductive approach is not capable of dealing with this sort of phenomena, and that a notion of 'rule equivalence' is needed, in contradistinction to our approach which according to Tranchini is based on 'formula equivalence'. A definition of harmony which is nearer to the level of rules and which would correspond to this notion of rule

variables' beyond $p_1, \ldots, p_n$ in the premisses of (cI) is another restriction, which is not relevant for the points we want to make here. We should like to remark, however, that introduction rules with extra variables in premisses are a neglected topic in proof-theoretic semantics. They represent an interesting and significant extension of the means of expression available, which corresponds to introducing existentially understood variables into the meaning of connectives.

There may be more than one introduction rule of the form (15.1) for c (as it is the case with disjunction). We assume that they are given as a finite list, where, as a limiting case, the empty list of introduction rules is permitted. This covers the absurdity constant $\bot$, which has no introduction rule. A connective which plays a prominent role in our investigations is the ternary operator $\star$ with the following two introduction rules

$$(\star\,\mathrm{I}) \quad \frac{\begin{array}{c}[p_1]\\ p_2\end{array}}{\star(p_1, p_2, p_3)} \qquad \frac{p_3}{\star(p_1, p_2, p_3)} \; . \tag{15.2}$$

As our general schema for an elimination rule for an n-ary connective c we propose the following:

$$(c\mathrm{E}) \quad \frac{c(p_1, \ldots, p_n) \qquad \begin{array}{c}[\Gamma_1]\\ s_1\end{array} \quad \ldots \quad \begin{array}{c}[\Gamma_\ell]\\ s_\ell\end{array}}{q} \; , \tag{15.3}$$

where $s_1, \ldots, s_\ell, q$ are propositional variables and the Γ_i are (possibly empty) lists of propositional variables. $c(p_1, \ldots, p_n)$ is called the *major premiss* of (cE), the remaining premisses are called the *minor premisses* of (cE). We allow for $\ell = 0$, in which case minor premisses are lacking. We do not impose any restriction on the propositional variables occurring in (cE). They may (and will normally) comprise $p_1, \ldots, p_n$, but any number of propositional variables beyond $p_1, \ldots, p_n$ may be present. This generalizes the fact that in elimination rules such as $\vee$-elimination

$$(\vee\,\mathrm{E}) \quad \frac{p_1 \vee p_2 \qquad \begin{array}{c}[p_1]\\ r\end{array} \qquad \begin{array}{c}[p_2]\\ r\end{array}}{r} \tag{15.4}$$

the additional propositional variable r is used as minor premiss and conclusion. Our motivation for proposing (15.3) as elimination schema is that we should be able to choose anything whatsoever as possible consequence of $c(p_1, \ldots, p_n)$, which means that the minor premisses and the conclusion should not be constrained in any way. The lack of this restriction makes our schema more general than elimination schemas derived from given introduction rules such as those in Prawitz (1979) and

equivalence, is proposed in Olkhovikov and Schroeder-Heister (2014b). In the German translation of Prawitz (1979), Prawitz himself considers (or at least mentions the possibility of) self-referential connectives, for example a connective defined in terms of its own negation.

Schroeder-Heister (1984) where $s_1, \ldots, s_\ell, q$ are identical, i.e., represented by a single variable r.[7]

It is absolutely crucial to realize that we are formulating (15.3) as an independent schema in its own right, i.e., without any reference to potential introduction rules for c. (15.3) is our general schema for an arbitrary elimination rule, not a general schema for elimination rules given certain introduction rules. All proposals for general elimination schemas that can be found in the literature on proof-theoretic semantics consider a schema that is generated from introduction rules given beforehand, thus (explicitly or implicitly) following Gentzen's (1934/35) idea that elimination rules are "functions" of introduction rules.

Evidently, the elimination rules for the standard intuitionistic connectives $\wedge$, $\vee$, $\to$, $\bot$ are of the form (15.3): The rule of $\vee$-elimination has just been stated. In the two $\wedge$-elimination rules

$$(\wedge \mathrm{E}) \quad \frac{p_1 \wedge p_2}{p_1} \qquad \frac{p_1 \wedge p_2}{p_2} \tag{15.5}$$

ℓ is zero and q is p_1 or p_2, respectively. In modus ponens

$$(\to \mathrm{E}) \quad \frac{p_1 \to p_2 \qquad\quad p_1}{p_2} \tag{15.6}$$

we have that ℓ is 1, s_1 is p_1 and q is p_2, with Γ_1 being empty. In the case of absurdity, the rule of *ex falso quodlibet*

$$(\bot \mathrm{E}) \ \frac{\bot}{q} \tag{15.7}$$

leaves q unchanged with ℓ being zero. Note that a general schema for elimination rules in which $s_1, \ldots, s_\ell, q$ are identical, cannot accommodate these rules.

There may be more than one elimination rule for c, as is the case with conjunction. We suppose that elimination rules for a connective c are given as a finite list, where, as a limiting case, we allow for the empty list of elimination rules. This covers the verity constant $\top$, which has no elimination rule.

Our elimination schema (cE) is restricted in a way similar to the introduction schema (cI): No operators are permitted to occur in it except the c in the major premiss. This restriction could be released, but in its given form the schema is sufficient for the points we want to make. Note, however, that in the elimination case, there is no restriction corresponding to the 'no-extra-variable' constraint. If we disallowed extra variables beyond $p_1, \ldots, p_n$ in (cE), we would not be able to formulate, e.g., the elimination rule for disjunction. This means that for the topic discussed in this paper it is crucial to consider variables in elimination rules which are understood universally. When translating elimination rules into standard logic, this will lead us to use not just intuitionistic propositional logic **PL**, but intuitionistic propositional logic with universal propositional quantification **PL2**.

[7] However, it is less general than these other schemas in that here Γ_i may only contain propositional variables.

A connective which will play a prominent role in the following, is the ternary operator $\circ$, which has a single elimination rule:

$$(\circ\,\mathrm{E})\quad \frac{\circ(p_1, p_2, p_3) \qquad \begin{matrix}[p_1]\\ p_2\end{matrix}}{p_3}\;. \tag{15.8}$$

15.3 Introduction Meaning and Elimination Meaning

In what follows, we take intuitionistic propositional logic based on the standard connectives $\wedge, \vee, \rightarrow, \bot$ for granted. We call this system *standard propositional logic* **PL**. We use formulas of this logic to express the intended meaning of n-ary connectives c, for which introduction or elimination rules are given. This means that we do not deal with the justification of the introduction and elimination rules for these standard connectives, nor with inversion and harmony principles and the like for them. Our enterprise is of a *reductive* kind, reducing problems associated with arbitrary n-ary connectives c to problems that only have to do with the standard connectives. If introduction and/or elimination rules for c are specified, by **PL+cI** we denote the system **PL** extended with the introduction rules for c, by **PL+cE** the system **PL** extended with the elimination rules for c and by **PL+cIE** the system **PL** extended with both the introduction and elimination rules for c. When we consider derivability in any of these systems, it will always be clear from the context, from which language the formulas involved are drawn, e.g., whether they contain c or not.

Suppose an introduction rule (cI) for an n-ary connective c is given according to (15.1). Then the intended meaning c^I of c according to this introduction rule, in short: the *introduction meaning* of c, can be expressed by translating the premisses of (cI) into a standard propositional formula. Let $\bigwedge \Gamma_i$ denote the conjunction of all elements of Γ_i. We define c^I to be the formula

$$(\bigwedge \Gamma_1 \rightarrow s_1) \wedge \;\ldots\; \wedge (\bigwedge \Gamma_\ell \rightarrow s_\ell) \tag{15.9}$$

Then the rule $\dfrac{c^I}{c(p_1, \ldots, p_n)}$ is derivable in **PL** extended with the rule (cI), and (cI) is derivable in **PL** extended with the rule $\dfrac{c^I}{c(p_1, \ldots, p_n)}$.[8] If we have k introduction rules $(c\mathrm{I})_1, \ldots, (c\mathrm{I})_k$ for c, then the introduction meaning c^I of c is defined to be

[8] We call a rule R *derivable* in a formal system K, if applications of R can be eliminated from all derivations in K, i.e., if $\Gamma \vdash_{K+R} \varphi$ implies, $\Gamma \vdash_K \varphi$ for any formula φ and any set of assumptions Γ. This corresponds to the usual definition of derivability of rules when R does not discharge assumptions, but includes the case of assumption-discharging rules. Note that we request the eliminability of R under arbitrary assumptions Γ. (Otherwise we would be defining the notion of admissibility of a rule.)

$$c_1^I \vee \ldots \vee c_k^I ,$$

where for each introduction rule $(c\mathrm{I})_i$, the formula c_i^I is specified as in (15.9). If $k = 0$, then c^I is absurdity $\bot$, and there is no introduction rule. We note as a fact:

Fact I1: *The rule* $\dfrac{c^I}{c(p_1, \ldots, p_n)}$ *is derivable in* **PL+cI**, *i.e. in* **PL** *extended with* $(c\mathrm{I})_1, \ldots, (c\mathrm{I})_k$, *and each introduction rule* $(c\mathrm{I})_i$ *is derivable in* **PL** *extended with the rule* $\dfrac{c^I}{c(p_1, \ldots, p_n)}$.

Using this fact we can conclude that in the context of introduction rules, c can be replaced with c^I. More precisely, let Γ' and φ' result from a set of formulas Γ and a formula φ by simultaneously replacing every occurrence of c with c^I. This is done by replacing every subformula of Γ and φ of the form $c(\varphi_1, \ldots, \varphi_n)$ with $c^I[p_1, \ldots, p_n/\varphi_1, \ldots, \varphi_n]$, where $[p_1, \ldots, p_n/\varphi_1, \ldots, \varphi_n]$ denotes the simultaneous substitution of $\varphi_1, \ldots, \varphi_n$ for $p_1, \ldots, p_n$. Then we can show that

$$\Gamma \vdash_{\mathbf{PL+cI}} \varphi \text{ implies } \Gamma' \vdash_{\mathbf{PL}} \varphi' .$$

In other words: In contexts, where only introductions for c are available, c and c^I behave identically. We note this as a fact:

Fact I2: *If* Γ' *and* φ' *result from* Γ *and* φ *by replacing* c *with* c^I, *then* $\Gamma \vdash_{\mathbf{PL+cI}} \varphi$ *implies* $\Gamma' \vdash_{\mathbf{PL}} \varphi'$.

Another way of exhibiting the equivalence between c^I and c with respect to introduction rules is by saying that for any set Γ of formulas not containing c,

$$\Gamma \vdash_{\mathbf{PL+cI}} c(p_1, \ldots, p_n) \text{ iff } \Gamma \vdash_{\mathbf{PL}} c^I .$$

This equivalence, which immediately follows from the two previous facts, expresses that the introduction meaning c^I of c is the weakest formula in the language without c which by using the introduction rules for c allows one to infer $c(p_1, \ldots, p_n)$. We might also say that in assertion position, i.e., on the right side of the turnstile, c^I is equally strong as c.[9] We note this as a fact:

Fact I3: *If* Γ *does not contain c, then:* $\Gamma \vdash_{\mathbf{PL+cI}} c(p_1, \ldots, p_n)$ iff $\Gamma \vdash_{\mathbf{PL}} c^I$.

In the elimination case the situation is slightly more complicated. In an elimination rule for c variables beyond $p_1, \ldots, p_n$ can be present. In the following, these variables are also called *extra variables*. A typical example of an extra variable is the variable

[9] Popper (1947, see Schroeder-Heister 2005) was the first to characterize logical constants in terms of maximality and minimality conditions. Tennant (1978, p. 74) used them as the basic ingredient of a principle of harmony.

r in the following formulation of the standard elimination rule for disjunction (15.4). It is obvious that the extra variables in elimination rules have a universal meaning. Correspondingly, we extend our means of expression by considering not just standard intuitionistic propositional logic, but this logic together with universal propositional quantification. We call this system **PL2**.[10] Correspondingly, we use the abbreviations **PL2+cI**, **PL2+cE** and **PL2+cIE**, if in addition introduction, elimination, or both introduction and elimination rules for c are available in the system.

Suppose an elimination rule (cE) is given for an n-ary connective c according to (15.3). Then the intended meaning c^E of c according to this elimination rule, in short: the *elimination meaning* of c, is obtained as follows. We remove the major premiss $c(p_1, \ldots, p_n)$ from (cE) and translate the 'rest' of the rule, which tells what can be inferred from $c(p_1, \ldots, p_n)$ according to (cE), into a formula of **PL2**. Thus we define c^E to be the formula

$$\overline{\forall}(((\bigwedge \Gamma_1 \to s_1) \wedge \ldots \wedge (\bigwedge \Gamma_\ell \to s_\ell)) \to q)\ . \tag{15.10}$$

Here $\overline{\forall}$ universally quantifies all extra variables in c^E.[11] Then the rule $\dfrac{c(p_1, \ldots, p_n)}{c^E}$ is derivable in **PL2** extended with the rule (cE), and (cE) is derivable in **PL2** extended with the rule $\dfrac{c(p_1, \ldots, p_n)}{c^E}$. If we have k elimination rules $(c\mathrm{E})_1, \ldots, (c\mathrm{E})_k$ for c, then the elimination meaning c^E of c is defined to be

$$c_1^E \wedge \ldots \wedge c_k^E\ ,$$

where for each elimination rule $(c\mathrm{E})_i$, the formula c_i^E is specified as in (15.10). If $k = 0$, then c^E is verity $\top$. We note as a fact:

Fact E1: *The rule* $\dfrac{c(p_1, \ldots, p_n)}{c^E}$ *is derivable in* **PL2+cE**, *i.e. in* **PL2** *extended with* $(c\mathrm{E})_1, \ldots, (c\mathrm{E})_k$, *and each elimination rule* $(c\mathrm{E})_i$ *is derivable in* **PL2** *extended with the rule* $\dfrac{c(p_1, \ldots, p_n)}{c^E}$.[12]

Using this fact we can conclude that in the context of elimination rules, c can be replaced with c^E. More precisely, let Γ' and φ' result from Γ and φ by simultaneously replacing every occurrence of c with c^E. This is done by replacing every subformula

[10] To express the meanings of elimination rules, we can restrict ourselves to the case of prenex formulas, i.e., formulas quantified only from outside. More involved forms of quantification might be considered, but are not needed here. We also do not use the fact that by using propositional quantification and implication, all intuitionistic connectives become definable (see Prawitz 1965).

[11] More precisely, $\overline{\forall}$ stands for $\forall r_1 \ldots \forall r_j$, where $\{r_1, \ldots, r_j\}$ is the set of those variables occurring in c^E, which are different from any variable in $\{p_1, \ldots, p_n\}$.

[12] Note that for this statement propositional quantification is not really needed, as we are treating all rules as schemas, which means that universal quantification could remain implicit just by the usage of free propositional variables.

of Γ and φ of the form $c(\varphi_1, \ldots, \varphi_n)$ with $c^E[p_1, \ldots, p_n/\varphi_1, \ldots, \varphi_n]$. Then we can show that

$$\Gamma \vdash_{\mathbf{PL2+cE}} \varphi \text{ implies } \Gamma' \vdash_{\mathbf{PL2}} \varphi' .$$

In other words: In contexts, where only eliminations for c are available, c and c^E behave identically. We note as a fact:

Fact E2: *If* Γ' *and* φ' *result from* Γ *and* φ *by replacing* c *with* c^E*, then:* $\Gamma \vdash_{\mathbf{PL2+cE}} \varphi$ implies $\Gamma' \vdash_{\mathbf{PL2}} \varphi'$.

Another way of exhibiting the equivalence between c^E and c with respect to elimination rules is by saying that for any set $\Gamma \cup \{\varphi\}$ of formulas not containing c,

$$c(p_1, \ldots, p_n), \Gamma \vdash_{\mathbf{PL2+cE}} \varphi \text{ iff } c^E, \Gamma \vdash_{\mathbf{PL2}} \varphi .$$

This equivalence, which immediately follows from the two previous facts, expresses that the elimination meaning c^E of c is the strongest formula in the language without c, which, by using the elimination rules for c, can be inferred from $c(p_1, \ldots, p_n)$. We might also say that in assumption position, i.e., on the left side of the turnstile, c^E is equally strong as c. We note this as a fact:

Fact E3: *If* Γ *and* φ *do not contain c, then:*
$c(p_1, \ldots, p_n), \Gamma \vdash_{\mathbf{PL2+cE}} \varphi$ iff $c^E, \Gamma \vdash_{\mathbf{PL2}} \varphi$.

15.4 Harmony, Conservativeness, Uniqueness

Given introduction rules for c, we have defined the introduction meaning of c to be the formula c^I, which is a formula in standard intuitionistic propositional logic **PL**. Given elimination rules for c, we have defined the elimination meaning of c to by the formula c^E, which is a prenex formula in **PL2**, i.e., in standard intuitionistic propositional logic with propositional quantification. If both introduction and elimination rules are provided for c, we say that they are in *harmony* with each other, if introduction meaning and elimination meaning of c match, i.e., if in **PL2** we can show:

$$c^I \dashv\vdash c^E . \tag{15.11}$$

Splitting up harmony into its two directions, we can say the following. Suppose the introduction meaning of c entails its elimination meaning:

$$c^I \vdash c^E . \tag{15.12}$$

Then we have conservativeness of the introduction and elimination rules for c over **PL**. We state this as a result:

Conservativeness Lemma *Suppose that in* **PL2** *it holds that* $c^I \vdash c^E$. *Suppose that in* **PL+cIE** *we have that* $\Delta \vdash \varphi$ *for a set of formulas* Δ *and a formula* φ *which do not contain* c. *Then* $\Delta \vdash \varphi$ *holds already in* **PL**.

Proof sketch We use a normalisation argument. In the introduction context we can, according to **Fact I2**, replace c with c^I, and in the elimination context we can, according to **Fact E2**, replace c with c^E. If c occurs both in introduction and elimination context, i.e., as a maximal formula, we replace it with c^I followed by a derivation in **PL2** of c^E from c^I. Thus $\Delta \vdash \varphi$ holds is **PL2**. Using the normalisability of **PL2** and the conservativeness of **PL2** over **PL**, we obtain our result. □

That this proof needs to rely on the heavy machinery of normalisation of **PL2** (i.e., Girard's system **F**; see, e.g., Girard et al. 1989, and Prawitz 1971) is due to the fact that we describe introduction and elimination meanings in abstract terms, and here the latter by means of a second-order formula. If we deal with a concrete system with harmonious rules, for example rules following a general schema for eliminations as in Prawitz (1979) or Schroeder-Heister (1984), then normalisation and conservativeness can be proven directly in the system under consideration.

Conversely, suppose the elimination meaning of c entails its introduction meaning:

$$c^E \vdash c^I . \tag{15.13}$$

Then c is uniquely characterised in the following sense.

Uniqueness Lemma *If we extend* **PL2** *with the introduction rules for c and with the elimination rules for a duplicate* c' *of c (in the joint language containing both c and* c'*), we can show*

$$c'(p_1, \ldots, p_n) \vdash c(p_1, \ldots, p_n).$$

If introduction and elimination rules for both c and c' *are available, this gives us the equivalence*

$$c'(p_1, \ldots, p_n) \dashv\vdash c(p_1, \ldots, p_n) .$$

Proof This follows immediately from **Fact I1** and **Fact E1** above, which give us the derivability of the rules $\dfrac{c'(p_1, \ldots, p_n)}{c^E}$ and $\dfrac{c^I}{c(p_1, \ldots, p_n)}$. □

Conservativeness and uniqueness are the two conditions Belnap (1962) considered to be crucial for the inferential definition of a connective (see also Došen and Schroeder-Heister 1985). Similar conditions appear under different names in proof-theoretic semantics, for example validity and stability (Dummett 1991) or (local) soundness and (local) completeness (Francez and Dyckhoff 2012). There are considerable differences between these and related notions, in particular as to whether they are understood locally (refer to applications of rules) or globally (refer to the behaviour of the logical system as a whole). In this paper, we understand the two conditions in an abstract way, by relating the logically coded introduction and elimination meanings of a connective, rather than using its introduction and elimination rules themselves (in contradistinction, for example, to the foundational analyses of Francez and Dyckhoff 2012, and Schroeder-Heister 2014a).

As we have defined harmony and, correspondingly, conservativeness and uniqueness in terms of derivability in **PL2**, one might ask[13] whether any of these relations is decidable. As **PL2** is undecidable, the immediate answer is negative. However, the conservativeness direction (15.12) is in fact decidable, as c^E is a prenex formula whose quantifiers can be represented by free variables, so that conservativeness becomes a derivability problem in the decidable system **PL**. The uniqueness direction (15.13) may contain quantifiers on the hypothesis side, which cannot be eliminated. There, as we would guess, the undecidability of **PL2** comes into effect.[14]

15.5 The Existence of Harmonious Rules

If the introduction and elimination rules for an n-ary connective c are in harmony with each other, we also say that the elimination rules are *harmonious* for the introduction rules and the introduction rules are *harmonious* for the elimination rules. For many sets of introduction rules there are harmonious elimination rules and vice versa. Table 15.1 gives the introduction and elimination rules of the standard connectives $\wedge$, $\vee$, $\rightarrow$, $\bot$ and $\top$ together with their respective introduction and elimination meanings. These rules are all in harmony with each other. For $\wedge$, $\rightarrow$ and $\top$ this is trivial, as the respective introduction and elimination meanings are identical. In the case of disjunction and absurdity, we can easily show in **PL2** that

$$p_1 \vee p_2 \dashv\vdash \forall r(((p_1 \rightarrow r) \wedge (p_2 \rightarrow r)) \rightarrow r)$$

and

$$\bot \dashv\vdash \forall r\, r\ .$$

In Table 15.2 we consider forms of conjunction and implication with alternative elimination rules. Here we have harmony as well. For example, in oder to prove it for the introduction and elimination rules for $\times$ we can easily show in **PL2** that

$$p_1 \times p_2 \dashv\vdash \forall r_1 r_2 r(((p_1 \rightarrow r_1) \wedge (p_2 \rightarrow r_2) \wedge ((r_1 \wedge r_2) \rightarrow r)) \rightarrow r)\ .$$

Table 15.3 presents some further connectives. The connectives c_1 and c_2 have the same introduction rules but different elimination rules. We nevertheless have harmony in both cases, since in **PL2** we can show that

[13] Heinrich Wansing posed this question.

[14] More precisely, we do not have an argument at hand showing that the derivability of a quantifier-free formula from a prenex formula in **PL2** is decidable.

Table 15.1 Introduction and elimination rules together with introduction and elimination meanings for the standard connectives

	Introduction rules	Elimination rules
	Introduction meaning	Elimination meaning
a	$\dfrac{p_1 \quad p_2}{p_1 \wedge p_2}$	$\dfrac{p_1 \wedge p_2}{p_1} \qquad \dfrac{p_1 \wedge p_2}{p_2}$
	$p_1 \wedge p_2$	$p_1 \wedge p_2$
b	$\dfrac{p_1}{p_1 \vee p_2} \qquad \dfrac{p_2}{p_1 \vee p_2}$	$\dfrac{p_1 \vee p_2 \quad \begin{matrix}[p_1]\\ r\end{matrix} \quad \begin{matrix}[p_2]\\ r\end{matrix}}{r}$
	$p_1 \vee p_2$	$\forall r(((p_1 \to r) \wedge (p_2 \to r)) \to r)$
c	$\dfrac{\begin{matrix}[p_1]\\ p_2\end{matrix}}{p_1 \to p_2}$	$\dfrac{p_1 \to p_2 \quad p_1}{p_2}$
	$p_1 \to p_2$	$p_1 \to p_2$
d	No I rule	$\dfrac{\bot}{r}$
	$\bot$	$\forall r \;\, r$
e	$\dfrac{}{\top}$	No E rule
	$\top$	$\top$

In all cases introduction and elimination rules are in harmony with each other

$$(p_1 \wedge p_2) \vee p_3 \dashv\vdash \forall r((((p_1 \wedge p_2) \to r) \wedge (p_3 \to r)) \to r)$$

as well as

$$(p_1 \wedge p_2) \vee p_3 \dashv\vdash \forall r(((p_1 \to r) \wedge (p_3 \to r)) \to r) \wedge \forall r(((p_2 \to r) \wedge (p_3 \to r)) \to r) \tag{15.14}$$

Note that we take the intuitionistic logic of the standard connectives for granted, which means in particular that we assume that the standard structural rules are at our disposal. Otherwise we would not, for example, be able to show (15.14), for which we essentially need distribution or $\vee$ over $\wedge$:

$$(p_1 \wedge p_2) \vee p_3 \dashv\vdash (p_1 \vee p_3) \wedge (p_2 \vee p_3)$$

which is not available as a general law when thinning or contraction are restricted.[15]

[15] It depends on whether these connectives are read additively or multiplicatively. This point is in particular relevant, if (like, e.g., Read 2010, 2014, this volume) one prefers more than one elimination rule in cases such as $\bullet$ (see Table 15.2).

Table 15.2 Introduction and elimination rules together with introduction and elimination meanings for various connectives related to the standard ones

	Introduction rules	Elimination rules
	Introduction meaning	Elimination meaning
a	$\dfrac{p_1 \qquad p_2}{p_1 \& p_2}$	$\dfrac{p_1 \& p_2 \qquad \begin{matrix}[p_1, p_2] \\ r\end{matrix}}{r}$
	$p_1 \wedge p_2$	$\forall r(((p_1 \wedge p_2) \to r) \to r)$
b	$\dfrac{p_1 \qquad p_2}{p_1 \bullet p_2}$	$\dfrac{p_1 \bullet p_2 \qquad \begin{matrix}[p_1] \\ r\end{matrix}}{r} \qquad \dfrac{p_1 \bullet p_2 \qquad \begin{matrix}[p_2] \\ r\end{matrix}}{r}$
	$p_1 \wedge p_2$	$\forall r((p_1 \to r) \to r) \wedge \forall r((p_2 \to r) \to r)$
c	$\dfrac{p_1 \qquad p_2}{p_1 \times p_2}$	$\dfrac{p_1 \times p_2 \qquad \begin{matrix}[p_1] \\ r_1\end{matrix} \quad \begin{matrix}[p_2] \\ r_2\end{matrix} \qquad \begin{matrix}[r_1, r_2] \\ r\end{matrix}}{r}$
	$p_1 \wedge p_2$	$\forall r_1 r_2 r(((p_1 \to r_1) \wedge (p_2 \to r_2) \wedge ((r_1 \wedge r_2) \to r)) \to r)$
d	$\dfrac{\begin{matrix}[p_1] \\ p_2\end{matrix}}{p_1 \supset p_2}$	$\dfrac{p_1 \supset p_2 \qquad p_1 \qquad \begin{matrix}[p_2] \\ r\end{matrix}}{r}$
	$p_1 \to p_2$	$\forall r((p_1 \wedge (p_2 \to r)) \to r)$

In all cases introduction and elimination rules are in harmony with each other

Obviously, the rules given for the connectives $\&\supset$ and *tonk* are not harmonious, since neither

$$p_1 \wedge (p_1 \to p_2) \dashv\vdash p_1 \to p_2$$

nor

$$p_1 \dashv\vdash p_2$$

holds in **PL2**.

The n-ary connectives *ci* and *ce* represent connectives of a general form. In the case of *ci* we have harmony, provided the introduction rules do not discharge any assumption, but are just productions. Likewise, the rules for *ce* are harmonious, provided the elimination rules do not discharge any assumption. This means that, if the introduction rules of *ci* are of the form stated in Table 15.3, then there is always a harmonious elimination rule. If the elimination rules for *ce* are of the form stated, then there is always a harmonious introduction rule.

Table 15.3 Introduction and elimination rules together with introduction and elimination meanings for further connectives

	Introduction rules / Introduction meaning	Elimination rules / Elimination meaning
a	$\dfrac{p_1 \quad p_2}{c_1(p_1, p_2, p_3)} \qquad \dfrac{p_3}{c_1(p_1, p_2, p_3)}$	$\dfrac{c_1(p_1, p_2, p_3) \quad \begin{matrix}[p_1, p_2]\\ r\end{matrix} \quad \begin{matrix}[p_3]\\ r\end{matrix}}{r}$
	$(p_1 \wedge p_2) \vee p_3$	$\forall r((((p_1 \wedge p_2) \to r) \wedge (p_3 \to r)) \to r)$
b	$\dfrac{p_1 \quad p_2}{c_2(p_1, p_2, p_3)} \qquad \dfrac{p_3}{c_2(p_1, p_2, p_3)}$	$\dfrac{c_2(p_1, p_2, p_3) \quad \begin{matrix}[p_1]\\ r\end{matrix} \quad \begin{matrix}[p_3]\\ r\end{matrix}}{r}$ $\dfrac{c_2(p_1, p_2, p_3) \quad \begin{matrix}[p_2]\\ r\end{matrix} \quad \begin{matrix}[p_3]\\ r\end{matrix}}{r}$
	$(p_1 \wedge p_2) \vee p_3$	$\forall r(((p_1 \to r) \wedge (p_3 \to r)) \to r)$ $\wedge \forall r(((p_2 \to r) \wedge (p_3 \to r)) \to r)$
c	$\dfrac{p_1 \quad \begin{matrix}[p_1]\\ p_2\end{matrix}}{p_1 \mathbin{\&\!\supset} p_2}$	$\dfrac{p_1 \mathbin{\&\!\supset} p_2 \quad p_1}{p_2}$
	$p_1 \wedge (p_1 \to p_2)$	$p_1 \to p_2$
d	$\dfrac{p_1}{p_1 \ tonk \ p_2}$	$\dfrac{p_1 \ tonk \ p_2}{p_2}$
	p_1	p_2
e	$\dfrac{\Delta_1}{ci(p_1, \ldots, p_n)} \ \ldots \ \dfrac{\Delta_m}{ci(p_1, \ldots, p_n)}$ Δ_i of the form $q_{i1} \ \ldots \ q_{i\ell_i}$ with $\{q_{i1}, \ldots, q_{i\ell_i}\} \subseteq \{p_1, \ldots, p_n\}$	$\dfrac{ci(p_1, \ldots, p_n) \quad \begin{matrix}[\Delta_1]\\ r\end{matrix} \ \ldots \ \begin{matrix}[\Delta_m]\\ r\end{matrix}}{r}$
	$\bigwedge \Delta_1 \vee \ldots \vee \bigwedge \Delta_m$	$\forall r((\bigwedge \Delta_1 \to r) \wedge \ldots \wedge (\bigwedge \Delta_m \to r) \to r)$
f	$\dfrac{\begin{matrix}[\Delta_1]\\ q_1\end{matrix} \quad \ldots \quad \begin{matrix}[\Delta_m]\\ q_m\end{matrix}}{ce(p_1, \ldots, p_n)}$	$\dfrac{ce(p_1, \ldots, p_n) \quad \Delta_1}{q_1} \ \ldots \ \dfrac{ce(p_1, \ldots, p_n) \quad \Delta_1}{q_m}$ Δ_i of the form $q_{i1} \ \ldots \ q_{i\ell_i}$ with $\{q_{i1}, \ldots, q_{i\ell_i}\} \cup \{q_1, \ldots, q_m\} \subseteq \{p_1, \ldots, p_n\}$
	$(\bigwedge \Gamma_1 \to q_1) \wedge \ldots \wedge (\bigwedge \Gamma_m \to q_m)$	$(\bigwedge \Gamma_1 \to q_1) \wedge \ldots \wedge (\bigwedge \Gamma_m \to q_m)$

However, not for every given set of introduction rules there are harmonious elimination rules, and not for every given set of elimination rules there are harmonious introduction rules. An example of a connective with given introduction rules, for which there are no harmonious elimination rules is the connective $\star$, whose introduction rules ($\star$ I) are given in (15.2). Its introduction meaning $\star^I$ is $(p_1 \to p_2) \lor p_3$. If there were harmonious elimination rules for $\star$, then, according to the definitions in Sect. 15.3, its elimination meaning $\star^E$ would have to be of the form $\star_1^E \land \ldots \land \star_k^E$, where each $\star_i^E$ is of the form $\overline{\forall}(((\bigwedge \Gamma_1 \to s_1) \land \ldots \land (\bigwedge \Gamma_\ell \to s_\ell)) \to q)$. However, in Olkhovikov and Schroeder-Heister (2014a) it could be demonstrated that no formula of this form is equivalent to $\star^I$ in **PL2**.

If we allow for connectives already defined to occur in introductions and eliminations, then there are harmonious elimination rules for $\star$. The trivial $\star$ elimination rule would be the single rule

$$\frac{\star(p_1, p_2, p_3)}{(p_1 \to p_2) \lor p_3}$$

which assumes that implication and disjunction are already being given. An alternative elimination rule only assumes that implication is available:

$$(\star\ \mathrm{E})\ \frac{\star(p_1, p_2, p_3) \qquad \begin{matrix}[p_1 \to p_2] \\ r\end{matrix} \qquad \begin{matrix}[p_3] \\ r\end{matrix}}{r}\ .$$

The elimination meaning $\star^E$ of $\star$ according to this elimination rule is

$$\forall r((((p_1 \to p_2) \to r) \land (p_3 \to r)) \to r)$$

which can easily be shown to be equivalent in **PL2** to $\star^I$:

$$(p_1 \to p_2) \lor p_3 \dashv\vdash \forall r((((p_1 \to p_2) \to r) \land (p_3 \to r)) \to r)\ .$$

Instead of assuming implication to be a connective already defined, we could extend the apparatus of natural deduction by using rules of higher levels, i.e., rules that can discharge not only formulas but also rules which are used as assumptions, as described in Schroeder-Heister (2014a). In such a framework the elimination rule for $\star$ would take the form

$$(\star\ \mathrm{E})\ \frac{\star(p_1, p_2, p_3) \qquad \begin{matrix}[p_1 \Rightarrow p_2] \\ r\end{matrix} \qquad \begin{matrix}[p_3] \\ r\end{matrix}}{r}\ .$$

Here $p_1 \Rightarrow p_2$ represents the *rule* which allows one to pass over from p_1 to p_2. It is assumed as an assumption in the subderivation of the left minor premiss and is discharged at the application of ($\star$ E). The result by Olkhovikov and Schroeder-Heister (2014a) can then be read as showing that $\star$ does not have flat elimination rules, where, following a terminology proposed by Read (2014, this volume), an elimination rule is called *flat*, if it does not allow one to discharge rules, but only formulas. Flat rules are rules of the kind considered in standard (non-extended) natural deduction.[16]

Non-flattening theorem for elimination rules *The connective* $\star$ *does not have flat elimination rules.*

If we allow for rules of higher levels, then every set of introduction rules for an n-ary connective c has harmonious elimination rules. In fact, only one single elimination rule is needed, which we call the *generalised* or *canonical* elimination rule. It is constructed as follows: We associate with every introduction rule for c of the form

$$(c\mathrm{I}) \quad \frac{\begin{matrix}[\Gamma_1] & & [\Gamma_\ell] \\ s_1 & \ldots & s_\ell\end{matrix}}{c(p_1, \ldots, p_n)}$$

a list Δ of rules

$$(\Gamma_1 \Rightarrow s_1), \ldots, (\Gamma_\ell \Rightarrow s_\ell)$$

representing the premisses of this introduction rule (note that the double arrow is used to linearly denote rules rather than implications). If there are m introduction rules for c, we obtain m such lists $\Delta_1, \ldots, \Delta_m$. Then the canonical elimination rule for c has the form

$$(c\mathrm{E})_{GEN} \quad \frac{c(p_1, \ldots, p_n) \quad \begin{matrix}[\Delta_1] \\ r\end{matrix} \quad \ldots \quad \begin{matrix}[\Delta_m] \\ r\end{matrix}}{r} \quad . \tag{15.15}$$

This schema is devised such as to guarantee that introduction and elimination meaning of c match. If Δ_i is $(\Gamma_1 \Rightarrow s_1), \ldots, (\Gamma_\ell \Rightarrow s_\ell)$, let Δ_i^{PROP} be its propositional translation $(\bigwedge \Gamma_1 \to s_1), \ldots, (\bigwedge \Gamma_\ell \to s_\ell)$. Then the elimination meaning of c is defined as

$$\forall r (((\bigwedge \Delta_1^{PROP} \to r) \wedge \ldots \wedge (\bigwedge \Delta_m^{PROP} \to r)) \to r) \ .$$

This corresponds to the definition of elimination meaning in Sect. 15.3 with the only difference that we cannot just form the conjunction of the elements of the Δ_i, as they are not necessarily formulas, but have to propositionally translate these elements into conjunction-implication-formulas, if they are rules. Then we can easily prove in **PL2** that introduction meaning (which is defined as before in Sect. 15.3) and elimination meaning of c match:

$$c_1^I \vee \ \ldots \ \vee c_m^I \quad \dashv\vdash \quad \forall r (((\bigwedge \Delta_1^{PROP} \to r) \wedge \ldots \wedge (\bigwedge \Delta_m^{PROP} \to r)) \to r)$$

[16] The general elimination schema Francez and Dyckhoff (2012) propose is flat and therefore not harmonious in the sense of the definition of harmony proposed in Sect. 15.4. However, their point on local soundness and completeness is independent of this schema and applies to harmonious rules in our sense.

We simply have to use that c_i^I is identical to $\bigwedge \Delta_i^{PROP}$.

In fact, if we allow for implication-conjunction formulas to occur as assumptions in elimination rules, we can obtain the same result without having to rely on rules as assumptions. Instead of the general elimination schema (15.15) we could use the following schema, which results from (15.15) by replacing lists of assumptions rules Δ_i with lists of their propositional translations Δ_i^{PROP}:

$$(c\mathrm{E})_P \quad \frac{c(p_1,\ldots,p_n) \qquad \begin{matrix}[\Delta_1^{PROP}]\\ r\end{matrix} \quad \ldots \quad \begin{matrix}[\Delta_m^{PROP}]\\ r\end{matrix}}{r} \;. \tag{15.16}$$

We call it the *Prawitz schema* for generalised elimination rules, as it corresponds to the schema proposed in Prawitz (1979). For further discussion of this issue see Sect. 15.7.

Our result even pertains to the case in which the introduction rules are not flat, i.e., may be of higher levels. In that case, the elements of Δ_i may be rules which discharge assumptions. For example, consider the following quaternary operator c with the following two introduction rules:

$$(c\,\mathrm{I}) \quad \frac{\begin{matrix}[p_1 \Rightarrow p_2]\\ p_3\end{matrix}}{c(p_1,p_2,p_3,p_4)} \qquad \frac{p_4}{c(p_1,p_2,p_3,p_4)} \;.$$

According to the general schema $(c\mathrm{E})_{GEN}$ the corresponding canonical (and thus harmonious) elimination rule is

$$(c\,\mathrm{E}) \quad \frac{c(p_1,p_2,p_3,p_4) \qquad \begin{matrix}[(p_1 \Rightarrow p_2) \Rightarrow p_3]\\ r\end{matrix} \qquad \begin{matrix}[p_4]\\ r\end{matrix}}{r} \;.$$

In general it holds that, if we pass from given introduction rules to the corresponding canonical elimination rule, the level always goes up by one step, as the premisses of the introduction rules then occur as dischargeable assumptions of minor premisses in the canonical elimination rule. This cannot be avoided, i.e., we can always construct a connective whose introduction rules are of level n, without there being harmonious elimination rules of level n or below. This generalised non-flattening result is proved in Olkhovikov and Schroeder-Heister (2014b, Theorem 1)]:

Generalised non-flattening theorem for elimination rules *Suppose the schema for introduction rules is limited to rules of maximal level* n*. Then we can always find a connective satisfying such a schema, whose elimination schema cannot be of level* n *or below, i.e. must be at least of level* $n+1$*. In fact, we can choose the* $(n+1)$*-place connective with the introduction meaning*

$$((\ldots(p_1 \to p_2)\ldots \to p_{n-1}) \to p_n) \vee p_{n+1},$$

which is a generalisation of $\star$.

If we start with elimination rules, we have an analogous situation: Not all connectives which are specified by given elimination rules have harmonious introduction rules. Consider negation $\neg$ with the elimination rule:

$$(\neg E)\ \frac{\neg p_1 \qquad p_1}{r}\ .$$

According to our definition, its elimination meaning $\neg^E$ is $\forall r(p_1 \to r)$. However, there is no set of introduction rules for $\neg$ such that the introduction meaning $\neg^I$ is equivalent to $\neg^E$. This is simply a consequence of the (almost) trivial fact that negation $\neg$ cannot be defined in terms of implication and conjunction.

However, if we allow for connectives already defined to occur in introduction rules, we can easily give an appropriate introduction rule for $\neg$:

$$(\neg \mathrm{I})\ \frac{\begin{array}{c}[p_1]\\ \bot\end{array}}{\neg p_1}\ .$$

The introduction meaning $\neg^I$ of $\neg$ is now $p_1 \to \bot$, which is interderivable in **PL2** with its elimination meaning $\neg^E$:

$$p_1 \to \bot \ \dashv\vdash\ \forall r(p_1 \to r)$$

by using the absurdity rule (*ex falso quodlibet*), which is the elimination rule for $\bot$. Another example is the ternary connective $\circ$ with the elimination rule ($\circ$ E) given in (15.8). Its elimination meaning $\circ^E$ is $(p_1 \to p_2) \to p_3$. If there were harmonious introduction rules for $\circ$, its introduction meaning $\circ^I$ could be described by a disjunction of formulas $\circ^I_1 \vee \ldots \vee \circ^I_k$, where each formula $\circ^I_i$ would be of the form $(\bigwedge \Gamma_1 \to s_1) \wedge \ldots \wedge (\bigwedge \Gamma_\ell \to s_\ell)$. It can be shown, however, that $\circ^E$ is never equivalent in **PL2** to a disjunction of formulas of this form. The proof of this fact can be found in Olkhovikov and Schroeder-Heister (2014a).

If we allow for connectives, which are already defined, to occur in introduction rules, we could equip $\circ$ with the trivial introduction rule

$$\frac{(p_1 \to p_2) \to p_3}{\circ(p_1, p_2, p_3)}$$

or alternatively with

$$(\circ\ \mathrm{I})\ \frac{\begin{array}{c}[p_1 \to p_2]\\ p_3\end{array}}{\circ(p_1, p_2, p_3)}\ .$$

The introduction meaning $\circ^I$ according to this introduction rule is $(p_1 \to p_2) \to p_3$, which is identical to its elimination meaning. If we use higher-level rules, we can write the introduction rule for c as

$$(\circ\,\mathrm{I})\ \frac{\begin{array}{c}[p_1 \Rightarrow p_2]\\ p_3\end{array}}{\circ(p_1, p_2, p_3)}$$

The result by Olkhovikov and Schroeder-Heister (2014a) then says that $\circ$ does not have flat introduction rules.

Non-flattening theorem for introduction rules *The connective $\circ$ does not have flat introduction rules.*

However, even if we allow for rules of higher levels, not every set of elimination rules for an n-ary connective c has corresponding harmonious introduction rules. This is due to the fact that in (cE) propositional variables beyond $p_1, \ldots, p_n$ may occur, which, as schematic variables, have a universal meaning and correspondingly enter the elimination meaning c^E as universally bound. If we want to turn the content of such an elimination inference into the premiss of an introduction rule, we have to devise a binding mechanism at the structural level. We need not only rules as assumptions, but also bound variables in the premisses of rules. For that to achieve we define the general schema of an introduction rule for c to be of the form

$$(c\mathrm{I})\ \frac{\begin{pmatrix}[\Gamma_1]\\ s_1\end{pmatrix}_{\overline{r_1}} \quad \ldots \quad \begin{pmatrix}[\Gamma_\ell]\\ s_\ell\end{pmatrix}_{\overline{r_\ell}}}{c(p_1, \ldots, p_n)}\ .$$

Here the $\overline{r_i}$ are lists of propositional variables different from $p_1, \ldots, p_n$, which cannot be substituted (as can $p_1, \ldots, p_n$), but which in the subproofs of s_i from Γ_i are treated like constants ('parameters' or 'free variables' in a different terminology).

Assuming this extension of natural deduction with quantified higher-level rules (described in detail in Schroeder-Heister 2014a), we can construct introduction rules, which are in harmony with given elimination rules for c of the form (cE) as given in (15.3). In fact, only one single introduction rule, is needed, which we call the *generalised* or *canonical* introduction rule. It is constructed as follows: We associate with every elimination rule of the form (15.3)

$$(c\mathrm{E})\ \frac{c(p_1, \ldots, p_n) \qquad \begin{array}{c}[\Gamma_1]\\ s_1\end{array} \quad \ldots \quad \begin{array}{c}[\Gamma_\ell]\\ s_\ell\end{array}}{q}\ ,$$

a list Δ of rules

$$(\Gamma_1 \Rightarrow s_1), \ldots, (\Gamma_\ell \Rightarrow s_\ell)$$

representing the premisses of this elimination rule. From this we construct the pattern

$$\left(\begin{array}{c}[\Delta]\\ q\end{array}\right)_{\{s_1,\ldots,s_\ell,q\}}$$

representing what can be inferred from $c(p_1, \ldots, p_n)$ using this elimination rule. Suppose c has m elimination rules and we have associated m patterns

$$\left(\begin{array}{c}[\Delta_1]\\ q_1\end{array}\right)_{Var_1} \quad \ldots \quad \left(\begin{array}{c}[\Delta_m]\\ q_m\end{array}\right)_{Var_m}$$

with them, respectively. Here Var_i are the sets of variables occurring in the respective patterns beyond $p_1, \ldots, p_n$. Then the canonical introduction rule for c has the following form:

$$(c\mathrm{I})_{GEN}\ \frac{\left(\begin{array}{c}[\Delta_1]\\ q_1\end{array}\right)_{Var_1} \quad \ldots \quad \left(\begin{array}{c}[\Delta_m]\\ q_m\end{array}\right)_{Var_m}}{c(p_1, \ldots, p_n)}\ .$$

It is constructed in such a way that introduction meaning and elimination meaning of c match. If Δ_i is $(\Gamma_1 \Rightarrow s_1),\ \ldots\ ,(\Gamma_\ell \Rightarrow s_\ell)$, let Δ_i^{PROP} be its propositional translation $(\bigwedge \Gamma_1 \to s_1),\ \ldots\ ,(\bigwedge \Gamma_\ell \to s_\ell)$. Then the introduction meaning of c is defined as

$$\overline{\forall}(\bigwedge \Delta_1^{PROP} \to q_1) \wedge \ldots \wedge \overline{\forall}(\bigwedge \Delta_m^{PROP} \to q_m)$$

(note that $\overline{\forall}$ binds all variables beyond $p_1, \ldots, p_n$). This corresponds to the definition of introduction meaning in Sect. 15.3 with the only difference that we cannot just take the conjunction of the elements of the Γ_i, as they are not necessarily formulas, but have to propositionally translate these elements into conjunction-implication-formulas, if they are rules. Then elimination meaning (which is defined as before in Sect. 15.3) and introduction meaning of c match, i.e., the following holds in **PL2**:

$$\overline{\forall}(\bigwedge \Delta_1^{PROP} \to q_1) \wedge \ldots \wedge \overline{\forall}(\bigwedge \Delta_m^{PROP} \to q_m) \quad \dashv\vdash \quad c_1^E \wedge \ldots \wedge c_k^E\ .$$

This is actually trivial since c_i^E is identical to $\overline{\forall}(\bigwedge \Delta_I^{PROP} \to q_m)$.

This result pertains to the case in which the elimination rules are not flat, i.e. may be of higher levels. In that case, the elements of Δ_i may be rules which discharge assumptions. Note however that when passing from eliminations to introductions, not only the level of the rule goes up by one step, but we have to use structural quantification in the premiss of the introduction rule, too, if the elimination rules contain extra variables. Going up one step cannot be avoided, i.e., we can always construct a connective whose elimination rules reach level n, without there being harmonious introduction rules of level n or below (Olkhovikov and Schroeder-Heister 2014b, Theorem 2):

Generalised non-flattening theorem for introduction rules *Suppose the schema for elimination rules is limited to rules of maximal level n. Then we can always find a connective satisfying such a schema, whose introduction schema cannot be of level n or below, i.e. must be at least of level* $n+1$. *In fact, we can choose the* $(n+1)$*-place connective with the elimination meaning* $(\ldots(p_1 \to p_2)\ldots \to p_n) \to p_{n+1}$, *which is a generalisation of* $\circ$.

The fact that in the canonical introduction rule we add some sort of structural quantification leads to a further generalisation. Once in the canonical introduction rule we allow for structural quantification in the premisses, there is no reason in principle why we should not specify introduction rules for a connective by using this sort of quantification in their premisses. In the corresponding harmonious canonical elimination rule this would lead to structural quantification in the assumptions of minor premisses. But if we allow for that, there is no reason why we should not iterate this process and use any sort of embedded (i.e. nested) universal quantification in the specification of connectives. In the end this means that, at the structural level, we would use means of expression which correspond to those available in **PL2** at the logical level.[17]

Concerning the negative results presented, the reader should keep in mind that we are working in an intuitionistic framework throughout. If we used classical second-order propositional logic $\mathbf{PL2}_c$ instead of the intuitionistic system **PL2**, we could always find harmonious rules, as emphasized by Read (2014, this volume). For example, $\star$ could be given the (flat) harmonious elimination rule

$$\frac{\star(p_1, p_2, p_3) \quad p_1 \quad \begin{matrix}[p_2] \\ r\end{matrix} \quad \begin{matrix}[p_3] \\ r\end{matrix}}{r} \ ,$$

since we can show in $\mathbf{PL2}_c$ that

$$(p_1 \to p_2) \vee p_3 \ \dashv\vdash \ \forall r((p_1 \wedge (p_2 \to r) \wedge (p_3 \to r)) \to r) \ .$$

15.6 Functional Completeness

From *Fact I1* (Sect. 15.3) we know that

[17] This is discussed in detail in Schroeder-Heister (2014a).—The fact that, as shown by Pitts (1992), **PL2** can be translated into **PL**, cannot be used here, as this translation uses all connectives of **PL** including disjunction, which does not have a structural analogue in our framework. However, this translation might become useful in the context of functional completeness. See Sect. 15.6 and footnote 19.—We have not discussed the issue of structural existential quantification, as this is not relevant for our central topic. In the framework discussed here we need to add universal quantification if from given elimination rules we want to construct harmonious introduction rules. If we allowed for extra variables and thus for implicit existential quantification in the premisses of introduction rules, we would need to add universal quantification when passing to harmonious elimination rules.

$$c^I \vdash c(p_1, \ldots, p_n) \tag{15.17}$$

holds in **PL+cI**, and from *Fact E1* we know that

$$c(p_1, \ldots, p_n) \vdash c^E \tag{15.18}$$

holds in **PL2+cE**. Now suppose that the introduction and elimination rules for c are in harmony with each other, i.e.,

$$c^I \dashv\vdash c^E \tag{15.19}$$

holds in **PL2**. This implies that both

$$c(p_1, \ldots, p_n) \dashv\vdash c^I \tag{15.20}$$

and

$$c(p_1, \ldots, p_n) \dashv\vdash c^E \tag{15.21}$$

hold in **PL2+cIE**, which means that we can regard (15.20) and (15.21) as two explicit definitions of c in **PL2**. Therefore c can be expressed by means of the connectives of **PL2**, which are $\wedge$, $\vee$, $\rightarrow$, $\bot$ and $\forall$. As $\wedge$, $\vee$ and $\bot$ are definable in **PL2** in terms of $\rightarrow$ and $\forall$, we obtain as a result that c can be expressed by using $\rightarrow$ and $\forall$ in the system that comprises both second-order propositional logic and the introduction and elimination rules for c.

For example, our connective $\star$ can be defined either by $(p_1 \rightarrow p_2) \vee p_3$ (its introduction meaning) or by $\forall r((((p_1 \rightarrow p_2) \rightarrow r) \wedge (p_3 \rightarrow r)) \rightarrow r)$ (its elimination meaning). From the latter formula we can eliminate conjunction by rewriting it as $\forall r(((p_1 \rightarrow p_2) \rightarrow r) \rightarrow ((p_3 \rightarrow r) \rightarrow r))$, obtaining a definition of $\star$ in terms of $\forall$ and $\rightarrow$.[18]

This is a reductive version of functional completeness in the sense that constants of standard second-order intuitionistic logic suffice to express all connectives definable by harmonious introduction and elimination rules. It is *reductive* as the standard constants (here $\rightarrow$ and $\forall$) are taken for granted and are conceptually not on the same level as the connective c.

Whereas (15.21) gives rise to a definition of c in the language of **PL2**, which can use propositional quantification in the definiens, the right hand side of (15.20) is a formula of **PL**, which can only contain $\wedge$, $\vee$, $\rightarrow$ and $\bot$ as connectives (see our definition of c^E and c^I in Sect. 15.3). As the derivability relation in (15.20) is that of **PL2+cIE**, (15.20) only yields a definition of c in **PL2**, even if no quantifier occurs in the definiens. However, the following observation shows that only derivability in **PL+cIE** is needed to establish (15.20), so that (15.20) actually is a definition of c in

[18] If we use the standard translation of $s_1 \wedge s_2$ into second-order logic, which is $\forall q((s_1 \rightarrow (s_2 \rightarrow q)) \rightarrow q)$, we would obtain the more complicated formula $\forall r(\forall q(((p_1 \rightarrow p_2) \rightarrow r) \rightarrow ((p_3 \rightarrow r) \rightarrow q) \rightarrow q) \rightarrow r)$. The formula $\forall r(((p_1 \rightarrow p_2) \rightarrow r) \rightarrow ((p_3 \rightarrow r) \rightarrow r))$ is actually the standard second-order translation of $(p_1 \rightarrow p_2) \vee p_3$ which uses the translation of $s_1 \vee s_2$ as $\forall r((s_1 \rightarrow r) \rightarrow ((s_2 \rightarrow r) \rightarrow r))$.

PL in terms of the connectives $\wedge$, $\vee$, $\rightarrow$ and $\bot$. In view of (15.17) we have to show that we do not need second-order quantification to establish $c(p_1, \ldots, p_n) \vdash c^I$. As c^E is a prenex formula of the form $\forall\forall\varphi$, where $\forall\forall$ represents a quantifier prefix and φ the quantifier-free kernel, we can, from the derivability of $c^E \vdash c^I$ in **PL2** [see (15.19)] and normalisation for **PL2**, conclude the derivability of $\varphi_1, \ldots, \varphi_n \vdash c^I$ in **PL**, where $\varphi_1, \ldots, \varphi_n$ are certain quantifier-free instances of φ. Furthermore, from $c(p_1, \ldots, p_n)$ we can derive each φ_i in **PL+cE**, which yields the required derivation of c^I from $c(p_1, \ldots, p_n)$ in **PL+cE**.

Therefore we have obtained the following result:

Functional completeness *Any connective c with harmonious introduction and elimination rules can be defined in* **PL** *by its introduction meaning* c^I *and also by its elimination meaning* c^E.

Here the same remark we made after the Conservativeness Lemma in Sect. 15.4 applies: That this proof needs to rely on the heavy machinery of normalisation of **PL2** is due to our description of introduction and elimination meanings in abstract terms by means of second-order formulas. In a concrete system with harmonious rules it would be replaced with a direct syntactic proof using the rules available (see Prawitz 1979; Schroeder-Heister 1984).

Pitts (1992) defines a translation * from **PL2** into **PL**, such that $\Gamma \vdash_{\mathbf{PL2}} \varphi$ entails $\Gamma^* \vdash_{\mathbf{PL}} \varphi^*$, where for every quantifier-free φ, φ^* is identical to φ. Thus, from (15.19), we can conclude that in **PL** the following holds:

$$c^I \dashv\vdash (c^E)^* \,.$$

This gives us another definition of c, namely as $(c^E)^*$. It might be interesting to check what $(c^E)^*$ looks like for various c^I.[19]

15.7 Prawitz's Account of Functional Completeness

Our reductive approach offers a plausible way of understanding Prawitz's (1979) proof of functional completeness of the standard intuitionistic constants $\wedge$, $\vee$, $\rightarrow$ and $\bot$. Prawitz starts from (15.1) as the general schema for introduction rules of an n-ary connective c. He then associates a corresponding elimination rule for c (there is only a single one) as follows. From an introduction rule of the form (cI) a list Δ^{PROP} of conjunction-implication formulas

$$\bigwedge \Gamma_1 \rightarrow s_1, \ \ldots \ , \bigwedge \Gamma_\ell \rightarrow s_\ell$$

[19] The reference to Pitts (1992) was brought to my attention by an anonymous reviewer of Olkhovikov and Schroeder-Heister (2014a).

is constructed, where $\bigwedge \Gamma_i$ denotes the conjunction of all elements of Γ_i, where $\bigwedge \Gamma_i \to s_i$ is identified with s_i, if Γ_i is empty. This list represents propositionally the premisses of (cI). If we have m introduction rules of the form (cI), we obtain m such lists $\Delta_1^{PROP}, \ldots, \Delta_m^{PROP}$. Then the elimination rule (cE) has the following form:

$$(c\mathrm{E})_P \ \frac{c(p_1, \ldots, p_n) \qquad \begin{matrix}[\Delta_1^{PROP}] \\ r\end{matrix} \qquad \ldots \qquad \begin{matrix}[\Delta_m^{PROP}] \\ r\end{matrix}}{r} \quad ,$$

which is exactly the schema (15.16). This schema is modelled by Prawitz after the pattern of the standard $\vee$ elimination rule. It expresses that everything that can be derived from the premisses of each introduction rule for c can be derived from its conclusion. To put it differently: $c(p_1, \ldots, p_n)$ is the strongest proposition that can be derived from the premisses of each introduction rule for c. In the case of absurdity $\bot$, which has no introduction rule, we obtain the *ex falso quodlibet* as the limiting case of (cE) ($m = 0$, i.e. no minor premisses).

Unfortunately, $(c\mathrm{E})_P$ already uses the connectives $\wedge$ and $\to$, which means that it cannot be used as a schema covering them. In fact, conjunction is not used in the elimination rule for conjunction, which according to $(c\mathrm{E})_P$ takes the form

$$(\wedge \mathrm{E}_{GEN}) \ \frac{p \wedge q \qquad \begin{matrix}[p, q] \\ r\end{matrix}}{r} \quad ,$$

but only in more complicated elimination rules. Thus one may view conjunction as defined by this general rule and later refer to it as an already defined connective. However, in the case of implication, Prawitz's elimination rule takes the form

$$\frac{p_1 \to p_2 \qquad \begin{matrix}[p_1 \to p_2] \\ r\end{matrix}}{r} \quad ,$$

which is trivial and therefore useless. From a foundational point of view, Prawitz's meaning-theoretical considerations as well as his proof that every connective can be defined in terms of the four standard connectives $\wedge$, $\vee$, $\to$ and $\bot$ misses out on implication.

However, if we adopt a reductive view, as we are doing in this paper, we can leave Prawitz's schema as it stands. We take the meaning of the standard connectives (in particular implication) for granted. Prawitz's schema then shows how the meaning of all connectives *except the standard ones* is reduced to the meaning of the standard ones. His completeness proof establishes that every connective which is characterised in a certain way is definable in terms of the standard connectives.

Therefore, from a reductive point of view, Prawitz's approach makes perfect sense. It is less general than the one advanced here in that he is proposing an introduction schema and generating a general elimination schema from it, rather than starting from independent introduction and elimination schemas and investigating their strength.

In defining harmonious elimination rules for given introduction rules, Prawitz does not need to use second-order quantification. In our terminology, Prawitz's work is a reductive approach focussing on the definability of connectives by their introduction meaning.

15.8 Outlook: The Foundational Approach

As mentioned in Sect. 15.2, the standard connectives $\wedge$, $\vee$, $\rightarrow$ and $\bot$ with their standard inference rules fall under the general schemas (15.1) and (15.3) for introduction and elimination inferences for an arbitrary connective c. In this sense they play no special role. However, they are needed and therefore taken for granted when formulating the introduction and elimination meaning of c and the corresponding notion of harmony. In order to establish harmony we need the logic of the standard operators. This is why our approach is reductive and not foundational.

This does not mean that our notion of harmony is not applicable to the standard operators. In fact, in Table 15.1 introduction and elimination meanings were associated with the standard connectives. For example, the formula $p_1 \vee p_2$ is the introduction meaning of $\vee$, the formula $\forall r(((p_1 \rightarrow r) \wedge (p_2 \rightarrow r)) \rightarrow r)$ its elimination meaning, and their equivalence

$$p_1 \vee p_2 \dashv\vdash \forall r(((p_1 \rightarrow r) \wedge (p_2 \rightarrow r)) \rightarrow r) \tag{15.22}$$

establishes that the standard introduction and elimination rules for disjunction as given in Table 15.1 are in harmony with each other. However, to show (15.22) we use the standard introduction and elimination rules for disjunction (plus those for $\wedge$, $\rightarrow$ and $\forall$), supposing that they are appropriate and therefore harmonious in some basic ('primordial') sense. If we chose different rules for the standard connectives which were not 'harmonious' in this basic sense, then (15.22) would perhaps no longer hold. This would not only affect disjunction but any other claim of reductive harmony.

At first sight one might think that this problem affects only disjunction and absurdity, as, in order to establish harmony for them, a real proof in **PL2** as a background logic must be given, using at least one logical rule for these connectives. In the case of conjunction and implication, introduction and elimination meanings are literally identical (see Table 15.1). To prove harmony we just need to rely on the identities

$$p_1 \wedge p_2 \dashv\vdash p_1 \wedge p_2 \qquad\qquad p_1 \rightarrow p_2 \dashv\vdash p_1 \rightarrow p_2$$

rather than on any logical rule of **PL2**. However, this impression is misleading. Let us consider implication. When defining the introduction meaning of a connective according to a given introduction rule, we translated the fact that a premiss depends on an assumption by an implication between the assumption and the premiss. That is, if an introduction rule for c is of the form

$$\frac{\ldots \quad \begin{matrix}[\boldsymbol{p_1}]\\ \boldsymbol{p_2}\end{matrix} \quad \ldots}{c(\ldots)}$$

the relation between assumption p_1 and premiss p_2 is interpreted as $p_1 \to p_2$. When defining the elimination meaning of a connective according to a given elimination rule, we translated the relation between minor premisses and conclusion by an implication as well. That is, in an elimination rule of the form

$$\frac{c(\ldots) \qquad \ldots \ \boldsymbol{p_1} \ \ldots}{\boldsymbol{p_2}} \tag{15.23}$$

the relationship between p_1 and p_2 was translated by the same implication $p_1 \to p_2$. This means that the dependence on an assumption and the relation between premiss and conclusion of a rule is given the same meaning. This is exactly what standard implication says:

$$\frac{\begin{matrix}[\boldsymbol{p_1}]\\ \boldsymbol{p_2}\end{matrix}}{p_1 \to p_2} \qquad \frac{p_1 \to p_2 \qquad \boldsymbol{p_1}}{\boldsymbol{p_2}} \ . \tag{15.24}$$

According to its introduction rule it expresses the dependence on an assumption, and according to its elimination rule (modus ponens) it expresses the relation between (minor) premiss and conclusion. In this way some fundamental harmony between implication introduction and modus ponens is built into the translation of rules for n-ary connectives to generate their introduction and elimination meanings. Something similar holds for conjunction, where we interpret the fact that p_1 and p_2 occur as two premisses in an introduction rule in the same way as the fact that p_1 and p_2 are the conclusions of two elimination rules, namely by conjunction $\wedge$:

$$\frac{\ldots \ \boldsymbol{p_1} \ \ldots \ \boldsymbol{p_2} \ \ldots}{c(\ldots)} \qquad \frac{c(\ldots) \qquad \ldots}{\boldsymbol{p_1}} \qquad \frac{c(\ldots) \qquad \ldots}{\boldsymbol{p_2}} \ . \tag{15.25}$$

This is exactly what standard conjunction says:

$$\frac{\boldsymbol{p_1} \qquad \boldsymbol{p_2}}{p_1 \wedge p_2} \qquad \frac{p_1 \wedge p_2}{\boldsymbol{p_1}} \qquad \frac{p_1 \wedge p_2}{\boldsymbol{p_2}} \ . \tag{15.26}$$

According to its introduction rule it expresses the association of two premisses, and according to its elimination rules it expresses the association of the conclusions of the two rules.

This shows that in our reductive approach we are implicitly relying on some fundamental harmony inherent in the rules of the standard connectives. How far it is possible to give a foundational analysis of this harmony is another matter. Any tool introduced to analyse and describe this harmony will possibly have to rely on some 'deeper' sort of harmony governing its own reasoning principles. This is a fundamental problem for approaches such as Lorenzen's (1955), von Kutschera's (1968) and our own (Schroeder-Heister 1984) that all deal with structural analogues of implication (especially higher-level rules) in order to deal with logical connectives.[20]

We should, however, already mention a point of specific interest: The standard connectives for implication and conjunction involved in describing the introduction and elimination meanings of logical constants are those with the two projections and modus ponens, respectively, as elimination rules. As explained above, the interpretation of the relation between p_1 and p_2 in (15.23) by means of implication corresponds to modus ponens in (15.24), and the interpretation of the association of the two elimination rules in (15.25) corresponds to the two projections in (15.26). This does not speak against generalised forms of implication or conjunction (in Table 15.2 denoted by & and $\supset$), but shows that *modus-ponens-based implication* and *projection-based conjunction* are not only connectives in their own right, but basic connectives that cannot be superseded by others.[21]

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[20] The problems and merits of a foundational analysis of logical constants in terms of quantified higher-level rules are discussed in (Schroeder-Heister 2014a).

[21] This is a point also reached by Dyckhoff (2009, 2015), on the basis of related considerations.

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