Strong Well–Quasi–Ordering Tree Theorems

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1. INTRODUCTION

Kruskal's tree theorem states that finite trees are well-quasi-ordered under the homeomeorphic embeddability (see [10]). Besides for its mathematical transparency, this theorem has applications in computer science (in the theory of rewriting, see [1], [3]). However, the most prominent feature of the Kruskal's theorem is its proof-theoretical strength that exceeds the one of the theory of predicative analysis, $\mathbf{ATR_0}$, which is well-known in mathematical logic and foundations of mathematics. Hence the Kruskal's theorem is not provable by predicative means. This fundamental result is due to Harvey Friedman (see [12]), who also invented finite trees labeled by bounded natural numbers under the homeomorphic embeddability with the asymmetrical gap-condition and proved (see [12]) that the corresponding well-quasi-ordering property is true but not provable in the theory of Π^1_1 comprehension axiom, $\mathbf{\Pi^1_1CA_0}$, which is stronger than $\mathbf{ATR_0}$ (in particular, $\mathbf{\Pi^1_1CA_0}$ easily proves the Kruskal's theorem). To put it more precisely, let T be a finite tree with a labeling function $\ell:\mathrm{V}(T)\to n$, where $\mathrm{V}(T)$ is the set of vertices in T, and $n\in\mathbb{N}$ is identified with $\{0,...,n-1\}$. Then $\langle T,\ell\rangle$ be the correlated n-bounded labeled finite tree. Now $\langle T_1,\ell_1\rangle$ is embeddable in $\langle T_2,\ell_2\rangle$ if there is a homeomorphic embedding $f: T_1\to T_2$ which preserves the labels while satisfying the following asymmetrical gap-condition

(a) If a vertex x is the immediate predecessor of a vertex y in T_1,

and u is any vertex between $f(x)$ and $f(y)$ in T_2,

then $\ell_1(y) = \ell_2(u)$.

THEOREM 1 (Friedman). *Let $n \in \mathbb{N}$ be fixed. The set of all n-bounded finite labeled trees is well-quasi-ordered by the above embeddability. Moreover, this statement is a theorem of* $\Pi^1_1\mathbf{CA}_0$. *However, the corresponding universal statement "For every $n \in \mathbb{N}$, the set of all n-bounded finite labeled trees is well-quasi-ordered by the above embeddability" is not provable in* $\Pi^1_1\mathbf{CA}_0$.

In order to generalize this result, the author invented (see [4], [5]) finite trees labeled by arbitrary (countable) ordinals under the homeomorphic embeddability which does not decrease the labels while satisfying the following symmetrical gap-condition

(s) If a vertex x is the immediate predecessor of a vertex y in T_1,

and u is any vertex between $f(x)$ and $f(y)$ in T_2,

then $\min\{\ell_1(x),\ \ell_1(y)\} \leq \ell_2(u)$.

It turns out that the corresponding generalized labeled tree theorem is accordingly stronger than the universal statement from Theorem 1. For let $\mathbf{ITR}_0$ be the theory of impredicative analysis that is defined analogously to its predicative counterpart, $\mathbf{ATR}_0$, but with respect to Π^1_1 transfinite recursion. That is, $\mathbf{ITR}_0$ enables to define sets by transfinite iteration of Π^1_1 comprehension on arbitrary (countable) ordinals, and hence it is essentially stronger than $\Pi^1_1\mathbf{CA}_0$. The following theorem is an immediate consequence of [5]. It shows that the generalized labeled tree theorem has exactly the strength of the impredicative analysis. (By contrast, [4] shows that its one-dimensional version dealing with finite intervals labeled by ordinals is just as strong as $\mathbf{ATR}_0$, thus being somewhat weaker than the Kruskal's theorem.)

THEOREM 2. *The set of all finite trees labeled by ordinals is well-quasi-ordered by the above embeddability. This statement is a theorem of* $\mathbf{ITR}_0$, *but it is not provable in a theory that arises from* $\mathbf{ITR}_0$ *by replacing* Π^1_1 *transfinite recursion axiom by the corresponding* Π^1_1 *transfinite recursion rule.*

Meanwhile, Harvey Friedman conjectured that finite trees with edges (not vertices!) labeled by ordinals are well-quasi-ordered by the homeomorphic embeddability with the following gap-condition

(e) Every edge x in T_1 is mapped by f onto a path in T_2 consisting of edges u such that $\ell_1(x) \leq \ell_2(u)$.

This conjecture was established by Igor Křiž (see [9]) at about the same time as the author's result (see Theorem 2 above) by using different ideas which are formalizable in the theory of Π^1_2 comprehension axiom, $\mathbf{\Pi^1_2CA}$, but certainly not in $\mathbf{ITR_0}$.

THEOREM 3 (Křiž). *The set of all finite trees with edges labeled by ordinals is well-quasi-ordered by the above embeddability. This statement is a theorem of* $\mathbf{\Pi^1_2CA}$.

It also follows from [9] that the generalized labeled tree theorem from Theorem 2 cannot be stronger than the one from Theorem 3. On the other hand, $\mathbf{ITR_0}$ is dramatically weaker than $\mathbf{\Pi^1_2CA}$, so that the proof from [9] can hardly estimate the strength of the Harvey Friedman's conjecture.

In this paper I prove by the "tree-priority" method of [5] that this conjecture is actually a theorem of $\mathbf{ITR_0}$, thus being proof-theoretically equivalent to the generalized labeled tree theorem from Theorem 2. Hence the Harvey Friedman's conjecture has exactly the strength of the impredicative analysis, $\mathbf{ITR_0}$. This answers a question posed by Harvey Friedman to the author at 1991 Joint Summer Research Conference on Graph Minors. The proof is purely combinatorial by nature and its formalization does not require any special knowledge in mathematical logic.

2. LABELING TREES BY ORDINALS

2.1. *Basic notations.* A (finite) tree is either the empty object, $\emptyset$, or a finite partial order $T = \langle S, \triangleleft \rangle$ with the minimal element, $\mathrm{r}(T) \in S$, for which $(\forall y \in S)(\mathrm{r}(T) \trianglelefteq y)$, and such that for any $x \in S$, $\{y \in S: y \triangleleft x\}$ is linearly ordered by "$\triangleleft$". We always assume $S \subset_{\mathrm{fin}} \mathbb{N}$ ($\mathbb{N}$ = the natural numbers) and $\mathrm{Card}(S) > 1$. Note that any $x, y \in S$ uniquely determine their infimum, $\inf(x,y) \in S$, for which $\inf(x,y) \trianglelefteq x$, $\inf(x,y) \trianglelefteq y$ and $(\forall z \in S)((z \trianglelefteq x \wedge z \trianglelefteq y) \rightarrow z \trianglelefteq \inf(x,y))$.

Let a tree $T=\langle S,\lhd\rangle$ be fixed. Elements of S are called vertices. r(T) is called the (uniquely determined) root of T. If $(\forall y\in S)\neg(x\lhd y)$ then x is called the end–vertex. If $x\lhd y$ then x is said to occur lower than y. If $x\lhd y$ and $(\forall z\in S)\neg(x\lhd z\lhd y)$, then we write pre($y$) = x and call x the (uniquely determined) predecessor of y. An edge is any pair $\langle \mathrm{pre}(y),y\rangle$. Any vertex x such that r(T) = pre(x) is called a root–neighbor. By V(T), End(T) and E(T) we respectively denote the set S of all vertices, the set of all end–vertices and the set of all edges.

For any two trees $T_1=\langle S_1,\lhd_1\rangle$ and $T_2=\langle S_2,\lhd_2\rangle$, let $f\in\mathrm{HEM}\colon T_1\to T_2$ express that f is the homeomorphic embedding of T_1 into T_1, i.e. a monomorphism preserving inf(–,–) as well as the order of each branching. The latter condition means that if $x=\mathrm{pre}(y)=\mathrm{pre}(z)$ in T_1, $f(x)=\mathrm{pre}(u)=\mathrm{pre}(v)$ in T_2, $u\unlhd_2 f(y)$ and $v\unlhd_2 f(z)$, then $y<z$ implies $u<v$ (in $\mathbb{N}$).

2.2. *Vertex-labeled trees.* Let $\mathcal{O}=\langle\mathcal{P},\prec\rangle$, $\mathcal{P}\subseteq\mathbb{N}$, be a fixed countable well-order. A vertex–labeled tree (v.t.) relative to $\mathcal{O}$ is a structure $V=\langle T,\ell\rangle$, T being the underlying tree, $\ell\colon \mathrm{V}(T)\to\mathcal{P}$ the labeling function. We define the embeddability "$\leq$" on v.t. (cf. [4], [5]).

Let $V_1=\langle S_1,\lhd_1,\ell_1\rangle=\langle T_1,\ell_1\rangle$ and $V_2=\langle S_2,\lhd_2,\ell_2\rangle=\langle T_2,\ell_2\rangle$ be nonempty v.t.. Let $f\colon V_1\leq V_2$ denote conjunction of the following assertions (2.2.1)–(2.2.3), where x,y and u range over S_1 and S_2, respectively, and $\mathrm{pre}_i(-)$ is pre(–) specified to T_i.

Then let $V_1\leq V_2$ abbreviate $\exists f(f\colon V_1\leq V_2)$.

(2.2.1) $$f\in\mathrm{HEM}\colon T_1\to T_2 \text{ and } f(\mathrm{End}(T_1))\subseteq\mathrm{End}(T_2).$$

(2.2.2) $$\ell_1(x)\preceq\ell_2\circ f(x).$$

(2.2.3) $$\text{If } \mathrm{pre}_1(y)=x \text{ and } f(x)\lhd_2 u\lhd_2 f(y), \text{ then } \min\{\ell_1(x),\ \ell_1(y)\}\preceq\ell_2(u).$$

The condition (2.2.3) is referred to as the symmetrical vertex–gap–condition.

The analogous "asymmetrical" embeddability "$\leq^*$" on v.t. arises by replacing (2.2.3) by the following stronger asymmetrical vertex–gap–condition

(2.2.4) $$\text{If } \mathrm{pre}_1(y)=x \text{ and } f(x)\lhd_2 u\lhd_2 f(y), \text{ then } \ell_1(y)\preceq\ell_2(u).$$

2.3. *Edge-labeled trees.* Let $\mathcal{O}=\langle\mathcal{P},\prec\rangle$ be as above. An edge–labeled tree (e.t.) relative to $\mathcal{O}$ is a structure $E=\langle T,\ell\rangle$, T being the underlying tree, $\ell\colon\mathrm{E}(T)\to\mathcal{P}$ the labeling function. The embeddability "$\leq$" on e.t. is defined as follows (cf. [9]).

Let $E_1 = \langle S_1, \lhd_1, \ell_1 \rangle = \langle T_1, \ell_1 \rangle$ and $E_2 = \langle S_2, \lhd_2, \ell_2 \rangle = \langle T_2, \ell_2 \rangle$ be nonempty e.t.. Let $f\colon E_1 \leq E_2$ denote conjunction of the following assertions (2.3.1) and (2.3.2), where x, y and u, v range over S_1 and S_2, respectively, and $\mathrm{pre}_i(-)$ is pre(−) specified to T_i.

Then let $E_1 \leq E_2$ be an abbreviation of $\exists f(f\colon E_1 \leq E_2)$.

(2.3.1): = (2.2.1).

(2.3.2) If $\mathrm{pre}_1(y) = x$, $\mathrm{pre}_2(v) = u$ and $f(x) \lhd_2 v \unlhd_2 f(y)$, then $\ell_1(\langle x,y \rangle) \preceq \ell_2(\langle u,v \rangle)$.

The condition (2.3.2) is referred to as the <u>edge-gap-condition</u>.

2.4. *Complex edge-labeled trees.* Let $\mathcal{O} = \langle \mathcal{P}, \prec \rangle$ be as above. A <u>complex edge-labeled tree</u> (c.e.t.) relative to $\mathcal{O}$ is a structure $C = \langle S, \lhd, M, \ell \rangle$, $T = \langle S, \lhd \rangle$ being the underlying tree, $M \subseteq S - \{\mathrm{r}(T)\}$ the set of distinguished vertices called marks, $\ell\colon \mathrm{E}(T) \to \mathcal{P}$ the labeling function, provided that the following conditions (2.4.1) and (2.4.2) hold.

(2.4.1) If $y \in M$ and $\mathrm{r}(T) \lhd x \lhd y$, then $\ell(\langle \mathrm{pre}(y), y \rangle) \prec \ell(\langle \mathrm{pre}(x), x \rangle)$.

(2.4.2) If $x, y \in M$, $x \lhd z \lhd y$ and $(\forall w \in M) \neg(x \lhd w \lhd y)$, then $\ell(\langle \mathrm{pre}(x), x \rangle) \preceq \ell(\langle \mathrm{pre}(z), z \rangle)$.

The (not necessarily transitive) embeddability "$\sqsubseteq$" on c.e.t. is defined as follows.

Let $C_1 = \langle S_1, \lhd_1, M_1, \ell_1 \rangle = \langle T_1, M_1, \ell_1 \rangle$ and $C_2 = \langle S_2, \lhd_2, M_2, \ell_2 \rangle = \langle T_2, M_2, \ell_2 \rangle$ be non-empty c.e.t.. Let $f\colon C_1 \sqsubseteq C_2$ denote conjunction of the following assertions (2.4.3)–(2.4.6), where x, y and u, v, w range over S_1 and S_2, respectively, and $\mathrm{pre}_i(-)$ is pre(−) specified to T_i.

Then let $C_1 \sqsubseteq C_2$ be an abbreviation of $\exists f(f\colon C_1 \sqsubseteq C_2)$.

(2.4.3): = (2.2.1)

(2.4.4): = (2.3.2).

(2.4.5) If $\mathrm{pre}_1(x) = \mathrm{r}(T_1)$, $\mathrm{pre}_2(v) = u$, $v \unlhd_2 f(x)$ and $x \in M_1$, then

$$\ell_1(\langle \mathrm{r}(T_1), x \rangle) \preceq \ell_2(\langle u, v \rangle).$$

(2.4.6) If $\mathrm{pre}_1(x) = \mathrm{r}(T_1)$, $\mathrm{pre}_2(v) = u$, $w \unlhd_2 v \unlhd_2 f(x)$ and $w \in M_2$, then

$$\ell_1(\langle \mathrm{r}(T_1), x \rangle) \preceq \ell_2(\langle u, v \rangle).$$

2.5. *Definitions.* Let $\mathfrak{S} = \{\mathfrak{S}(0), \mathfrak{S}(1), \ldots, \mathfrak{S}(n), \ldots\}$ be an infinite sequence of non-empty v.t. relative to $\mathcal{O}$. $\mathfrak{S}$ is called "$\leq$"-bad if $\mathfrak{S}(i) \leq \mathfrak{S}(j)$ does not hold for any $i < j$. The notion of a "$\leq^*$"-bad sequence of v.t., the notion of a "$\leq$"-bad sequence of e.t. and the notion of a "$\sqsubseteq$"-bad sequence of c.e.t., relative to $\mathcal{O}$, are specified accordingly.

Let Propositions A, B, C and D say that for any countable well-order $\mathcal{O}$, there is no infinite "$\leq$"-bad sequence of v.t relative to $\mathcal{O}$, no infinite "$\leq^*$"-bad sequence of v.t relative to $\mathcal{O}$, no "$\leq$"-bad sequence of e.t relative to $\mathcal{O}$ and no "$\sqsubseteq$"-bad sequence of c.e.t. relative to $\mathcal{O}$, respectively. Note that Propositions A, B and C respectively express that the corresponding structures of vertex-labeled trees and edge-labeled trees are well-quasi-orders. Recall that A and C are true propositions according to [5] and [9], respectively.

Let $\mathbf{ITR_0}$ (also known as $\mathbf{\Pi^1_1TR_0}$) be formal subsystem of analysis that extends the elementary analysis, $\mathbf{ACA_0}$ (being the $2nd$-order conservative extension of Peano Arithmetic), by the axiom standardly expressing in the $2nd$-order language that for any well-order O, there exists the hyperjump-hierarchy of sets along O (cf. [4], [5], see also [2], [11], [12] for more information on subsystems of analysis in question). By [5], Proposition A is provable in $\mathbf{ITR_0}$.

2.6. THEOREM. *Propositions* A, B *and* C *are all provable in* $\mathbf{ITR_0}$.

In the next two sections, we shall prove that Proposition C is a theorem of $\mathbf{ITR_0}$ (and in the final section, we shall prove in $\mathbf{ITR_0}$ Proposition D). The remainder of the theorem is easy. Namely, C implies B provably in $\mathbf{ACA_0}$. For let $\mathfrak{S}$ be any "$\leq^*$"-bad infinite sequence of non-empty v.t. relative to $\mathcal{O} = \langle\mathcal{P},\preceq\rangle$. Let $\rho(n)\in\mathcal{P}$ be the root-label of $\mathfrak{S}(n)$. Arguing in $\mathbf{ACA_0}$, there is a strictly increasing function $\mathfrak{f}$ on $\mathbb{N}$ such that $\rho\circ\mathfrak{f}(i)\preceq\rho\circ\mathfrak{f}(j)$ holds for all $i<j$. Let $\mathfrak{S}' = \mathfrak{S}\circ\mathfrak{f}$ be the correlated infinite "$\leq^*$"-bad subsequence of $\mathfrak{S}$. Let $\mathcal{O}' = \langle\mathcal{P}',\preceq\rangle$ be the minimal well-ordered proper extension of $\mathcal{O}$, and let $\mathcal{O}_1 = \langle\mathcal{P}_1,\preceq_1\rangle$ be the disjoint well-ordered sum $\mathcal{O}'+\mathcal{O}$. Let σ be the order-type of $\mathcal{O}$, then the order-type of $\mathcal{O}_1$ be $(\sigma+1)+\sigma$. Let $\mathfrak{S}_1$ be the sequence of v.t. relative to $\mathcal{O}_1$ that arises by replacing in every v.t. $\mathfrak{S}'(n)$, labels $\pi_k\in\mathcal{P}$ of all its root-neighbors by the labels corresponding to $\sigma+\pi_k$ in $\mathcal{P}_1$. Clearly, $\mathfrak{S}_1$ is "$\leq^*$"-bad. Let $\mathfrak{S}_2$ be the sequence of e.t. relative to $\mathcal{O}_1$ such that each e.t. $\mathfrak{S}_2(n)$ arises by setting $\ell(\langle \mathrm{pre}(x),x\rangle) := \ell(x)$ for every edge in $\mathfrak{S}_1(n)$. Assume that $f:\mathfrak{S}_2(i)\leq\mathfrak{S}_2(j)$ holds for $i<j$. Clearly, f must preserve the roots. By definition of $\mathfrak{S}'$, this implies $f:\mathfrak{S}_1(i)\leq^*\mathfrak{S}_1(j)$, i.e. $\mathfrak{S}_1$ is not "$\leq^*$"-bad – a contradiction.

2.7. *Remark.* It is readily seen from the above arguments (see also [9]) that Propositions B and C are in fact equivalent, provably in $\mathbf{ACA_0}$. Since A has the proof-theoretical strength of $\mathbf{ITR_0}$ (see [5], [6], [7]), while being obviously implied by B, this shows that C has the proof-theoretical strength of $\mathbf{ITR_0}$ as well. To put it more exhaustive, the following are the case.

(2.7.1) C is a theorem of $\mathbf{ITR_0}$, but it is not provable in a theory that arises from $\mathbf{ITR_0}$ by replacing Π^1_1 transfinite recursion axiom by the corresponding Π^1_1 transfinite recursion rule.

(2.7.2) $\mathbf{ITR_0}$ has the same Π^1_1-theorems, and hence the same arithmetical theorems, as $\mathbf{ACA_0}$ extended by C.

(2.7.3) The consistency of $\mathbf{ITR_0}$ is provable in $\mathbf{ACA}$ (= $\mathbf{ACA_0}$ extended by full induction schema) extended by C.

2.8. *Note.* In the sequel we use Higman's well-quasi-ordering theorem in the following form (which has the same proof-theoretical strength as the familiar result from [8]).

THEOREM (Higman). *Let* $\mathcal{R} = \langle \mathcal{S}, \sqsubseteq \rangle$ *be any binary relation that admits no infinite "*$\sqsubseteq$*"-bad sequence. Let* $\mathcal{R}^{<\omega} = \langle \mathcal{S}^{<\omega}, \sqsubseteq^{<\omega} \rangle$, *where* $\mathcal{S}^{<\omega}$ *consists of all tuples* $\langle x_0, \ldots x_n \rangle$, $x_i \in \mathcal{S}$, *and*

$$\langle x_0, \ldots x_n \rangle \sqsubseteq^{<\omega} \langle y_0, \ldots y_m \rangle \Leftrightarrow (\exists\, 0 \leq f(0) < f(1) < \ldots < f(n) \leq m)(\forall i \leq n)(x_i \sqsubseteq y_{f(i)}).$$

Then $\mathcal{R}^{<\omega}$ *admits no infinite "*$\sqsubseteq^{<\omega}$*"-bad sequence.*

3. PROOF OF THEOREM 2.6. PART I

3.1. *Definitions.* Let a well-order $\mathcal{O} = \langle \mathcal{P}, \preceq \rangle$ be fixed. For the sake of brevity, in the sequel we drop "relative to $\mathcal{O}$" when speaking on c.e.t.. A non-empty c.e.t. $C = \langle T, M, \ell \rangle$ is called <u>open</u> if it has some root-neighbor $x \notin M$. A non-open and non-empty c.e.t. is called <u>nopen</u>. A nopen c.e.t. $C = \langle T, M, \ell \rangle$ is called <u>closed</u> if it has the unique root-neighbor x (hence $x \in M$ holds). In this case, the corresponding (uniquely determined) lowest edge $\langle \mathrm{r}(T), x \rangle$ is denoted by $\mathrm{e}(T)$. For any $\nu \in \mathcal{P}$, a non-empty c.e.t. $C = \langle T, M, \ell \rangle$ is called <u>closed before ν</u>, respectively <u>behind ν</u>, if it is closed and (its lowest label) $\ell(\mathrm{e}(T))$ is $\prec \nu$, respectively $\succeq \nu$ (in $\mathcal{O}$). For the sake of brevity we often write $\mathrm{r}(C)$, $\mathrm{e}(C)$, etc. instead of $\mathrm{r}(T)$, $\mathrm{e}(T)$, etc., respectively, provided that $C = \langle T, M, \ell \rangle$.

A c.e.t. $C_1 = \langle S_1,\triangleleft_1,M_1,\ell_1\rangle = \langle T_1,M_1,\ell_1\rangle$ is a subtree of a c.e.t. $C_2 = \langle S_2,\triangleleft_2,M_2,\ell_2\rangle = \langle T_2,M_2,\ell_2\rangle$ if $\triangleleft_1 = \triangleleft_2 \restriction S_1$, $M_1 = M_2 \cap S_1$ and $\ell_1 = \ell_2 \restriction T_1$, where S_1 satisfies (3.1.1)-(3.1.3.).

(3.1.1) $$S_1 \subseteq S_2.$$

(3.1.2) $$\text{If } x,y \in S_1,\ z \in S_2 \text{ and } x \triangleleft_2 z \triangleleft_2 y, \text{ then } z \in S_1.$$

(3.1.3) $$\text{If } x,y \in S_1 \text{ and } z = \inf(x,y) \text{ in } T_2, \text{ then } z \in S_1.$$

This relation we denote $C_1 \underline{\ll} C_2$. Clearly, "$\underline{\ll}$" and its sharp version "$\ll$" are both transitive.

The (finite) collection of all subtrees of a given a c.e.t. C is thought to be lexicographically well ordered such that $C_1 \ll C_2$ always implies that C_1 is smaller than C_2.

A c.e.t. $C_1 = \langle T_1,M_1,\ell_1\rangle$ occurs in the root–piece of $C_2 = \langle S_2,\triangleleft_2,M_2,\ell_2\rangle = \langle T_2,M_2,\ell_2\rangle$ if $C_1 \underline{\ll} C_2$ and for some root-neighbor y in T_1 there is no $x \in M_2$ with $x \trianglelefteq_2 y$. This relation we denote $C_1 \in (C_2)_{rp}$. Thus in particular, an edge $\langle \mathrm{pre}(y),y\rangle$ occurs in the root piece of $C = \langle S,\triangleleft,M,\ell\rangle$ if there is no $x \in M$ with $x \trianglelefteq y$.

3.2. *Definition.* Let $\mathbb{C}[\mathcal{O}]$ be the set of all c.e.t. relative to $\mathcal{O}$, and let $\mathfrak{F}:\mathbb{N}\to\mathbb{C}[\mathcal{O}]$ be "$\sqsubseteq$"–bad. We define a suitable minimal cofinal c.e.t.–sequence $\mathsf{M}_{cf}(\mathfrak{F})$: $\mathbb{N}\to\mathbb{C}[\mathcal{O}]$, and the correlated strictly increasing index-sequence $\mathfrak{f}:\mathbb{N}\to\mathbb{N}$, by the following simultaneous recursion.

(3.2.1) Let $\mathfrak{f}(0) := \min\{\mathrm{i}: \mathfrak{F}(\mathrm{i}) \neq \emptyset\}$. For every $\mathrm{m} < \mathfrak{f}(0)$, let $\mathsf{M}_{cf}(\mathfrak{F})(\mathrm{m}) := \emptyset$.

(3.2.2) Suppose that $\mathsf{M}_{cf}(\mathfrak{F})(\mathrm{j})$ and $\mathfrak{f}(\mathrm{i})$ are defined for all $\mathrm{i} \leq \mathrm{k}$ and $\mathrm{j} < \mathfrak{f}(\mathrm{k})$.

(3.2.2.1) Let $\mathsf{M}_{cf}(\mathfrak{F})\circ\mathfrak{f}(\mathrm{k})$ be the minimal subtree $C \underline{\ll} \mathfrak{F}\circ\mathfrak{f}(\mathrm{k})$ for which there is a strictly increasing $\mathfrak{g}:\mathbb{N}\to\mathbb{N}$ with $\mathfrak{g}(0) = \mathfrak{f}(\mathrm{k})$ and a $\mathfrak{G}:\mathbb{N}\to\mathbb{C}[\mathcal{P}]$ such that $\mathfrak{G}(0) = C$ and the following conditions hold for all $\mathrm{n,m,j} \in \mathbb{N}$.

(a) $$\mathfrak{F}\circ\mathfrak{g}(\mathrm{n}) \neq \emptyset \neq \mathfrak{G}(\mathrm{n}).$$

(b) If $\mathrm{j} < \mathfrak{f}(\mathrm{k})$ and $\mathrm{n} < \mathrm{m}$, then $\neg(\mathsf{M}_{cf}(\mathfrak{F})(\mathrm{j}) \sqsubseteq \mathfrak{G}(\mathrm{n}))$ and $\neg(\mathfrak{G}(\mathrm{n}) \sqsubseteq \mathfrak{G}(\mathrm{m}))$.

(c) $$\mathfrak{G}(\mathrm{n}) \underline{\ll} \mathfrak{F}\circ\mathfrak{g}(\mathrm{n}).$$

(d) If $\mathfrak{G}(\mathrm{n}) \neq \mathfrak{F}\circ\mathfrak{g}(\mathrm{n})$, then $\mathfrak{F}\circ\mathfrak{g}(\mathrm{n})$ is open and $\mathfrak{G}(\mathrm{n}) \in (\mathfrak{F}\circ\mathfrak{g}(\mathrm{n}))_{rp}$.

(3.2.2.2) Let $\mathfrak{f}(\mathrm{k}+1)$ be the minimal $\mathrm{i} > \mathfrak{f}(\mathrm{k})$ for which there are $\mathfrak{g}$ and $\mathfrak{G}$ as in (2.2.2.1) where $C = \mathsf{M}_{cf}(\mathfrak{F})\circ\mathfrak{f}(\mathrm{k})$ and $\mathfrak{g}(1) = \mathrm{i}$. For every $\mathfrak{f}(\mathrm{k}) < \mathrm{m} < \mathfrak{f}(\mathrm{k}+1)$, let $\mathsf{M}_{cf}(\mathfrak{F})(\mathrm{m}) := \emptyset$.

3.3. *Definition.* Under the same assumptions as in 3.2 (see above), let $\mathfrak{F}:\mathbb{N}\to\mathbb{C}[\mathcal{O}]$ be "$\sqsubseteq$"-bad, $\mathcal{O}=\langle\mathcal{P},\preceq\rangle$ and $\nu\in\mathcal{P}$. We define a suitable minimal limit behind ν c.e.t.-sequence $\mathsf{M}_{lim}(\nu,\mathfrak{F})$: $\mathbb{N}\to\mathbb{C}[\mathcal{O}]$, and the correlated strictly increasing index-sequence $\mathfrak{f}:\mathbb{N}\to\mathbb{N}$, by the same clauses as in 3.2, except replacing in (3.2.2.1) clause (d) by the following clause (e).

(e) If $\mathfrak{G}(n)\neq\mathfrak{F}\circ\mathfrak{g}(n)$ then $\mathfrak{G}(n)$ is closed behind ν.

3.4. LEMMA. *Let $\mathfrak{F}:\mathbb{N}\to\mathbb{C}[\mathcal{O}]$ be "$\sqsubseteq$"-bad, let $\mathcal{O}=\langle\mathcal{P},\preceq\rangle$ and $\nu\in\mathcal{P}$ be as above. Then both $\mathsf{M}_{cf}(\mathfrak{F})$ and $\mathsf{M}_{lim}(\nu,\mathfrak{F})$ are "$\sqsubseteq$"-bad, and the following are the case.*

(3.4.1) *Suppose that for all $m<n$, if $\mathfrak{F}(m)=\langle T_m,M_m,\ell_m\rangle$ and $\mathfrak{F}(n)=\langle T_n,M_n,\ell_n\rangle$ then $\ell_m(\langle r(T_m),x\rangle)\preceq\ell_n(a)$ holds for any root-neighbor $x\in M_m$ and for any edge $a\in(\mathfrak{F}(n))_{rp}$. Then for any "$\sqsubseteq$"-bad $\mathfrak{G}:\mathbb{N}\to\mathbb{C}[\mathcal{P}]$ there is no strictly increasing function $\mathfrak{g}$ on $\mathbb{N}$ such that for every $n\in\mathbb{N}$,*

$$\mathfrak{G}(n)\ll\mathsf{M}_{cf}(\mathfrak{F})\circ\mathfrak{g}(n)\ \textit{and}\ \mathfrak{G}(n)\in(\mathsf{M}_{cf}(\mathfrak{F})\circ\mathfrak{g}(n))_{rp}.$$

(3.4.2) *For any "$\sqsubseteq$"-bad $\mathfrak{G}:\mathbb{N}\to\mathbb{C}[\mathcal{O}]$ there is no strictly increasing function $\mathfrak{g}$ on $\mathbb{N}$ such that for every $n\in\mathbb{N}$,*

$$\mathfrak{G}(n)\ll\mathsf{M}_{lim}(\nu,\mathfrak{F})\circ\mathfrak{g}(n)\ \textit{and}\ \mathfrak{G}(n)\ \textit{is closed behind}\ \nu.$$

(3.4.3) *Suppose that the set $\{C:(\exists n\in\mathbb{N})(C\underline{\ll}\mathfrak{F}(n)$ and C is closed before $\nu)\}$ admits no infinite "$\sqsubseteq$"-bad sequence. Then $\mathsf{M}_{lim}(\nu,\mathfrak{F})$ contains at most finitely many components which are neither open nor closed behind ν.*

Proof. That $\mathsf{M}_{cf}(\mathfrak{F})$ and $\mathsf{M}_{lim}(\nu,\mathfrak{F})$ are "$\sqsubseteq$"-bad follows immediately from definition. (3.4.1): Suppose that there is a "$\sqsubseteq$"-bad sequence $\mathfrak{G}:\mathbb{N}\to\mathbb{C}[\mathcal{O}]$ such that for every n, $\mathfrak{G}(n)$ is a proper subtree occurring in the root-piece of $\mathsf{M}_{cf}(\mathfrak{F})\circ\mathfrak{g}(n)$, where $\mathfrak{g}$ is fixed. Since $\mathsf{M}_{cf}(\mathfrak{F})\circ\mathfrak{g}(n)$ is a nonempty subtree of $\mathfrak{F}\circ\mathfrak{g}(n)$, we have $\mathfrak{F}\circ\mathfrak{g}(n)\neq\emptyset$. Since $\mathfrak{G}(n)\neq\mathsf{M}_{cf}(\mathfrak{F})\circ\mathfrak{g}(n)$, we have $\mathfrak{G}(n)\neq\mathfrak{F}\circ\mathfrak{g}(n)$, and hence $\mathsf{M}_{cf}(\mathfrak{F})\circ\mathfrak{g}(n)\in(\mathfrak{F}\circ\mathfrak{g}(n))_{rp}$ holds by the clause (d) of (3.2.2.1). Therefore, $\mathfrak{G}(n)\in(\mathsf{M}_{cf}(\mathfrak{F})\circ\mathfrak{g}(n))_{rp}$ implies $\mathfrak{G}(n)\in(\mathfrak{F}\circ\mathfrak{g}(n))_{rp}$. Finally, for any $m<n$, $\mathsf{M}_{cf}(\mathfrak{F})\circ\mathfrak{g}(m)\sqsubseteq\mathfrak{G}(n)$ implies $\mathsf{M}_{cf}(\mathfrak{F})\circ\mathfrak{g}(m)\sqsubseteq\mathsf{M}_{cf}(\mathfrak{F})\circ\mathfrak{g}(n)$, since the only nontrivial condition (2.4.5) holds by the assumption of (3.4.1) – a contradiction with "$\sqsubseteq$"-badness of $\mathsf{M}_{cf}(\mathfrak{F})$. As a result, $\mathfrak{G}$ satisfies all the requirements of (3.2.2.1), and hence it could be chosen in the definition of $\mathsf{M}_{cf}(\mathfrak{F})\circ\mathfrak{g}(0)$ – but this contradicts the minimality of $\mathsf{M}_{cf}(\mathfrak{F})\circ\mathfrak{g}(0)$, since $\mathfrak{G}(n)$ is smaller than $\mathsf{M}_{cf}(\mathfrak{F})\circ\mathfrak{g}(0)$.
(3.4.2): This is proved analogously.

(3.4.3): Suppose that there is an increasing function $\mathfrak{h}$ on $\mathbb{N}$ such that $\mathsf{M}_{\mathrm{lim}}(\nu,\mathfrak{F})\circ\mathfrak{h}(\mathrm{n})$ are all nopen, but not closed behind ν. Let $\mathfrak{H}$ abbreviate $\mathsf{M}_{\mathrm{lim}}(\nu,\mathfrak{F})\circ\mathfrak{h}$. So for any $\mathfrak{H}(\mathrm{n}) = \langle T,M,\ell\rangle$, if $\mathrm{r}(T) = \mathrm{pre}(x)$ then $x\in M$. Now by the assumption of (3.4.3), $\mathfrak{H}$ has no infinite subsequence consisting of closed c.e.t.. Therefore, we may just as well assume that every $\mathfrak{H}(\mathrm{n})$ has at least two root-neighbors. Consider the collection of all c.e.t. $C' = \langle T',M\restriction S',\ell\restriction S'\rangle$, where $T' = \langle S',\lhd\restriction S'\rangle$, $\mathfrak{H}(\mathrm{n}) = \langle T,M,\ell\rangle = \langle S,\lhd,M,\ell\rangle$, $\mathrm{r}(T) = \mathrm{pre}(x)$ and $S' = \{\mathrm{r}(T)\}\cup\{v\in \mathrm{S}: x\unlhd v\}$ hold for some $x\in \mathrm{V}(T)$. Since $\mathfrak{H}$ is "$\sqsubseteq$"-bad, it is clear by an obvious specification of the Higman's theorem that there is an infinite "$\sqsubseteq$"-bad sequence $\mathfrak{H}'$ consisting of such c.e.t. C'. By the assumption of (3.4.3), there is an infinite "$\sqsubseteq$"-bad $\mathfrak{H}'$-subsequence $\mathfrak{G}$ consisting of such C' which are closed behind ν. Now that all these C' are closed proper subtrees of the correlated components of $\mathfrak{H}$, this leads to a contradiction as in the previous case (3.4.2). □

3.5. *Definition.* Let $\mathfrak{F}:\mathbb{N}\to\mathbb{C}[\mathcal{O}]$ be any infinite sequence in $\mathbb{C}[\mathcal{O}]$, let $\mathcal{O} = \langle\mathcal{P},\preceq\rangle$ and $\tau\in\mathcal{P}$. Then $\mathfrak{F}$ is called $\underline{\tau\text{-bad}}$ in $\mathbb{C}[\mathcal{O}]$ if the following hold.

(3.5.1) $\mathfrak{F}$ is "$\sqsubseteq$"-bad.

(3.5.2) $\{C{:}(\exists \mathrm{n}\in\mathbb{N})(C\underline{\ll}\mathfrak{F}(\mathrm{n})$ *and* C *is closed*$)\}$ admits no infinite "$\sqsubseteq$"-bad sequence.

(3.5.3) If $\mathfrak{F}(\mathrm{n}) = \langle T,M,\ell\rangle$ and $x\in M$, then $\ell(\langle \mathrm{pre}(x),x\rangle)\prec\tau$.

(3.5.4) If $\mathfrak{F}(\mathrm{n}) = \langle T,M,\ell\rangle$ and $\langle \mathrm{pre}(x),x\rangle\in(\mathfrak{F}(\mathrm{n}))_{\mathrm{rp}}$, then $\tau\preceq\ell(\langle \mathrm{pre}(x),x\rangle)$.

3.6. LEMMA. *For any well-order* $\mathcal{O} = \langle\mathcal{P},\preceq\rangle$ *and any* $\nu\in\mathcal{P}$, *there is no* ν*-bad sequence in* $\mathbb{C}[\mathcal{O}\restriction\nu]$.

Proof. Suppose that $\mathfrak{F}:\mathbb{N}\to\mathbb{C}[\mathcal{O}\restriction\nu]$ is ν-bad. Since all labels occurring in $\mathfrak{F}(\mathrm{n})$, $\mathrm{n}\in\mathbb{N}$, are $\prec\nu$, (3.5.4) implies $(\mathfrak{F}(\mathrm{n}))_{\mathrm{rp}} = \emptyset$, i.e. if $\mathfrak{F}(\mathrm{n}) = \langle T,M,\ell\rangle$ and $\mathrm{r}(T) = \mathrm{pre}(x)$, then $x\in M$. Now by (3.5.2), the set of all lowest closed subtrees of $\mathfrak{F}(\mathrm{n})$, $\mathrm{n}\in\mathbb{N}$, admits no infinite "$\sqsubseteq$"-bad sequence. Hence there is no infinite "$\sqsubseteq$"-bad sequence consisting of c.e.t. $C' = \langle T',M\restriction S',\ell\restriction S'\rangle$, where $T' = \langle S',\lhd\restriction S'\rangle$, $\mathfrak{F}(\mathrm{n}) = \langle T,M,\ell\rangle = \langle S,\lhd,M,\ell\rangle$, $\mathrm{r}(T) = \mathrm{pre}(x)$ and $S' = \{\mathrm{r}(T)\}\cup\{v\in \mathrm{S}: x\unlhd v\}$ hold for some $\mathrm{n}\in\mathbb{N}$ and $x\in S$. Since all these C' are closed (before ν), it is clear from definition of "$\sqsubseteq$" for closed c.e.t. that, by an obvious specification of the Higman's theorem, there is no "$\sqsubseteq$"-bad sequence whose components are such C' – a contradiction with (3.5.1). □

The remainder of Theorem 2.6. now readily follows from the above lemma and the assertion that the following theorem is provable in $\mathbf{ITR_0}$.

3.7. THEOREM. *For any well-order* $\mathcal{O} = \langle \mathcal{P}, \preceq \rangle$ *and any* $\nu \in \mathcal{P}$, *if there exists an infinite* "$\leq$"*-bad sequence of e.t. relative to* $\mathcal{O}\restriction\nu$, *then there exists a* ν*-bad sequence in* $\mathbb{C}[\mathcal{O}\restriction\nu]$.

Proof. Part I (Construction). We define the operator R which, for any $\sigma \preceq \tau \preceq \nu$ in $\mathcal{O}$, and for any given σ-bad sequence $\mathfrak{S}: \mathbb{N} \to \mathbb{C}[\mathcal{O}\restriction\nu]$, produces the appropriate τ-bad sequence $\mathsf{R}(\sigma,\tau,\mathfrak{S})$: $\mathbb{N} \to \mathbb{C}[\mathcal{O}\restriction\nu]$. In particular, if $\mathfrak{S}$ is any given "$\leq$"-bad sequence of e.t. relative to $\mathcal{O}\restriction\nu$, then $\mathsf{R}(0,\nu,\mathfrak{S})$ will be the required ν-bad sequence in $\mathbb{C}[\mathcal{O}\restriction\nu]$.

$\mathsf{R}(\sigma,\tau,\mathfrak{S})$ arises by the following transfinite recursion on τ. For the sake of brevity we assume that the previous operators $\mathsf{M}_{\mathrm{cf}}(-)$ and $\mathsf{M}_{\mathrm{lim}}(\nu,-)$ are everywhere defined on the whole domain $\mathbb{N} \to \mathbb{C}[\mathcal{O}\restriction\nu]$.

(3.7.1) *Preliminary clause.* Set $\mathsf{R}(\sigma,\sigma,\mathfrak{S}) := \mathfrak{S}$.

(3.7.2) *Successor clause.* Let $\tau = \rho+1$ in $\mathcal{O}\restriction\nu$, where $\sigma \preceq \rho$. Let $\mathsf{R}(\sigma,\tau,\mathfrak{S}) := \mathfrak{G}_3$ for the following auxiliary sequences $\mathfrak{G}_1$, $\mathfrak{G}_2$, $\mathfrak{G}_3$: $\mathbb{N} \to \mathbb{C}[\mathcal{O}\restriction\nu]$.

(S.1) Set $\mathfrak{G}_1 := \mathsf{R}(\sigma,\rho,\mathfrak{S})$.

(S.2) Set $\mathfrak{G}_2 := \mathsf{M}_{\mathrm{cf}}(\mathfrak{G}_1)$.

(S.3) $\mathfrak{G}_3$ arises as follows. Take any $n \in \mathbb{N}$, and let $\mathfrak{G}_2(n) = \langle T,M,\ell \rangle$. Set $\mathfrak{G}_3(n) := \langle T,N,\ell \rangle$, $M \subseteq N$, where N extends M by adding every lowest vertex $x \in V(T)$ with $\langle \mathrm{pre}(x),x \rangle \in (\mathfrak{G}_2(n))_{\mathrm{rp}}$ and $\ell(\langle \mathrm{pre}(x),x \rangle) = \rho$.

(3.7.3) *Limit clause.* Let τ be limit in $\mathcal{O}\restriction\nu$, where $\sigma \prec \tau$. Let $\{\tau\}: \mathbb{N} \to \mathcal{P}$ be the correlated canonical τ-fundamental sequence, i.e. a sequence that satisfies $(\forall i<j \in \mathbb{N})(\{\tau\}(i) \prec \{\tau\}(j) \prec \tau)$ and $(\forall \alpha \prec \tau)(\exists i \in \mathbb{N})(\alpha \prec \{\tau\}(i))$. Let $\{\tau\}_\sigma: \mathbb{N} \to \mathcal{P}$ be the analogous τ-fundamental sequence behind σ that is defined by $\{\tau\}_\sigma(0) := \sigma$ and $\{\tau\}_\sigma(i+1) := \{\tau\}(i+\min\{j : \sigma \prec \{\tau\}(j)\})$.

Let $\mathsf{R}(\sigma,\tau,\mathfrak{S}) := \mathfrak{H}_7$, where the auxiliary sequences $\mathfrak{D}: \mathbb{N}^2 \to \mathbb{C}[\mathcal{O}\restriction\nu]$ and $\mathfrak{R}$, $\mathfrak{H}_1, \ldots, \mathfrak{H}_7$: $\mathbb{N} \to \mathbb{C}[\mathcal{O}\restriction\nu]$ are as follows.

(L.0) $\mathfrak{R}: \mathbb{N} \to \mathbb{C}[\mathcal{O}\restriction\nu]$ and $\mathfrak{D}: \mathbb{N}^2 \to \mathbb{C}[\mathcal{O}\restriction\nu]$ are defined recursively as follows. Let $\mathfrak{R}(0) := \mathfrak{S}(n)$ and $\mathfrak{R}(i+1) := \mathsf{R}(\{\tau\}_\sigma(i), \{\tau\}_\sigma(i+1), \mathfrak{R}(i))$, and let $\mathfrak{D}(i,n) := \mathfrak{R}(i)(n)$.

(L.1) Let $\mathfrak{H}_1:\mathbb{N}\to\mathbb{C}[\mathcal{O}\upharpoonright\nu]$ be the corresponding diagonal sequence that is formally definad by $\mathfrak{H}_1(n):=\mathfrak{D}(J(n),n)$, where $J(n):=\min\{i:(\forall j>i)\mathfrak{D}(i,n)=\mathfrak{D}(j,n)\}$ and $\mathfrak{D}(\infty,n):=\mathfrak{S}(n)$. (In the next section we prove that $J(n)\in\mathbb{N}$ holds for all $n\in\mathbb{N}$, so that the latter clause is unimportant.)

(L.2) Let $\mathfrak{H}_2:=\mathsf{M}_{lim}(\sigma,\mathfrak{H}_1)$.

(L.3) $\mathfrak{H}_3$ arises as follows. Put $\mathfrak{H}_3(0):=\mathfrak{H}_2(0)$. In order to determine $\mathfrak{H}_3(k+1)$, consider $\mathfrak{H}_2(k+1)=\langle T,M,\ell\rangle$. For any $m\leq k$, let $\mathfrak{H}_3(m)=\langle T_m,M_m,\ell_m\rangle$, and let $J(-)$ be as in (L.1) above. Now set

$$\mathfrak{H}_3(k+1):=\begin{cases}\emptyset, & \text{if } \mathfrak{H}_2(k+1) \text{ is closed behind } \sigma \text{ but } (\star) \text{ fails (see below)},\\ \mathfrak{H}_2(k+1), & \text{otherwise.}\end{cases}$$

$(\star)$: $$(\forall m\leq k)(\forall a\in E(T_m))(\ell_m(a)\prec\tau\to\max\{\ell_m(a),\{\tau\}_\sigma(J(m))\}\preceq\ell(e(T))).$$

(L.4) $\mathfrak{H}_4$ arises as follows. Take any $n\in\mathbb{N}$. If $\mathfrak{H}_3(n)=\langle T,M,\ell\rangle$ then let

$$\mathfrak{H}_4(n):=\begin{cases}\langle T,M-\{x\},\ell\rangle, & \text{if } \mathfrak{H}_3(n) \text{ is closed behind } \sigma \text{ and } r(T)=\mathrm{pre}(x),\\ \mathfrak{H}_3(n), & \text{otherwise}.\end{cases}$$

(L.5) $\mathfrak{H}_5$ arises as follows. Take any $n\in\mathbb{N}$. If $\mathfrak{H}_4(n)=\langle T,M,\ell\rangle$ then let

$$\mathfrak{H}_5(n):=\begin{cases}\mathfrak{H}_4(n), & \text{if } \mathfrak{H}_4(n)=\mathfrak{H}_3(n),\\ \langle T,M,\ell'\rangle, & \text{otherwise.}\end{cases}$$

where

$$\ell'(a):=\begin{cases}\max\{\ell(b)\prec\tau: b\in E(T)\cap(\mathfrak{H}_4(n))_{rp}\}, & \text{if } a\in E(T)\cap(\mathfrak{H}_4(n))_{rp},\\ \ell(a), & \text{otherwise.}\end{cases}$$

(L.6) Set $\mathfrak{H}_6:=\mathsf{M}_{cf}(\mathfrak{H}_5)$.

(L.7) $\mathfrak{H}_7$ arises as follows. Take any $n\in\mathbb{N}$. If $\mathfrak{H}_6(n)=\langle T,M,\ell\rangle$ then set $\mathfrak{H}_7(n):=\langle T,N,\ell\rangle$, $M\subseteq N$, where N extends M by adding every lowest vertex $x\in V(T)$ with $\langle\mathrm{pre}(x),x\rangle\in(\mathfrak{H}_6(n))_{rp}$ and $\ell(\langle\mathrm{pre}(x),x\rangle)=\mu$, where we set

$$\mu=\max\{\ell(b)\prec\tau: b\in E(T)\cap(\mathfrak{H}_6(n))_{rp}\}.$$

This yields $\mathsf{R}(\sigma,\tau,\mathfrak{S})$ for limit τ, and thereby completes the construction of R. □

4. PROOF OF THEOREM 3.7. SOUNDNESS

Soundness of the above construction (3.7.1)-(3.7.3), together with the auxiliary assertions (4.2.1)-(4.2.6), is proved by simultaneous transfinite recursion on τ in 4.4 below. Let us first introduce some new notations.

4.1. *Notations.* For any c.e.t. $C=\langle T,M,\ell\rangle$, its underlying (unlabeled) tree T is called the skeleton of C. Previously defined relations on c.e.t. are naturally specified to the skeletons so that e.g. $C_1 \ll C_2$, $C_1 \underline{\ll} C_2$ or $C_1 \in (C_2)_{rp}$ strengthen $T_1 \ll T_2$, $T_1 \underline{\ll} T_2$ or $T_1 \in (C_2)_{rp}$, respectively, provided that $C_1=\langle T_1,M_1,\ell_1\rangle$ and $C_2=\langle T_2,M_2,\ell_2\rangle$.

For purely technical reasons, in every $C=\langle T,M,\ell\rangle$ we extend ℓ onto M by setting $\ell(x)=\ell(\langle \mathrm{pre}(x),x\rangle)$ for each $x\in M$.

For any $\pi\in \mathrm{P}$, let $(C)_{d(\pi)}:=\langle T,N,\ell\rangle$ for $N:=M-\{x\in M: \pi\preceq\ell(x)\}$, and in particular $(\emptyset)_{d(\pi)}:=\emptyset$. We say that $(C)_{d(\pi)}$ arises from C by dropping all marks behind π.

Finally, we say that an edge $\langle \mathrm{pre}(y),y\rangle\in \mathrm{E(T)}$ occurs in the root-piece behind π in a given c.e.t. $C=\langle T,M,\ell\rangle=\langle S,\triangleleft,M,\ell\rangle$, if there is no $x\in \mathrm{M}$ with $x\trianglelefteq y$ and $\ell(x)\prec\pi$. This relation we abbreviate by $\langle \mathrm{pre}(y),y\rangle\in(C)_{rp(\pi)}$.

4.2. *Definition.* Let $\mathfrak{S},\mathfrak{F}\colon \mathbb{N}\to\mathbb{C}[\mathcal{O}\restriction\nu]$ be any infinite sequences in $\mathbb{C}[\mathcal{O}\restriction\nu]$, and let $\sigma\preceq\tau\preceq\nu$ in $\mathcal{O}$. We say $\mathfrak{S}$ and $\mathfrak{F}$ cohere with respect to σ and τ if the following conditions hold for any $\mathrm{n,m}\in\mathbb{N}$.

(4.2.1) Let $\mathfrak{S}(\mathrm{n})=\langle T_\sigma,M_\sigma,\ell_\sigma\rangle$ and $\mathfrak{F}(\mathrm{n})=\langle T_\tau,M_\tau,\ell_\tau\rangle$. Then $T_\tau\underline{\ll}T_\sigma$. Now assume that $\mathfrak{F}(\mathrm{n})\neq\emptyset$. Then if $\mathfrak{S}(\mathrm{n})$ is open then $T_\tau\in(\mathfrak{S}(\mathrm{n}))_{rp}$, and if $\mathfrak{S}(\mathrm{n})$ is nopen then $\mathfrak{S}(\mathrm{n})=\mathfrak{F}(\mathrm{n})$.

(4.2.2) Let $\mathfrak{S}(\mathrm{n})=\langle T_\sigma,M_\sigma,\ell_\sigma\rangle$ and $\emptyset\neq\mathfrak{F}(\mathrm{n})=\langle T_\tau,M_\tau,\ell_\tau\rangle$, and let $x\in \mathrm{V}(T_\tau)$. Then $x\in M_\sigma$ implies both $x\in M_\tau$ and $\ell_\sigma(x)=\ell_\tau(x)$.

(4.2.3) Let $\mathfrak{S}(\mathrm{n})=\langle T_\sigma,M_\sigma,\ell_\sigma\rangle$, $\emptyset\neq\mathfrak{F}(\mathrm{n})=\langle S_\tau,\triangleleft_\tau,M_\tau,\ell_\tau\rangle=\langle T_\tau,M_\tau,\ell_\tau\rangle$, $x\in M_\tau$ and let $\ell_\tau(x)\prec\sigma$. Assume $x\trianglelefteq_\tau y$. Then $\ell_\sigma(\langle \mathrm{pre}(y),y\rangle)=\ell_\tau(\langle \mathrm{pre}(y),y\rangle)$. Moreover, $y\in M_\tau$ implies $y\in M_\sigma$, and hence $y\in M_\tau$ implies $\ell_\sigma(y)=\ell_\tau(y)$.

(4.2.4) Let $\mathfrak{S}(\mathrm{n})=\langle T_\sigma,M_\sigma,\ell_\sigma\rangle$, $\emptyset\neq\mathfrak{F}(\mathrm{n})=\langle T_\tau,M_\tau,\ell_\tau\rangle$ and $a\in \mathrm{E}(T_\tau)\cap(\mathfrak{F}(\mathrm{n}))_{rp(\sigma)}$. Then $\ell_\sigma(\mathrm{a})=\ell_\tau(a)$ provided that $\tau\preceq\ell_\tau(a)$. Otherwise, $\ell_\tau(a)\prec\tau$ implies $\ell_\sigma(a)\preceq\ell_\tau(a)=\ell_\sigma(b)$ for some $b\in \mathrm{E(T)}\cap(\mathfrak{S}(\mathrm{n}))_{rp}$.

(4.2.5) If $m<n$, $\sigma\preceq\pi$ and $\mathfrak{S}(m)=\mathfrak{F}(m)$, then $\neg(\mathfrak{S}(m)\sqsubseteq(\mathfrak{F}(n))_{d(\pi)})$.

(4.2.6) Let $\emptyset\neq\mathfrak{S}(n)=\langle T_n,M_n,\ell_n\rangle$, and for all $m<n$ let $\mathfrak{S}(m)=\mathfrak{F}(m)=\langle T_m,M_m,\ell_m\rangle$ (possibly including $\langle T_m,M_m,\ell_m\rangle=\emptyset$). Then $\mathfrak{F}(n)\neq\emptyset$ provided that (*) below holds.

(*) $(\forall m<n)(\forall a\in E(T_m))(\forall b\in E(T_n)\cap(\mathfrak{S}(n))_{rp})(\ell_m(a)\prec\tau\rightarrow\ell_m(a)\preceq\ell_n(b))$.

4.3. LEMMA. *For any* $\mathfrak{S}:\mathbb{N}\rightarrow\mathbb{C}[\mathcal{O}]$, *if* $\{C:(\exists n\in\mathbb{N})(C\underline{\ll}\mathfrak{S}(n)$ *and* C *is closed*$)\}$ *admits no infinite "*$\sqsubseteq$*"-bad sequence, then so is* $\{C:(\exists n\in\mathbb{N})(C\underline{\ll}\mathfrak{S}(n)$ *and* C *is nopen*$)\}$*. The conclusion also holds under the root-preserving strengthen of the embeddability "*$\sqsubseteq$*".*

Proof. This is an easy consequence of the Higman's theorem (cf. above the analogous passage in the proof of the Lemma 3.6). □

In order to complete the proof of Theorem 3.7, it suffices to establish soundness of the operator R constructed according to the above clauses (3.7.1)-(3.7.3). This, in turn, is an obvious consequence of the following theorem.

4.4. THEOREM. *Let* $\sigma\preceq\tau\preceq\nu$ *in* $\mathcal{O}$, *let* $\mathfrak{S}:\mathbb{N}\rightarrow\mathbb{C}[\mathcal{O}\upharpoonright\nu]$ *be* σ*-bad, and let* $\mathfrak{F}:\mathbb{N}\rightarrow\mathbb{C}[\mathcal{O}\upharpoonright\nu]$ *denote the sequence* $\mathsf{R}(\sigma,\tau,\mathfrak{S})$ *defined by* (3.7.1)-(3.7.3)*. Then* $\mathfrak{F}$ *is* τ*-bad, and* $\mathfrak{S}$ *and* $\mathfrak{F}$ *cohere with respect to* σ *and* τ*.*

Proof. This is proved as follows by simultaneous transfinite induction on τ. Since the required conditions (3.5.3), (3.5.4) and (4.2.1)-(4.2.4) are readily seen by definition of $\mathsf{R}(\sigma,\tau,\mathfrak{S})$, we verify the remaining conditions (3.5.1), (3.5.2), (4.2.5) and (4.2.6) only. So let $\mathfrak{F}=\mathsf{R}(\sigma,\tau,\mathfrak{S})$ and consider the corresponding defining clauses (3.7.1)-(3.7.3).

(3.7.1): Clear.

(3.7.2): Let $\tau=\rho+1$ where $\sigma\preceq\rho$, and let the sequences $\mathfrak{G}_1$, $\mathfrak{G}_2$ and $\mathfrak{G}_3=\mathfrak{F}$ be as above.

(3.5.1): That $\mathfrak{G}_1=\mathsf{R}(\sigma,\rho,\mathfrak{S})$ and $\mathfrak{G}_2=\mathsf{M}_{cf}(\mathfrak{G}_1)$ are "$\sqsubseteq$"-bad is readily seen by the induction hypothesis and by definition of $\mathsf{M}_{cf}(-)$, respectively. But then $\mathfrak{G}_3$ is "$\sqsubseteq$"-bad as well, since adding new marks can only destroy the embeddability "$\sqsubseteq$".

(3.5.2): Since closed subtrees occurring in the range of $\mathfrak{G}_2=\mathsf{M}_{cf}(\mathfrak{G}_1)$ are the ones occurring in the range of $\mathfrak{G}_1=\mathsf{R}(\sigma,\rho,\mathfrak{S})$, by the induction hypothesis all these closed subtrees admit no "$\sqsubseteq$"-bad infinite sequence, i.e. $\mathfrak{G}_2$ satusfies (3.5.2). It remains to show that so are the new closed subtrees which arise by adding marks in the range of $\mathfrak{G}_3$ according to (S.3).

Suppose this does not apply and consider an infinite "$\sqsubseteq$"-bad sequence $\mathfrak{G}:\mathbb{N}\to\mathbb{C}[\mathcal{O}\restriction\nu]$ such that for every $n\in\mathbb{N}$ there is an $m\in\mathbb{N}$ for which $\mathfrak{G}(n)$ is a closed subtree of $\mathfrak{G}_3(m)$ with the lowest label ρ. Hence $(\mathfrak{G}(n))_{d(\rho)}\in(\mathfrak{G}_2(m))_{rp}$. Moreover, the corresponding sequence of c.e.t. $\{(\mathfrak{G}(n))_{d(\rho)}:n\in\mathbb{N}\}$ is "$\sqsubseteq$" bad as well, since ρ is the miminal label occurring in the root-piece of $\mathfrak{G}_2(m)$. By Lemma 3.4 (3.4.1), there are finitely many proper subtrees of $\mathfrak{G}_2(m)$ among these c.e.t. $(\mathfrak{G}(n))_{d(\rho)}$, i.e. such that $(\mathfrak{G}(n))_{d(\rho)}\ll\mathfrak{G}_2(m)$. So $\mathfrak{G}(n)=\mathfrak{G}_3(m)$ and $(\mathfrak{G}(n))_{d(\rho)}=\mathfrak{G}_2(m)$ hold for almost all $n\in\mathbb{N}$. For every such n, take the correlated m and let $\mathfrak{G}_2(m)=\langle T,M,Q\rangle=\langle S,\lhd,M,\ell\rangle$. Since $\mathfrak{G}(n)$ is closed, $\mathfrak{G}_2(m)$ has the unique root-neighbor $x\in S$. Let $\mathfrak{G}_2(m)^*=\langle T^*,M\restriction S^*,\ell\restriction T^*\rangle$ for $T^*=\langle S^*,\lhd\restriction S^*\rangle$ where $S^*=\{v\in S:x\unlhd v\}$. So $\mathfrak{G}_2(m)^*\ll\mathfrak{G}_2(m)$. Consider two cases.

(a): Suppose that there are infinitely many open subtrees among these $\mathfrak{G}_2(m)^*$, i.e. such that $\mathfrak{G}_2(m)^*\in(\mathfrak{G}_2(m))_{rp}$. Then by Lemma 3.4 (3.4.1), the collection of all these open c.e.t. $\mathfrak{G}_2(m)^*$ admits no "$\sqsubseteq$"-bad infinite sequence, and hence $\mathfrak{G}_2(m)^*\sqsubseteq\mathfrak{G}_2(m')^*$ holds for some $m<m'$. But then $\mathfrak{G}_2(m)\sqsubseteq\mathfrak{G}_2(m')$ holds as well by sending the root of $\mathfrak{G}_2(m)$ to the root of $\mathfrak{G}_2(m')$. For by the induction hypothesis (3.5.4), at least one among the root-neighboring edges in $\mathfrak{G}_2(m)^*$ is labeled behind ρ, and so are labeled all edges occurring in the root-piece of $\mathfrak{G}_2(m')$. Hence by the clause (2.4.5), all labels occurring in $\mathfrak{G}_2(m')^*$ lower than the image of the root of $\mathfrak{G}_2(m)^*$ are $\succeq\rho=\ell(e(\mathfrak{G}_2(m)))$. Now let n and n$'$ be the correlated coordinates such that $(\mathfrak{G}(n))_{d(\rho)}=\mathfrak{G}_2(m)$ and $(\mathfrak{G}(n'))_{d(\rho)}=\mathfrak{G}_2(m')$. Then $\mathfrak{G}(n)\sqsubseteq\mathfrak{G}(n')$, and hence $\mathfrak{G}_3(n)\sqsubseteq\mathfrak{G}_3(n')$.

(b): Otherwise, there are infinitely many nopen subtrees $\mathfrak{G}_2(m)^*\in(\mathfrak{G}_2(m))_{rp}$ in question. Then by the induction hypothesis (3.5.2) together with Lemma 4.3, there are $m<m'$ such that $\mathfrak{G}_2(m)^*\sqsubseteq\mathfrak{G}_2(m')^*$ holds by sending the root of $\mathfrak{G}_2(m)$ to the root of $\mathfrak{G}_2(m')$. But then $\mathfrak{G}_2(m)\sqsubseteq\mathfrak{G}_2(m')$ holds as well by sending the root of $\mathfrak{G}_2(m)$ to the root of $\mathfrak{G}_2(m')$, since $\rho=\ell(e(\mathfrak{G}_2(m)))=\ell(e(\mathfrak{G}_2(m')))$. As above, this implies $\mathfrak{G}_3(n)\sqsubseteq\mathfrak{G}_3(n')$.

(4.2.5): Let $m<n$, $\sigma\preceq\pi$ and $\mathfrak{S}(m)=\mathfrak{G}_3(m)=\mathfrak{F}(m)$. By the induction hypothesis (4.2.1)-(4.2.4) and by an obvious monotonicity of $\mathfrak{G}_1$-$\mathfrak{G}_3$ this equation just as well implies $\mathfrak{S}(m)=\mathfrak{G}_1(m)=\mathsf{R}(\sigma,\rho,\mathfrak{S})(m)=\mathfrak{G}_2(m)=\mathsf{M}_{cf}(\mathfrak{G}_1)(m)$. Consider three cases.

(a): Let $\pi \prec \rho$. Then clearly $(\mathfrak{G}_2(n))_{d(\pi)} = (\mathfrak{G}_3(n))_{d(\pi)}$, while $\neg(\mathfrak{S}(m) \sqsubseteq (\mathfrak{G}_1(n))_{d(\pi)})$ holds by the induction hypothesis (4.2.5). By the induction hypothesis (3.5.3), every mark in $\mathfrak{S}(m)$ is $\prec \sigma$, and every label in the root-piece of $(\mathfrak{G}_1(n))_{d(\pi)}$ is $\succeq \sigma$. Since $(\mathfrak{G}_2(n))_{d(\pi)}$ is a subtree of $(\mathfrak{G}_1(n))_{d(\pi)}$, this also yields $\neg(\mathfrak{S}(m) \sqsubseteq (\mathfrak{G}_2(n))_{d(\pi)})$, i.e. $\neg(\mathfrak{S}(m) \sqsubseteq (\mathfrak{G}_3(n))_{d(\pi)})$.

(b): Let $\pi = \rho$. Then $\mathfrak{G}_2(n) = (\mathfrak{G}_3(n))_{d(\pi)}$ and $\neg(\mathfrak{G}_2(m) \sqsubseteq \mathfrak{G}_2(n))$, since $\mathfrak{G}_2$ is "$\sqsubseteq$"-bad. Since $\mathfrak{S}(m) = \mathfrak{G}_2(m)$, this yields $\neg(\mathfrak{S}(m) \sqsubseteq (\mathfrak{G}_3(n))_{d(\pi)})$.

(c): Let $\rho \prec \pi$. Then $\mathfrak{G}_3(n) = (\mathfrak{G}_3(n))_{d(\pi)}$ and $\neg(\mathfrak{G}_3(m) \sqsubseteq \mathfrak{G}_3(n))$, since $\mathfrak{G}_3$ is "$\sqsubseteq$"-bad. Since $\mathfrak{S}(m) = \mathfrak{G}_3(m)$, this yields $\neg(\mathfrak{S}(m) \sqsubseteq (\mathfrak{G}_3(n))_{d(\pi)})$.

(4.2.6): Let $\emptyset \neq \mathfrak{S}(n)$, let $\mathfrak{S}(m) = \mathfrak{G}_3(m) = \mathfrak{F}(m)$ for all $m<n$, and let (*) be the case. As in the previous case, this also implies $\mathfrak{S}(m) = \mathfrak{G}_1(m) = \mathsf{R}(\sigma,\rho,\mathfrak{S})(m) = \mathfrak{G}_2(n) = \mathsf{M}_{cf}(\mathfrak{G}_1)(m)$. Hence $\mathfrak{G}_1(n) \neq \emptyset$ holds by the induction hypothesis (4.2.6). But then $\mathfrak{G}_2(n) = \mathsf{M}_{cf}(\mathfrak{G}_1)(n) \neq \emptyset$, since $(\forall m<n)(\mathfrak{G}_1(m) = \mathsf{M}_{cf}(\mathfrak{G}_1)(m))$ and $\mathfrak{G}_1(n) \neq \emptyset$ together easily imply that $\mathsf{M}_{cf}(\mathfrak{G}_1)(n)$ is a non-empty subtree of $\mathfrak{G}_1(n)$ by definition of $\mathsf{M}_{cf}(-)$ (see 3.2 above).

(3.7.3): Let $\tau \succ \sigma$ be limit, and consider the corresponding sequences $\mathfrak{D}$, $\mathfrak{R}$, $\mathfrak{H}_1,\ldots, \mathfrak{H}_7 = \mathfrak{F}_\tau$.

(3.5.1,2): Note that $\mathfrak{H}_1(n) = \mathfrak{D}(J(n),n)$ and $J(n) = \min\{i:(\forall j>i)\mathfrak{D}(i,n) = \mathfrak{D}(j,n)\}$ holds for every $n \in \mathbb{N}$. Indeed, consider an infinite sequence of c.e.t. $\mathfrak{D}(0,n),\ldots, \mathfrak{D}(i,n),\ldots$. By the induction hypothesis (4.2.1)-(4.2.4), the skeletons of $\mathfrak{D}(i,n)$ are decreasing (or remain the same), and their labels and marks are increasing (or remain the same) while being bounded e.g. by the maximal label occurring in $\mathfrak{D}(0,n)$. So, clearly, this sequence must somewhere stabilize. Hence $J(n) \in \mathbb{N}$ is well defined on the whole domain.

(L.1): We first prove that the diagonal sequence $\mathfrak{H}_1$, where $\mathfrak{H}_1(n) = \mathfrak{D}(J(n),n)$, contains infinitely many non empty-components. Suppose this is not the case. Then arguing by induction on i, it follows from the induction hypothesis (4.2.1)-(4.2.4) that every single sequence $\mathfrak{R}(i)$, i.e. $\mathfrak{D}(i,0),\ldots, \mathfrak{D}(i,n),\ldots$, has infinitely many non-empty components. For any m, let $\mathfrak{H}_1(m) = \langle T_m, M_m, \ell_m \rangle$ (possibly including the empty triple). Now set

$$n_0 := \min\{n:(\forall m>n)\mathfrak{H}_1(m) = \emptyset\}, \quad i_0 := \max\{J(0),\ldots,J(n_0)\},$$

$$n_1 := \min\{n>n_0:(\exists i \geq i_0)(\mathfrak{D}(i,n)\neq\emptyset \wedge (\forall m<n_0)(\forall a \in E(T_m))(\ell_m(a) \prec \tau \rightarrow \ell_m(a) \preceq \{\tau\}_\sigma(i)))\},$$

$$i_1 := \min\{i \geq i_0:\mathfrak{D}(i,n_1)\neq\emptyset \wedge (\forall m<n_0)(\forall a \in E(T_m))(\ell_m(a) \prec \tau \rightarrow \ell_m(a) \preceq \{\tau\}_\sigma(i))\}.$$

(The existence of n_1 follows from the above remark that every $\mathfrak{R}_1(i)$ has infinitely many non-empty components.) So for any $j,j' \geq i_1$, $m<n_0$ implies $\mathfrak{D}(j,m) = \mathfrak{D}(j',m) = \mathfrak{D}(i_0,m) = \mathfrak{H}_1(m)$, and $n_0 \leq m<n_1$ implies $\mathfrak{D}(j,m) = \emptyset = \mathfrak{D}(j',m) = \mathfrak{H}_1(m)$. Hence $\mathfrak{D}(j,m) = \mathfrak{D}(j',m)$ holds for all $j,j' \geq i_1$ and $m<n_1$. On the other hand, $\mathfrak{D}(i_1,n_1) \neq \emptyset = \mathfrak{D}(J(n_1),n_1) = \mathfrak{H}_1(n_1)$. Moreover, for any $j \geq i_1$, $\mathfrak{R}(j)$ is $\{\tau\}_\sigma(j)$-bad by the induction hypothesis (4.2.1)-(4.2.4), and hence every label occurring in the root piece of $\mathfrak{D}(j,n_1)$ is $\succeq \{\tau\}_\sigma(j) \succeq \{\tau\}_\sigma(i_1)$. By the above definitions of i_1, this shows that for any $j \geq i_1$ and $m<n_1$, any label $\prec \tau$ occurring in $\mathfrak{D}(j,m)$ is not smaller than any label occurring in the root piece of $\mathfrak{D}(j,n_1)$, which proves the corresponding instance of the condition (*) of (4.2.6). Hence by successively applying the induction hypothesis (4.2.6) for all $i_1 \leq j \leq J(n_1)$, we finally arrive at $\mathfrak{D}(J(n_1),n_1) \neq \emptyset$ - a contradiction. So there are infinitely many non-empty c.e.t. $\mathfrak{H}_1(n) = \mathfrak{D}(J(n),n)$. Since every single sequence $\mathfrak{R}(i)$ is "$\sqsubseteq$"-bad, this obviously implies $\mathfrak{H}_1$ is "$\sqsubseteq$"-bad as well.

Moreover, we claim that for any fixed $\pi \prec \tau$ there is no infinite "$\sqsubseteq$"-bad sequence of closed subtrees occurring in the range of $\mathfrak{H}_1$ whose lowest labels are all $\preceq \pi$. To this effect, arguing by the induction hypothesis (3.5.2), it will suffice to prove that any such bounded closed subtree occurs (as closed subtree) in the range of $\mathfrak{R}(j)$ for any j with $\pi \prec \{\tau\}_\sigma(j)$. So let $C = \langle T,M,\ell \rangle$ be closed and $\ell(e(T)) \preceq \pi \prec \tau$ where $C \underline{\ll} \mathfrak{D}(J(n),n)$ for some n. By the induction hypothesis (3.5.3)-(3.5.4), we may just as well assume $\sigma \preceq \ell(e(T))$. Now let $e(T) = \langle r(T),x \rangle$ where $x \in M$. By the induction hypothesis (4.2.1), T is a skeleton-subtree of $\mathfrak{D}(j,n) = \langle T_j,M_j,Q_j \rangle$ for every $j \leq J(n)$. By the induction hypothesis (4.2.1)-(4.2.2), there exists the uniquely determined $j_0 \leq J(n)$ such that $x \in (M_{j_0} - M_{j_0}\text{-}1)$, and hence $e(T)$ occurs in the root-piece of $\mathfrak{D}(j_0-1,n)$, which implies $\{\tau\}_\sigma(j_0-1) \preceq \ell(e(T))$ by the induction hypothesis (3.5.4) implies, and hence $\{\tau\}_\sigma(j_0-1) \preceq \ell(e(T)) \preceq \pi$. Hence for any j, $\pi \prec \{\tau\}_\sigma(j)$ implies $j_0 \leq j$, and hence $C \underline{\ll} \mathfrak{D}(j,n)$ holds by the induction hypothesis (4.2.3).

(L.2): That $\mathfrak{H}_2 = \mathsf{M}_{\mathrm{lim}}(\sigma,\mathfrak{H}_1)$ is "$\sqsubseteq$"-bad immediately follows by Lemma 3.4. It also follows from the lemma that closed proper subtrees of the range of $\mathfrak{H}_2$ admit no infinite "$\sqsubseteq$"-bad sequence. But we cannot exclude the remaining case that there is an infinite "$\sqsubseteq$"-bad subsequence of $\mathfrak{H}_2$ consisting of c.e.t. closed behind σ, which by the above arguments implies that the correlated subsequence of the lowest labels is not bounded before τ. (Therefore, in order to fulfill (3.5.2), we defined more sophisticated sequences $\mathfrak{H}_3$–$\mathfrak{H}_7$.)

(L.3): It is readily seen from the previous observations that $\mathfrak{H}_3$ still contains infinitely many non-empty components, and hence it is "$\sqsubseteq$"-bad. Moreover, its closed proper subtrees still admit no infinite "$\sqsubseteq$"-bad sequence. In addition, for every component $\mathfrak{H}_3(n)$ which is closed behind σ, $\mathfrak{H}_3(n) = \mathfrak{H}_2(n)$ actually satisfies the condition ($\star$) of (L.3). Now $\mathfrak{H}_3$ clearly satisfies the required condition (3.5.3), as well as (3.5.4), since by definition an open component $\mathfrak{H}_3(n)$ must coincide with $\mathfrak{H}_2(n)$ for which (3.5.4) is readily seen.

(L.4): Recall that $\mathfrak{H}_4$ is obtained by dropping lowest marks in all c.e.t. $\mathfrak{H}_3(n)$ which are closed behind σ. So $\mathfrak{H}_4$ satisfies (3.5.2) by (L.2) above. We prove that $\mathfrak{H}_4$ is "$\sqsubseteq$"-bad. Take any $m<n$ and assume $\mathfrak{H}_4(m)\sqsubseteq\mathfrak{H}_4(n)$. This trivially contradicts (L.3) above if $\mathfrak{H}_3(n)$ is not closed behind σ. So suppose that $\mathfrak{H}_3(n)$ is closed behind σ, and consider two cases.

(a): Let $\mathfrak{H}_3(m)$ be nopen. Then $\mathfrak{H}_3(m) = \mathfrak{H}_2(m)$, and $\mathfrak{H}_3(n) = \mathfrak{H}_2(n)$ while satisfying ($\star$). Hence every label $\prec\tau$ occurring in the root-piece of $\mathfrak{H}_3(n)$ is not smaller than any label from $\mathfrak{H}_3(m)$, and hence the underlying embedding $f{:}\,\mathfrak{H}_4(m)\sqsubseteq\mathfrak{H}_4(n)$ also yields $f{:}\,\mathfrak{H}_3(m)\sqsubseteq\mathfrak{H}_3(n)$ – a contradiction with (L.3) above.

(b): Otherwise, let $\mathfrak{H}_3(m)$ be open. Denote $\mathfrak{H}_3(n) = \langle T,M,\ell\rangle$, $e(T) = \langle r(T),x\rangle$ and $\sigma\preceq\pi = \ell(e(T))\prec\tau$. Then clearly $\mathfrak{H}_4(m) = \mathfrak{H}_3(m) = \mathfrak{D}(J(m),m)$ and $\mathfrak{H}_4(n) = (\mathfrak{H}_3(n))_{d(\pi)}$. Let $\mathfrak{D}(j,n) = \langle T_j,M_j,Q_j\rangle$ for any $j\leq J(n)$. Then arguing as in (L.1) above, there exists the uniquely determined $j_0\leq J(n)$ such that $x\in(M_{j_0}-M_{j_0-1})$, which implies $\{\tau\}_\sigma(j_0-1)\preceq\pi\prec\{\tau\}_\sigma(j_0)$. On the other hand, the condition ($\star$) of (L.3) yields $\{\tau\}_\sigma(J(m))\preceq\pi$, and hence $J(m)<j_0\leq J(n)$ holds. Therefore $\mathfrak{D}(J(m),m) = \mathfrak{D}(J(n)-1,m) = \mathfrak{D}(J(n),m)$. Since $\mathfrak{D}(J(n)-1,-) = \mathfrak{R}(J(n)-1)$ and $\mathfrak{D}(J(n),-) = R(\{\tau\}_\sigma(J(n)-1),\{\tau\}_\sigma(J(n)),\mathfrak{R}(J(n)-1))$, then by the induction hypothesis (4.6.5) this yields $\neg(\mathfrak{D}(J(n)-1,m)\sqsubseteq(\mathfrak{D}(J(n),n))_{d(\pi)})$, i.e. $\neg(\mathfrak{H}_3(m)\sqsubseteq(\mathfrak{H}_1(n))_{d(\pi)})$. However, our assumption $\mathfrak{H}_4(m)\sqsubseteq\mathfrak{H}_4(n)$, i.e. $\mathfrak{H}_3(m)\sqsubseteq(\mathfrak{H}_3(n))_{d(\pi)}$, easily implies $\mathfrak{H}_3(m)\sqsubseteq(\mathfrak{H}_1(n))_{d(\pi)}$ – a contradiction.

(L.5): Since $\mathfrak{H}_4$ satisfies (3.5.2)–(3.5.3), but not necessarily (3.5.4), so does $\mathfrak{H}_5$. We prove that $\mathfrak{H}_5$ is "$\sqsubseteq$"-bad. Take any $m<n$ and assume $\mathfrak{H}_5(m)\sqsubseteq\mathfrak{H}_5(n)$. Since labels of $\mathfrak{H}_5(m)$ majorize the ones of $\mathfrak{H}_4(m)$, this obviously implies $\mathfrak{H}_4(m)\sqsubseteq\mathfrak{H}_5(n)$. By definition $\mathfrak{H}_4(m) = \mathfrak{H}_3(n)$ implies $\mathfrak{H}_4(m) = \mathfrak{H}_5(n)$, and hence $\mathfrak{H}_4(m)\sqsubseteq\mathfrak{H}_4(n)$ – a contradiction to (L.4) above.

Otherwise, let $\mathfrak{H}_5(n)$ be closed behind σ, and hence $(\star)$ of (L.3) be satisfied. But then $\mathfrak{H}_4(m)\sqsubseteq\mathfrak{H}_5(n)$ would also imply $\mathfrak{H}_4(m)\sqsubseteq\mathfrak{H}_4(n)$, since all labels occurring in the root-piece of $\mathfrak{H}_4(n)$ are not smaller than the lowest label of $\mathfrak{H}_3(n)$, and hence the crucial condition (2.4.5) would be preserved according to (3.5.3) and $(\star)$.

(L.6): That $\mathfrak{H}_6 = \mathsf{M}_{cf}(\mathfrak{H}_5)$ still satisfies (3.5.1)-(3.5.3) easily follows by Lemma 3.4.

(L.7): Since adding new marks can only destroy the embeddability "$\sqsubseteq$", $\mathfrak{H}_7 = \mathfrak{F}$ is still $\sqsubseteq$"-bad, while obviously satisfying (3.5.3)-(3.5.4). In order to prove the remaining clause (3.5.2), we adopt the previous arguments in the analogous case (3.5.2) of the successor clause. So suppose there is an infinite "$\sqsubseteq$"-bad sequence $\mathfrak{H}:\mathbb{N}\rightarrow\mathbb{C}[\mathcal{O}\restriction\nu]$ such that for every $n\in\mathbb{N}$ there is an $m\in\mathbb{N}$ for which $\mathfrak{H}(n)$ is the new closed subtree of $\mathfrak{H}_7(m)$ that arises according to the definition (L.7). Since $\mathcal{O}$ is well-ordered, we can just as well assume that for any $n<n'$, the lowest label of $\mathfrak{H}(n')$ is not smaller than the one of $\mathfrak{H}(n)$. But by using the condition $(\star)$ of (L.3), the existence of such $\mathfrak{H}$ leads to a contradiction, via Lemma 3.4 (3.4.1), in the same manner as the existence of the analogous sequence $\mathfrak{G}$ in the analogous successor-clause (3.5.2).

This completes the proof of (3.5.1,2).

(4.2.5): Let $m<n$, $\sigma\preceq\pi$ and $\mathfrak{S}(m) = \mathfrak{H}_7(m) = \mathfrak{F}(m)$. By the induction hypothesis (4.2.1)-(4.2.4), and an obvious monotonicity of $\mathfrak{R}(i),\mathfrak{H}_1$-$\mathfrak{H}_7$, this implies $\mathfrak{S}(m) = \mathfrak{D}(j,m) = \mathfrak{H}_i(m) = \mathfrak{F}(m)$ for all j and $0<i<7$. Hence by iteration, the induction hypothesis (4.2.5) proves the negation of $\mathfrak{R}(J(n))(m)\sqsubseteq(\mathfrak{R}(J(n))(n))_{d(\pi)}$. Hence $\neg(\mathfrak{H}_1(m)\sqsubseteq(\mathfrak{H}_1(n))_{d(\pi)})$ holds. Now suppose that there is an embedding $f:\mathfrak{S}(m)\sqsubseteq(\mathfrak{F}(n))_{d(\pi)}$, and consider the following cases.

(a): Assume $\mathfrak{H}_7(n) = \mathfrak{H}_6(n)$. Then $f:\mathfrak{F}(m)\sqsubseteq(\mathfrak{H}_7(n))_{d(\pi)}$ implies $f:\mathfrak{S}(m)\sqsubseteq(\mathfrak{H}_4(n))_{d(\pi)}$. Consider two cases.

(a.a): Assume $\mathfrak{H}_4(n) = \mathfrak{H}_3(n)$. Then by definition $\mathfrak{H}_3(n) = \mathfrak{H}_2(n) = \mathfrak{H}_1(n) = \mathfrak{D}(J(n),n)$, and hence $\mathfrak{S}(m)\sqsubseteq(\mathfrak{H}_4(n))_{d(\pi)}$ implies $\mathfrak{R}(J(n))(m)\sqsubseteq(\mathfrak{R}(J(n))(n))_{d(\pi)}$ - a contradiction.

(a.b): Otherwise, let $\mathfrak{H}_4(n)\neq\mathfrak{H}_3(n)$. Then by definition $\mathfrak{H}_3(n)$ is closed behind σ, and hence the condition $(\star)$ of (L.3) holds, while $\mathfrak{H}_4(n)$ arises from $\mathfrak{H}_3(n)$ by dropping its lowest mark. Let ρ be the lowest label in $\mathfrak{H}_3(n)$. Thus $\sigma\preceq\rho\prec\tau$ holds. By our assumption, there is an embedding $f:\mathfrak{H}_4(m)\sqsubseteq(\mathfrak{H}_4(n))_{d(\pi)}$. If $f(r(\mathfrak{H}_4(m)))\in(\mathfrak{H}_4(n))_{r\rho}$ then, by $(\star)$, $f:\mathfrak{H}_4(m)\sqsubseteq\mathfrak{H}_4(n)$ holds as well - a contradiction with (L.4) above. For the same reason, since f has no extension

$g{:}\mathfrak{H}_4(m)\sqsubseteq(\mathfrak{H}_4(n))_{d(\pi)}$ with $g(r(\mathfrak{H}_4(m)))\in(\mathfrak{H}_4(n))_{rp}$, there are marks which are placed in $\mathfrak{H}_4(m)$ below $f(r(\mathfrak{H}_4(m)))$ and whose labels are thereby all $\prec\rho$. Let $\alpha\prec\rho$ be maximal such label. Now if $\alpha\prec\pi$ then clearly $(\mathfrak{H}_4(n))_{d(\pi)} = \mathfrak{H}_4(n)$ and hence, as above, $\mathfrak{H}_4(m)\sqsubseteq\mathfrak{H}_4(n)$ is the case – a contradiction with (L.4) above. On the other hand, if $\pi\preceq\alpha\prec\rho$ then our embedding $f{:}\mathfrak{H}_1(m)\sqsubseteq(\mathfrak{H}_4(n))_{d(\pi)}$ also yields $f{:}\mathfrak{H}_1(m)\sqsubseteq(\mathfrak{H}_1(n))_{d(\pi)}$ – a contradiction (see above).

(b): Suppose that $\mathfrak{H}_7(n)\neq\mathfrak{H}_6(n)$, and let μ be as in the defining clause (L.7). If $\mu\prec\pi$ then $(\mathfrak{H}_7(n))_{d(\pi)} = \mathfrak{H}_7(n)$, and hence $\mathfrak{H}_7(m)\sqsubseteq\mathfrak{H}_7(n)$ holds – a contradiction with (L.7) above. If $\mu = \pi$ then $(\mathfrak{H}_7(n))_{d(\pi)} = \mathfrak{H}_6(n)$, and hence $\mathfrak{H}_6(m)\sqsubseteq\mathfrak{H}_6(n)$ holds – a contradiction with (L.6) above. In the remaining case, $\pi\prec\mu$, our assumption $f{:}\mathfrak{S}(m)\sqsubseteq(\mathfrak{H}_7(n))_{d(\pi)}$ easily implies $f{:}\mathfrak{S}(m)\sqsubseteq(\mathfrak{H}_5(n))_{d(\pi)}$. Since $\mathfrak{H}_3(n)$ is closed behind σ and $(\star)$ holds, this also implies $f{:}\mathfrak{S}(m)\sqsubseteq(\mathfrak{H}_4(n))_{d(\pi)}$. The desired contradiction is obtained as in the previous subcase (a.b).

(4.2.6): Let $\emptyset\neq\mathfrak{S}(n)$, let $\mathfrak{S}(m) = \mathfrak{H}_7(m) = \mathfrak{F}(m)$ for all $m<n$, and let (*) be satisfied. By the induction hypothesis (4.2.1)–(4.2.4) and by an obvious monotonicity of $\mathfrak{R}(i),\mathfrak{H}_1-\mathfrak{H}_7$, this implies $\mathfrak{S}(m) = \mathfrak{D}(j,m) = \mathfrak{H}_i(m) = \mathfrak{F}(m)$ for all j and $0<i<7$. In particular, $J(m) = 0$ for all $m<n$. Hence by iteration, $\mathfrak{D}(J(n),n) = \mathfrak{H}_1(n)\neq\emptyset$ holds by the induction hypothesis (3.2.1)–(4.2.4) and (4.2.6). Hence, arguing as in the analogous proof of the succesor clause (see above), $\mathfrak{H}_2(n) = \mathsf{M}_{lim}(\mathfrak{H}_1)(n)\neq\emptyset$. We claim that $\mathfrak{H}_3(n)\neq\emptyset$ holds as well. Indeed, by the condition $(\star)$ of (L.3), $\mathfrak{H}_3(n)$ can disappear only when $\mathfrak{H}_2(n)$ is closed behind σ and, for some $m<n$ and for some label $\alpha\prec\tau$ in $\mathfrak{S}(m)$, either $\sigma\preceq\ell(e(\mathfrak{H}_2(n)))\prec\alpha$ or $\sigma\preceq\ell(e(\mathfrak{H}_2(n)))\prec\{\tau\}_\sigma(J(n))$. However, by the the induction hypothesis (4.2.1)–(4.2.4), the first option implies that $\ell(e(\mathfrak{H}_2(n)))$ is not smaller than the corresponding label occurring in the root-piece of in $\mathfrak{S}(n)$ – a contradiction with (*). By the above observation, the second option implies $\sigma\preceq\ell(e(\mathfrak{H}_2(n)))\prec\ \{\tau\}_\sigma(J(0)) = \sigma$ – a contradiction. Thus $\mathfrak{H}_3(n)\neq\emptyset$ and $\mathfrak{H}_4(n)\neq\emptyset\neq\mathfrak{H}_5(n)$. But then $\mathfrak{G}_6(n) = \mathsf{M}_{cf}(\mathfrak{G}_5)(n)\neq\emptyset$, and hence $\mathfrak{G}_7(n) = \mathfrak{F}(n)\neq\emptyset$ as in the analogous succesor clause(see above).

This completes the proof of Theorem 4.4, and thereby completes the whole proof of Theorem 3.7. □

4.5. *Formalization.* In order to complete the proof of Theorem 2.6, is will suffice to formalize in $\mathbf{ITR_0}$ all the previous arguments. To this effect, first note that the Higman's theorem is provable well within $\mathbf{ACA_0}$ (see e.g. [4]). Consider operators $\mathsf{M}_{cf}(-)$ and $\mathsf{M}_{lim}(-)$.

It is redily seen that both sequences $\mathsf{M}_{cf}(\mathfrak{F})$ and $\mathsf{M}_{lim}(\nu,\mathfrak{F})$, as $2nd$-order objects, are definable by Σ^1_1-formulas with $2nd$-order parameter $\mathfrak{S}$, and hence are recursive in the universal Π^1_1-predicate over $\mathfrak{S}$.

Now consider the first part of the proof of Theorem 3.7, i.e. the construction of the sequence $\mathsf{R}(\sigma,\tau,\mathfrak{S})$ by transfinite recursion on τ. Arguing in $\mathbf{ACA_0}$, we can surely extend our well-order $\mathcal{O} = \langle \mathcal{P},\preceq \rangle$ to a suitable well-order $\mathcal{O}_1 = \langle \mathcal{P}_1,\preceq_1 \rangle$ so large that it admits ordinal exponentiation $\omega^{(\alpha+1)}$, where $\alpha \preceq \nu$ in the sense of $\mathcal{O}$, and ordinal sum $\alpha+\beta$ for all $\alpha,\beta \preceq_1 \omega^{(\nu+1)}$. We claim that for every $\sigma \preceq \tau \preceq \nu$, the sequence $\mathsf{R}(\sigma,\tau,\mathfrak{S})$ is recursive in $\omega^{(\tau+1)}th$ iteration of the universal Π^1_1-predicate over $\mathfrak{S}$. This is easily verified by transfinite induction on τ. If $\tau = 0$ then $\mathsf{R}(\sigma,\tau,\mathfrak{S}) = \mathfrak{S}$ and we are done. Let $\tau = \rho+1$. Then either $\mathsf{R}(\sigma,\tau,\mathfrak{S}) = \mathfrak{S}$, if $\sigma = \tau$, or by the above estimate, $\mathsf{R}(\sigma,\tau,\mathfrak{S})$ is arithmetical in the universal Π^1_1-predicate over $\mathsf{R}(\sigma,\rho,\mathfrak{S})$. Since $\omega^{\rho}+1 \prec_1 \omega^{(\rho+1)}$, we are done. For limit τ, by the same token, either $\mathsf{R}(\sigma,\tau,\mathfrak{S}) = \mathfrak{S}$, or $\mathsf{R}(\sigma,\tau,\mathfrak{S})$ is arithmetical in second iteration of the universal Π^1_1-predicate over the collection of all sequences $\mathfrak{R}(\mathrm{i})$, $\mathrm{i} < \omega$, where $\mathfrak{R}(0) = \mathfrak{S}(\mathrm{n})$ and $\mathfrak{R}(\mathrm{i}+1) = \mathsf{R}(\{\tau\}_\sigma(\mathrm{i}),\{\tau\}_\sigma(\mathrm{i}+1),\mathfrak{R}(\mathrm{i}))$. Since for all $\alpha,\beta \preceq \tau$, $\omega^{(\alpha+1)}+\omega^{(\beta+1)} \prec_1 \omega^{(\max(\alpha,\beta)+2)} \prec_1 \omega^{\tau}$, then by the induction hypothesis, this collection is arithmetical in $\omega^{\tau}th$ iteration of the universal Π^1_1-predicate over $\mathfrak{S}$. Hence, surely, $\mathsf{R}(\sigma,\tau,\mathfrak{S})$ is recursive in the $\omega^{(\tau+1)}th$ iteration. Summing up, the first part of the proof is explicitly formalizable in $\mathbf{ACA_0}$.extended by transfinite Π^1_1-recursion along $\mathcal{O}_1 \upharpoonright \omega^{(\nu+1)}$.

In the second part of the proof, i.e. in the proof of Theorem 4.4, we argued in $\mathbf{\Pi^1_1CA_0}$ (= $\mathbf{ACA_0}$ plus Π^1_1-comprehension axiom) extended by Π^1_1-transfinite induction along $\mathcal{O} \upharpoonright \nu$, while having all $2nd$-order parameters definable by transfinite Π^1_1-recursion along $\mathcal{O}_1 \upharpoonright \omega^{(\nu+1)}$ (see above). Since Π^1_1-comprehension is obviously included in $\mathbf{ITR_0}$, both the transfunute induction and all nested numerical inductions involved can be replaced by their restricted to sets variants, which are easily derivable in $\mathbf{ITR_0}$,. Thereby the whole proof is formalizable in $\mathbf{ITR_0}$. This completes the proof of Theorem 2.6.

5. PROOF OF PROPOSITION D

5.4. *Definition.* Let $\mathcal{Q} = \langle \mathcal{R}, \leq \rangle$ be any well-quasi-order, and, as before, let $\mathcal{O} = \langle \mathcal{P}, \prec \rangle$ be any well-order. <u>Generalized e.t.</u> (g.e.t.) and <u>generalized c.e.t.</u> (g.c.e.t.), relative to $\mathcal{O}$ and $\mathcal{Q}$, are structures $F = \langle S, \triangleleft, \ell, \varrho \rangle$ and $F = \langle S, \triangleleft, M, \ell, \varrho \rangle$, respectively, that extend the corresponding e.t. $E = \langle S, \triangleleft, \ell \rangle = \langle T, \ell \rangle$ and c.e.t. $C = \langle S, \triangleleft, M, \ell \rangle$, by the new labeling function $\varrho : \mathrm{End}(T) \to \mathcal{R}$, which is referred to as the quasi-labeling function. So ϱ assigns to end-vertices of T labels from $\mathcal{Q}$, which we call quasi-labels. The corresponding relations of embeddability "$\leq^{\#}$" and "$\sqsubseteq^{\#}$" on g.e.t. and g.c.e.t. arise by extending $f : E_1 \leq E_2$ and $f : C_1 \sqsubseteq C_2$ by the new assertion

(5.4.1) $$\text{If } x \in \mathrm{End}(T_1) \text{ then } \varrho_1(x) \leq \varrho_2 \circ f(x).$$

Let E and F be the corresponding modified propositions about g.e.t. and g.c.e.t. obtained by replacing in C and D the underlying embeddability "$\leq$" by "$\leq^{\#}$" and "$\sqsubseteq$" by "$\sqsubseteq^{\#}$", respectively. Clearly, E implies C, and F implies D, provably in $\mathbf{ACA_0}$.

5.5. THEOREM. $\mathbf{ITR_0}$ *proves Proposition* E. *Hence* E *has the same proof-theoretical strength as each Proposition* A, B *and* C.

Proof. By an obvious specification of the previous proof of Theorem 2.6. □

Below by adding superscript "#" to the previous lemmas, definitions, etc. we refer to their obvious "$\sqsubseteq^{\#}$"-counterparts. So, in particular, Lemma 3.4# deals with minimal "$\sqsubseteq^{\#}$"-sequences that arise by applying Definitions 3,2# and 3.3# which, in turn, are the same as the corresponding Definitions 3.2 and 3.3, except replacing everywhere "$\sqsubseteq$" by "$\sqsubseteq^{\#}$".

5.6. THEOREM. $\mathbf{ITR_0}$ *proves Proposition* F. *Hence* F *has the same proof-theoretical strength as each Proposition* A, B, C, D *and* E.

Proof. Let $\mathcal{O} = \langle \mathcal{P}, \preceq \rangle$ and $\mathcal{Q} = \langle \mathcal{R}, \leq \rangle$ be any well-order and any well-quasi-order, respectively. Let $\mathbb{C}[\mathcal{O}, \mathcal{Q}]$ denote the the set of all g.c.e.t. relative to $\mathcal{O}$ and $\mathcal{Q}$. Suppose $\mathfrak{F} : \mathbb{N} \to \mathbb{C}[\mathcal{O}, \mathcal{Q}]$ be "$\sqsubseteq^{\#}$"-bad in $\mathbb{C}[\mathcal{O}, \mathcal{Q}]$. Let $\mathfrak{F}_1$ be the correlated minimal sequence $\mathsf{M}_{\lim}(0, \mathfrak{F}) : \mathbb{N} \to \mathbb{C}[\mathcal{O}, \mathcal{Q}]$. By Lemma 3.4#, $\mathfrak{F}_1$ is "$\sqsubseteq^{\#}$"-bad, and there is no infinite "$\sqsubseteq^{\#}$"-bad sequence whose components are closed proper subtrees occurring in the range of $\mathfrak{F}_1$. Moreover, by (3.4.2)#, every infinite subsequence of nopen components $\mathfrak{F}_1(\mathrm{n})$ must include an infinite subsequence of closed components $\mathfrak{F}_1(\mathrm{n})$. Note that "$\sqsubseteq^{\#}$" restricted to the set of closed (in fact, nopen) g.c.e.t.

is transitive. Hence the set of all closed proper subtrees in question is well-quasi-ordered by "$\sqsubseteq$". A desired contradiction is proved by cases as follows.

(a) Assume that $\mathfrak{F}_1$ has an infinite subsequence $\mathfrak{F}_1\circ\mathfrak{f}$ such that $\mathfrak{F}_1\circ\mathfrak{f}(n)$ is open for all $n\in\mathbb{N}$, where $\mathfrak{f}:\mathbb{N}\to\mathbb{N}$ is strictly increasing. Let $\mathfrak{F}_2$ denote $\mathfrak{F}_1\circ\mathfrak{f}$. Let $\mathcal{Q}_1$ be the well-quasiorder consisting of all lowest closed (proper!) subtrees occurring in the range of $\mathfrak{F}_2$ under the embeddability "$\sqsubseteq^{\#}$". Let $\mathcal{Q}_2$ be the disjoint well-quasiordered sum $\mathcal{Q}+\mathcal{Q}_1$. Observe that with every g.c.e.t. $\mathfrak{F}_2(n)$ is naturally associated a g.e.t. $\mathfrak{T}(n)$ relative to $\mathcal{O}$ and $\mathcal{Q}_2$, which is defined as follows. Let x be any lowest mark in $\mathfrak{F}_2(n)$, let $F(x)$ be the correlated maximal closed subtree whose lowest edge is $\langle \mathrm{pre}(x),x\rangle$, and let $F_0(x)$ be obtained by deleting the lowest edge in $F(x)$ (including both vertices $\mathrm{pre}(x)$ and x). Now by definition $\mathfrak{T}(n)$ arises by deleting from $\mathfrak{F}_2(n)$ every subgraph $F_0(x)$ in question while supplying the resultinng new end-vertex x with the new quasi-label $F(x)$, viewed as an element of $\mathcal{Q}_1$. It is easily verified that for any $i,j\in\mathbb{N}$, $\mathfrak{T}(i)\leq^{\#}\mathfrak{T}(j)$ implies $\mathfrak{F}_2(i)\sqsubseteq^{\#}\mathfrak{F}_2(j)$. But by Theorem 5.5, there are $i<j$ such that $\mathfrak{T}(i)\leq^{\#}\mathfrak{T}(j)$, and hence $\mathfrak{F}_2(i)\sqsubseteq^{\#}\mathfrak{F}_2(j)$ – a contradiction.

(b) Assume that (a) does not apply. But then $\mathfrak{F}_1$ has an infinite subsequence $\mathfrak{F}_1\circ\mathfrak{g}$ such that $\mathfrak{F}_1\circ\mathfrak{g}(n)$ is closed for all $n\in\mathbb{N}$, where $\mathfrak{g}:\mathbb{N}\to\mathbb{N}$ is strictly increasing. Moreover, since $\mathcal{O}$ is well-order, we may just as well assume that the lowest labels in $\mathfrak{F}_1\circ\mathfrak{g}$ are weakly increasing. Now for every $n\in\mathbb{N}$, let $\mathfrak{F}_2(n)$ be the g.c.e.t. that arises by dropping the lowest mark in $\mathfrak{F}_1\circ\mathfrak{g}(n)$. Consider the resulting infinite sequence $\mathfrak{F}_2$. Clearly, $\mathfrak{F}_2(n)$ are all open. Moreover, by the monotonicity of the lowest labels in $\mathfrak{F}_1\circ\mathfrak{g}$, $\mathfrak{F}_2(i)\sqsubseteq^{\#}\mathfrak{F}_2(j)$ implies $\mathfrak{F}_1\circ\mathfrak{g}(i)\sqsubseteq^{\#}\mathfrak{F}_1\circ\mathfrak{g}(j)$ for all $i,j\in\mathbb{N}$. Hence $\mathfrak{F}_2$ is $\sqsubseteq^{\#}$-bad. Arguing as in the previous case, this leads to a contratiction. □

4.7. *Remark.* Finally, we further generalize edge-labeled trees by extending the quasi-labeling function ϱ onto the whole vertex-domain, while replacing the previous embedding-condition (5.4.1) accordingly by

(5.7.1) For all $x\in V(T_1)$, $\varrho_1(x)\leq\varrho_2\circ f(x)$.

Let G be the corresponding strenghtening of Proposition F.

Then by a slight modification of the above arguments, $\mathbf{ITR_0}$ proves Proposition G. So in the proof-theoretical sense, G is not stronger than F, and hence it is not stronger than A.

5.8. *Remark.* Recall that so far we considered vertex-labeled embeddability under the symmetrical gap-condition (2.2.3)

$$\text{If } \mathrm{pre}_1(y) = x \text{ and } f(x) \lhd_2 u \lhd_2 f(y), \text{ then } \min\{\ell_1(x),\ \ell_1(y)\} \preceq \ell_2(u),$$

and under the asymmetrical gap-condition (2.2.4)

$$\text{If } \mathrm{pre}_1(y) = x \text{ and } f(x) \lhd_2 u \lhd_2 f(y), \text{ then } \ell_1(y) \preceq \ell_2(u).$$

Consider the alternative asymmetrical gap-condition

(5.8.1) $$\text{If } \mathrm{pre}_1(y) = x \text{ and } f(x) \lhd_2 u \lhd_2 f(y), \text{ then } \ell_1(x) \preceq \ell_2(u).$$

It is clear that both (2.2.4) and (5.8.1) imply (2.2.3). Note that [5] shows that $\mathbf{ITR_0}$ actually proves Proposition A under the stronger condition (5.8.1). By Theorem 2.6, this also holds for (2.2.4). Hence both possible asymmetrical gap-conditions are proof-theoretically equivalent to the symmetrical gap-condition (2.2.3).

On the other hand, consider the alternative symmetrical gap-condition

(5.8.2) $$\text{If } \mathrm{pre}_1(y) = x \text{ and } f(x) \lhd_2 u \lhd_2 f(y), \text{ then } \max\{\ell_1(x),\ \ell_1(y)\} \preceq \ell_2(u).$$

By contrast to (2.2.3), this condition does not provide a well-quasi-order. An easy counter-example is as follows. For any $n \in \mathbb{N}$, let $\mathfrak{S}(n)$ be an interval (i.e. a tree without branching) $\langle 0 \lhd 1 \lhd \ldots \lhd n+2 \rangle$ whose vertices are labeled by the following function $\ell(0) = \ell(n+2) = 2$, $\ell(2i) = 0$ and $\ell(2j+1) = 1$ for all $i < [\frac{1}{2}n]+1$ and $j \leq [\frac{1}{2}(n+1)]$. So all labels are natural numbers smaller than 3. Clearly, there is no $i \neq j$ such that $\mathfrak{S}(i)$ is embeddable into $\mathfrak{S}(j)$ with the gap-condition (5.8.2).

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