

# Advanced Mathematical Methods

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## 4 Mathematical Statistics

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# Outline: Mathematical Statistics

- 4.5 Specific probability distributions
- 4.6 Distribution of a function of a random variable
- 4.7 Moment generating functions (MGF)

# Readings

A. Papoulis and A. U. Pillai. *Probability, Random Variables and Stochastic Processes*.

Mc Graw Hill, fourth edition, 2002 Chapter 5

# Online References

MIT Course on Probabilistic Systems Analysis and Applied Probability (by John Tsitsiklis)

- ▶ Discrete RVs II: Functions of RV, conditional probabilities, specific distribution, total expectation theorem, joint probabilities

<https://www.youtube.com/watch?v=-qCEoqpwjf4>

## 4.5 Specific probability distributions

Some distributions stem from experimental situations.

### Existence theorem

For  $F_X(x)$  to be a distribution function, it must hold that

$$(i) F_X(x) = \int_{-\infty}^x f(u)du$$

$$(ii) f(x) \text{ non-negative and } \int_{-\infty}^{\infty} f(x)dx = 1$$

(iii)  $F_X(x)$  continuous from the right and

(iv) monotonically increasing from 0 to 1 as  $x$  goes from  $-\infty$  to  $\infty$

## 4.5 Specific probability distributions

### The normal distribution

$X$  is a gaussian or normal random variable with parameters  $\mu$  and  $\sigma^2$  if its density function is given by

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)$$

denoted  $X \sim N(\mu, \sigma^2)$

**linear transformation is also normally distributed:**

if  $X \sim N(\mu, \sigma^2)$ , then  $a + bX \sim N(a + b\mu, b^2\sigma^2)$

## 4.5 Specific probability distributions

standardization of  $X$  leads to standard normal distribution:

$$a = -\frac{\mu}{\sigma} \quad , \quad b = \frac{1}{\sigma}$$

$$z = \frac{x - \mu}{\sigma} \sim N(0, 1)$$

$$\Phi(z) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{z^2}{2}\right)$$

Thus , if  $X \sim N(\mu, \sigma)$ , then  $f(x) = \frac{1}{\sigma} \Phi\left(\frac{x-\mu}{\sigma}\right)$

## 4.5 Specific probability distributions

The  $\chi^2$  distribution:

$X$  is said to be  $\chi^2(n)$  with  $n$  degrees of freedom if

$$f_X(x) = \begin{cases} \frac{x^{\frac{n}{2}-1} e^{-\frac{x}{2}}}{2^{\frac{n}{2}} \Gamma(\frac{n}{2})} & x \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

if  $z \sim N(0, 1)$ , then  $x = z^2 \sim \chi^2(1)$

if  $z_i$  are iid  $N(0, 1)$ , then  $\sum_{i=1}^n z_i^2 \sim \chi^2(n)$



## 4.6 Distribution of a function of a random variable

transformation of the random variable  $X$  to a new random variable  $Y$  using a measurable function  $g(\cdot)$ :

$$Y = g(X)$$

requirements:

- ▶  $g(\cdot)$  needs to be invertible (monotonic function)
- ▶  $g(\cdot)$  needs to be continuously differentiable

## 4.6 Distribution of a function of a random variable

### Transformation theorem

$X$  is continuous with pdf  $f_X(x)$ .  $Y = g(X)$  is strictly monotonous and continuously differentiable,

$$f_Y(y) = \begin{cases} f_X(g^{-1}(y)) \cdot \left| \frac{d}{dy}(g^{-1}(y)) \right| & \text{for } y \in \{y : y = g(x); x \in \mathbb{W}_X\} \\ 0 & \text{else} \end{cases}$$

$\mathbb{W}_X$  is the domain of  $X$

## 4.7 Moment generating functions (MGF)

for a random variable  $X$  with pdf  $f_X(x)$ , the MGF is

$$M_X(t) = E[e^{tX}]$$
$$= \begin{cases} \sum_i e^{tx_i} f_X(x_i) & \text{if } X \text{ is discrete} \\ \int_{-\infty}^{\infty} e^{tx} f_X(x) dx & \text{if } X \text{ is continuous} \end{cases}$$

if the MGF exists, the  $k$  - *th* uncentered moment of  $X$  is given as

$$M_X^{(k)}(0) = \left. \frac{d^k M_X(t)}{dt^k} \right|_{t=0} = \mu'_k = E[X^k]$$

## 4.7 Moment generating functions (MGF)

if  $M_X(t)$  exists, then the MGF of  $Y = a + bX$  is

$$\begin{aligned}M_Y(t) &= E[e^{t(aX+b)}] \\&= e^{tb} \cdot E[e^{t(aX)}] \\&= e^{tb} \cdot E[e^{(ta)X}] \\&= e^{tb} \cdot M_X(at)\end{aligned}$$

if  $X$  and  $Y$  are independent, then the MGF of  $X + Y$  is  
 $M_X(t) \cdot M_Y(t)$

**Note:** the MGF of sums of random variables does not always exist