Proof lab for first year students

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The goal of this proof lab is that you get more experienced and more confident in doing proofs. Here are some general recommendations. Feel free to add your own!

Coming up with a proof

1. Read the claim carefully.
2. Do you know how all terms in the statement of the claim are defined? If not, find out!
3. Do you have one or more examples in mind that satisfy all or most of the assumptions? If not, try to find one (by experimentation, looking up in the text book or lecture notes, or by asking others)!
4. Does your examples satisfy the claim? Why?
5. Do you see a pattern in which the examples work out to satisfy the claim? How do the assumptions come into play?
6. Try if you understand how things work in a special case, maybe low dimension or such. Does this insight help you for the proof of the general claim?
7. Have you seen any similar claims before? If yes, how were they proved? Can you maybe imitate or modify those proofs?
8. What facts do you know about the objects satisfying the assumptions in the claim? Check the lemmas, propositions, and theorems in the lecture notes and/or the book: does any of them help link the assumptions to the claim? (What can you "deduce" from the assumptions?)
9. What facts do you know about objects satisfying the conclusion of the claim? Check the lemmas, propositions, and theorems in the lecture notes and/or the book: does any of them help link the assumptions to the claim? (By what would the conclusion of the claim be implied?)
10. What methods of proof could you use: proof by contradiction, proof by proving the contrapositive, proof by example, proof by counterexample, proof by induction (we will see that soon)
11.
12.

When you get stuck

- have a break
- talk to others, trying to explain what you do and what you do not understand
- do another problem before re-attacking the original one
- try to take another perspective, for example: "does it help to think about equations or is it better to think about matrices?" or "does it help to think about maps rather than matrices" etc.
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Writing up a proof

Once you came up with a proof, you need to write it down or otherwise communicate it to other people. It turns out that even when you know how a proof works there is still work necessary to work out how to communicate it (present it at the board or write it down). In particular, it is important to write the steps of the proof in the right order, starting with the assumption(s) and finishing with the conclusion(s). You should always try to write your proofs down in a way that allows others – and yourself at a later point of time – to understand what you are doing. Here are some suggestions:

- State clearly what your assumptions are ("if..." or "let...").
- State clearly what your conclusions are ("then...").
- Define and quantify all variables your using ("Let $\vec{x} \in \mathbb{R}^n$ such that ..." or "for all $c \in \mathbb{R}$ ..." or "there exists $p \in \mathcal{P}$ such that ...").
- In each (or at least in each slightly involved) step of you proof, be it a manipulation of equations or a logical argument, explain
  1. what you are doing
  2. why you are doing it
  3. why you are allowed to do it (e.g. state that you use a fact from class or from the text book or something you proved earlier, say in part a) of the same exercise), for example "by associativity" or "by the Det-Theorem" etc.
- Read your proof again after writing it:
  1. Did you prove everything you claimed?
  2. Did you prove both directions in an iff-claim?
  3. Did you prove existence and uniqueness in "there is a unique ..."?
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The more complicated the proof is, the more important it becomes to make an effort to write it up understandably.

Checking a proof

- Can you identify what the assumptions and the conclusions are?
- Do you understand how to get from each step in the proof to the next?
- Are all steps correct?
- Are the steps well-justified (by citing theorems or facts etc.)?
- Do you know why the author of the proof is doing what they are doing?
- Do you understand the proof as a whole?
- Could you put it aside and reproduce it?
- Do you find a shortcut?
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Instruction

Please form groups of 2-4 students. As a group, choose one of the problems. Please make sure that not all groups work on the same problem. You have **30mins** to come up with a proof for the problem you chose as a group. If you finish early, begin with a second exercise. Don't forget to take notes. Then take another **15mins** to write up your proof carefully (one written proof per group).

Now exchange proofs between the different groups. Read the proof you obtained from the other group carefully, maybe using the check-list above. Take notes. You have **20mins** for this. Then sit down with the group whose proof you read and give constructive feedback. Please be friendly and careful not to offend your class mates! Also, sit down with the group who read your proof and carefully listen what they have to say about your proof.

If there is time, repeat the entire process with another problem, preferably not the one you have proof-read. I hope the lab is enjoyable and helpful!

Exercises (generic[1])

1. injectivity and surjectivity in finite sets
   A map $f : \mathcal{A} \to \mathcal{B}$ between two sets is called *injective* if $f(a_1) = f(a_2)$ implies $a_1 = a_2$ for all $a_1, a_2 \in \mathcal{A}$. It is called *surjective* if for each $b \in \mathcal{B}$ there is $a \in \mathcal{A}$ such that $f(a) = b$. It is called *bijective* if it is injective and surjective. Show that
   a) f can only be injective if $\mathcal{A}$ has at most as many elements as $\mathcal{B}$.
   b) f can only be surjective if $\mathcal{A}$ has at least as many elements as $\mathcal{B}$.
   c) f can only be surjective if $\mathcal{A}$ has exactly as many elements as $\mathcal{B}$.
   d) True or false: If $\mathcal{A}$ has exactly as many elements as $\mathcal{B}$ then f is bijective.
   e) True or false: If $\mathcal{A}$ has at least as many elements as $\mathcal{B}$ then f is surjective.
   f) True or false: If $\mathcal{A}$ has at most as many elements as $\mathcal{B}$ then f is injective.
2. Pick one or several proof exercises from the text book and prove them.

[1]Copyright is not violated if you add your own exercises.