Proof-theoretic conservations of weak weak intuituonistic constructive set theories

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- Despite proof-theoretic weakness, these intuitionistic set theories have great expressive power.

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- However intuitionistically **Exp** is weaker than **Pow**.
    - Hint: think of ${}^x y$ as (possibly **enumerable**) set of constructive functions from x to y (e. g. algorithms).

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- T_4 contains full separation and is not really constructive.

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- Meaning: if S is **i**-*conservative*, then its arithmetical part is "correct", i.e. based on standard intuitionistic principles only.

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4. T_3 includes both **Ind** and **RDC**, as well as full $\in$-induction (**Ind**$_\in$).

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$\left|\mathbf{Basic}^{(\mathbf{i})} + \mathbf{Ext}\right| = \left|\mathbf{Basic}^{(\mathbf{i})} + \mathbf{Ext} + \Delta_0\text{-}\mathbf{Sep}\right| = \varepsilon_0$.
Moreover $\mathbf{Basic}^{(\mathbf{i})} + \mathbf{Ext} + \Delta_0\text{-}\mathbf{Sep}$ *is* **i**-*conservative,*
and hence conservative extension of HA.

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  - **Clps** $\equiv \mathrm{Ord}(x) \wedge (\forall s, t \in x)(\langle s, t\rangle \in r \leftrightarrow \langle s, t\rangle \notin r') \wedge \mathrm{WF}(x, r) \to (\exists f, y)\, \mathrm{TrClps}(f, x, r, y)$

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- Stronger results to be discussed later.

§3. More on **Basic + Ext +** Δ_0-**Sep**

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K. Sato [2009]: $|\mathbf{Basic} + \mathbf{Ext} + \Delta_0\text{-}\mathbf{Sep}| \geq \Gamma_0$ *(in fact* $= \Gamma_0$*).*

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§3. More on **Basic** + **Ext** + Δ_0-**Sep**

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* Classical inconstructive tautology:
$\forall x (A(x) \vee B(x)) \wedge (\exists x B(x) \to C) \to \forall x A(x) \vee C$, where $x \notin \mathrm{FV}(C)$ □

Theorem

§4. Stronger results

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$$(\exists z)\left((\forall r \in z)\,\mathrm{Tot}\,(r, x, y) \wedge \forall r \left(\begin{array}{c} \mathrm{Tot}\,(r, x, y) \to (\exists s \in z) \\ (s \subset r \wedge \mathrm{Tot}\,(s, x, y)) \end{array}\right)\right),$$

§4. Stronger results

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  Cpl is equivalent to **Anti-Reg**.

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- Within $\mathbf{Basic}^{(\mathbf{i})} + \Delta_0\text{-}\mathbf{Sep} + \mathbf{Enm}$:
  Cpl is equivalent to **Anti-Reg**.
- For brevity we use standard constructive version of $\mathrm{Ord}(x)$:

$$\mathrm{Ord}(x) \equiv \begin{array}{c} \mathrm{POrd}(x) \wedge \emptyset \in x \wedge \\ (\forall u)\left((\forall y \in x)(y \subset u \leftrightarrow y \in u) \rightarrow x \subset u\right) \end{array} .$$

§5. Academic question

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(cf. L. G. [1987]).

- **Q**. Wanted ? with $\left|\mathbf{Basic}^{(\mathbf{i})} + \mathbf{Ext} + \Delta_0\text{-}\mathbf{Sep} + ?\right| = \Gamma_0$.
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$\left|\mathbf{Basic}^{(\mathbf{i})} + \mathbf{Ext} + \Delta_0\text{-}\mathbf{Sep} + \mathbf{EHT}\,(+\mathbf{Exp})\right| =$
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$\left|\mathbf{Basic}+\mathbf{Ext}+\Delta_0\text{-}\mathbf{Sep}+\mathbf{EHT}\right|=\Gamma_0$.
Moreover $\mathbf{Basic}^{(\mathbf{i})}+\mathbf{Ext}+\Delta_0\text{-}\mathbf{Sep}+\mathbf{EHT}\,(+\mathbf{Exp})$ *is* **i**-*conservative, i.e. conservative over* $\mathrm{HA}+\mathbf{TI}_{\mathrm{Ar}}\,(<\Gamma_0)$.

- The proofs run along the lines of L. G. [1982,1988] in 3 steps:

§6. On proofs -1-

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2. Constructive cut elimination in AFC (most difficult part of proof).
3. Realizability elimination via forcing in *explicit* intuitionistic arithmetic AHA (along the lines of M. Beeson [1979]).

On proofs -2-

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- This yield for any arithmetical sentence A :

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2. $\mathrm{AFC} \vdash (A \ \textit{realizable}) \Rightarrow \mathrm{AHA} \vdash (A \ \textit{realizable})$,
3. $\mathrm{AHA} \vdash (A \ \textit{realizable}) \Rightarrow \mathrm{HA} \vdash A$, as desired.

Appendix: Stronger constructive set theories

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Consider Aczel-Rathjen's CZF [2000/2001] possibly extended by Sato's **Clps**.

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Consider Aczel-Rathjen's CZF [2000/2001] possibly extended by Sato's **Clps**.

Theorem

$\left|\mathrm{CZF}^{(\mathbf{i})}(+\mathbf{Clps})\right| = $ (*Howard ordinal*) $\varphi_{\varepsilon_{\Omega+1}}(0) = \left|\mathrm{T}_3(+\mathbf{Clps})\right|$.
Moreover $\mathrm{CZF}^{(\mathbf{i})}(+\mathbf{Clps})$ *and* $\mathrm{T}_3(+\mathbf{Clps})$ *are* **i**-*conservative.*

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