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UNIVERSITÄT
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MATHEMATISCH-
NATURWISSENSCHAFTLICHE
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Advanced Labwork in Astronomy and Astrophysics

# Stellar oscillations

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Version 08.10.2020

https://uni-tuebingen.de/de/4203

# 1 Introduction

The aim of this exercise is to examine the equilibrium configuration and oscillations of spherically-symmetric, self-gravitating stars. For that, the student has to solve numerically the equilibrium structure equation (the Lane-Emden equation), and then, use a one-dimensional hydrodynamics code to compute perturbations to the equilibrium configuration and study the stellar oscillations. As an additional, optional practice, the student can also use the code developed in the first part of this exercise to compute the Chandrasekhar limit.

During the execution of the exercise, the student will encounter several aspects of computational astrophysics, namely, the discretization of the problem, the implementation of the solution of a second-order ordinary differential equation, a general understanding of the solution of the one-dimensional hydrodynamic equations through the use of a given code, the interpolation of results into a grid, and the plotting and analysis of the final results.

Up to the 20th century, only geometrical methods were known for measuring distances in space. In this way, only objects that were located at less than approx. 300 ly had their distances well measured. This excluded the possibility of measuring the distance to distant stars and galaxies (whose distances are from millions to billions of light years). This changed thanks to the discovery of variable stars of a certain type, the *Delta Cepheids*. The brightness of such stars undergoes periodic pulsations, in which the period and the luminosity depend from one another. That is, the bigger the absolute brightness of such a star, the longer lasts the oscillation in brightness. Through the measurement of the period of this kind of Cepheids, the absolute magnitude can be known, which then is used to determine the distance.

# 2 Equilibrium configuration

## 2.1 Stellar structure equations

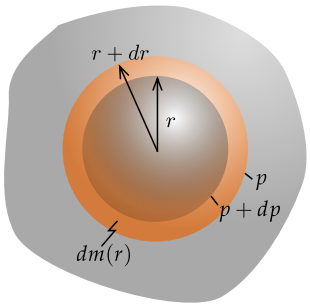


Figure 1: Spherical shell (mass differential)

Consider a static, spherically symmetric star. The mass enclosed in a spherical shell of thickness $dr$ (see Fig. 1) is $dm(r) = 4\pi r^2 \rho(r) dr$, which implies that

$$\frac{dm(r)}{dr} = 4\pi r^2 \rho. \tag{1}$$

This is the **mass continuity equation**. Now, let us consider the forces that act on the spherical shell. Gravity exerts a force

$$F_{\text{grav},r} = -\frac{Gm(r)}{r^2}(\rho dr A),$$

where $m(r) = \int_0^r \rho A dr$, and $A = 4\pi r^2$. Gravity is balanced by the gas pressure, which must be *higher* in the *inner* shell, and thus, it decreases with radius:

$$F_{\text{p},r} = [p - (p + dp)]A = -dpA.$$

The sum of the forces acting on the shell have to be zero:

$$F_{\mathrm{p},r} + F_{\mathrm{grav},r} = 0$$

$$\implies -dpA - \frac{Gm(r)}{r^2}(\rho dr A) = 0$$

eliminating $A$, and dividing by $dr$, we obtain the **stellar equilibrium equation**:

$$\frac{1}{\rho}\frac{dp}{dr} = -\frac{Gm(r)}{r^2} \tag{2}$$

## 2.2 Lane-Emden equation

If we take the eq. 2 and differentiate it with respect to $r$, we obtain

$$\frac{d}{dr}\left(\frac{1}{\rho}\frac{dp}{dr}\right) = -\frac{G}{r^2}\frac{dm}{dr} + 2\frac{Gm}{r^3}. \tag{3}$$

Now, in the right hand side of this equation, we use the mass continuity equation to replace the first term and the stellar equilibrium equation to replace the second term:

$$\frac{d}{dr}\left(\frac{1}{\rho}\frac{dp}{dr}\right) = -4\pi G\rho - \frac{2}{r}\frac{1}{\rho}\frac{dp}{dr}. \tag{4}$$

It is very easy to show that

$$\frac{1}{r^2}\frac{d}{dr}(r^2 f) = \frac{df}{dr} + \frac{2f}{r}. \tag{5}$$

We use this relation to rewrite the left hand side and the second term of the right hand side of eq. 4 to obtain

$$\frac{1}{r^2}\frac{d}{dr}\left(\frac{r^2}{\rho}\frac{dp}{dr}\right) = -4\pi G\rho. \tag{6}$$

Now, we will use the *polytropic equation of state*, which relates the pressure and the density via

$$p = K\rho^{\gamma}, \tag{7}$$

where $K$ is a constant, and $\gamma$ is the *adiabatic exponent*. We want now to put this equation in an adimensional form, for which we use the following transformations:

$$\rho = \rho_c w^n \tag{8}$$

and

$$z = Ar, \quad \text{with } A^2 = \frac{4\pi G}{(n+1)K}\rho_c^{1-1/n}; \tag{9}$$

where $\rho_c$ is the central density of the star [i.e., $\rho(r=0)$], and

$$n = \frac{1}{\gamma - 1} \tag{10}$$

is the *polytropic index*. After using the transformations, we obtain the **Lane-Emden equation**:

$$\frac{1}{z^2}\frac{d}{dz}\left(z^2\frac{dw}{dz}\right) + w^n = 0. \tag{11}$$

This equation is used to obtain the structure of a non-rotating star, that is, its density and pressure as a function of the radial distance. From eq. 11 it follows that, in order for the density

to stay finite at the center of the star, the derivative radial derivative of the density has to vanish at $r = 0$, that is, one of the boundary conditions for solving the Lane-Emden equation is

$$\frac{dw}{dz} = 0 \qquad \text{at } z = 0, \tag{12}$$

and the boundary condition on $w$ is given by the definition in eq. 8, which implies that

$$w = 1 \qquad \text{at } z = 0. \tag{13}$$

Given eq. 11 and the boundary conditions of eqs. 12 and 13, there are known analytical solutions for only three values of the polytropic index:

$$n = 0: \quad w(z) = 1 - \frac{1}{6}z^2 \tag{14}$$

$$n = 1: \quad w(z) = \frac{\sin z}{z} \tag{15}$$

$$n = 5: \quad w(z) = \frac{1}{(1 + z^2/3)^{1/2}}. \tag{16}$$

Only the solutions for $n = 0$ and $n = 1$ are finite; the solution for $n = 5$ has an infinite radius and mass.

### 2.3 Numerical solution of the Lane-Emden equation

For other (more realistic) values of the polytropic index, the Lane-Emden equation has to be solved numerically. An ideal, monoatomic gas, for example, has $\gamma = c_p/c_v = 5/3 \implies n = 3/2 = 1.5$, that is, the numerical solution is required.

The Lane-Emden equation is a second order ordinary differential equation. In order to write a program that solves it numerically, we need to transform it into a system of two first order ordinary differential equations for the functions $w$ and $\xi := dw/dz$. This system of equations is

$$\begin{cases} \dfrac{dw}{dz} = \xi \\ \dfrac{d\xi}{dz} = -\dfrac{2}{z}\xi - w^n \end{cases} \tag{17}$$

and the boundary conditions transform into initial conditions:

$$w = 1 \quad \text{and} \quad \xi = 0. \tag{18}$$

## 3 Stellar oscillations

### 3.1 Hydrodynamical equations

The equations of hydrodynamics are commonly written in two forms: the Lagrangian form, and the Eulerian or conservative form. In the Lagrangian form, the aim is to follow a fluid element along its motion, and see how its properties (density, pressure, etc.) change along the the path. In the Eulerian, or conservative form, a fixed "grid" is defined in such a way that the properties of the fluid are studied in every point of the grid. For this exercise, we use the conservative form of the fluid equations.

The continuity equation is

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot (\rho \vec{u}) = 0 \tag{19}$$

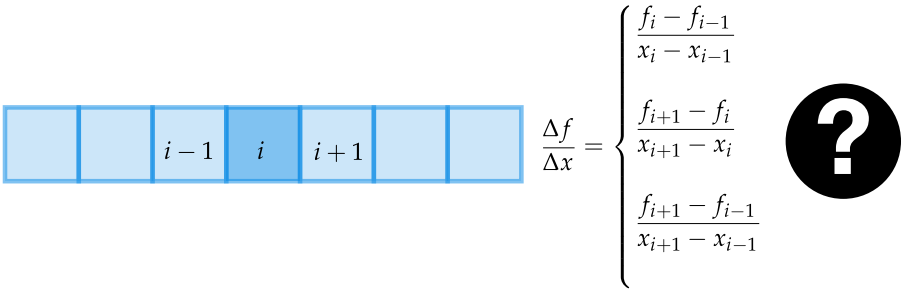


Figure 2: Different ways of discretizing a gradient

After expressing this equation in spherical coordinates and considering spherical symmetry, that is, $\partial/\partial\phi = 0$ and $\partial/\partial\theta = 0$, we get

$$\frac{\partial \rho}{\partial t} + \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 \rho u) = 0. \tag{20}$$

We call the quantity $F^m = \rho u$, the *mass flux*.

The momentum conservation equation is

$$\frac{\partial}{\partial t}(\rho \vec{u}) + \vec{\nabla} \cdot (\rho \vec{u} \otimes \vec{u}) = -\vec{\nabla} p + \rho \vec{f} \tag{21}$$

where $\vec{f}$ is the specific external force (force per unit mass) experienced by the fluid. Again, writing the operators in spherical coordinates and applying the symmetry simplifications, we get

$$\frac{\partial}{\partial t}(\rho u) + \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 \rho u u) = -\frac{\partial p}{\partial r} - \frac{G m(r) \rho}{r^2}. \tag{22}$$

The *momentum density* is $w := \rho u$, and the *momentum flux*, $F^w = wu$.

As it can be seen from the definition of the fluxes, the left hand sides of both equations have similar forms, and can be interpreted (and derived) from the analysis of the mass or momentum that enters one fixed volume (time derivative), and the mass or momentum that is transported out of the volume through the surfaces of the fixed volume (the divergence of the flux). This means, the left hand side of both equations express conservation laws.

In a similar way to the equilibrium configuration, we consider the polytropic equation of state

$$p = K \rho^{\gamma} \tag{23}$$

In the adiabatic case, the speed of sound is defined as

$$c_s = \sqrt{\frac{\gamma p}{\rho}}. \tag{24}$$

## 3.2 Numerical solution

### 3.2.1 Finite differences

For the numerical solution of the hydrodynamical equations, the basic idea is to divide the space and time in discrete steps; the division of space is called a *grid*. Once this division has been made, then, we approximate the differential operators with finite differences ($d \to \Delta$).

Not all discretizations lead to a stable approximation of the solution; a detailed analysis is required in many cases. For example, consider the approximation of a gradient according to fig. 2. Which discretization is more convenient? The last one, that combines information in equal manner from the left and the right of the analyzed cell, seems to be well motivated. However, it can be shown that in an advection problem, like the one we are trying to solve, and if combined by a straightforward time approximation, this approximation leads to an unstable solution for any time.

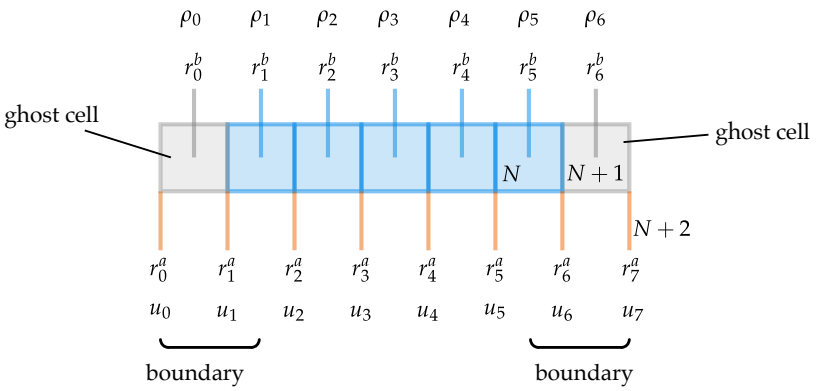


Figure 3: Example of the grid used in the solution of the hydrodynamics equations.

### 3.2.2 Grid

In the solution of the problem, we use a *staggered* grid (fig. 3), where some quantities are defined at the cell centers, and others, at the cell walls. As it is shown in the figure, we define the grid $r_i^a$ at the cell walls, and the grid $r_i^b$ at the cell centers (the actual grid should have an appropriately high number of cells). The grid should extend from $r_1^a = 0$ to $r_N^a \approx 1.4R_\star$, where $R_\star$ is the radius of the star determined in the first part, i.e., with the Lane-Emden equation. The cells we use here are in the shape of a spherical shell.

In this staggered grid, we define the scalar quantities, i.e., the density and the pressure, at the cell centers, and the vector quantities, that is, velocity, force and momentum, at the cell walls. In fig. 3, we see some cells in gray, labeled *ghost cells*. They are meant to facilitate the setup and application of the boundary conditions, but they do not form part of the solution of the problem.

### 3.2.3 Overview of the numerical method

In this section, we will have a short overview of the numerical method used for the solution of the hydrodynamics equations. The independent variables of the system of equations formed by eqs. 20 and 22 are the velocity $u$ and the momentum $\rho$. The pressure can always be determined by the equation of state. The conservation form of the equations allows us to notice that the left hand side of both equations has the same form:

$$\frac{\partial}{\partial t} f + \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 F^f) = \text{sources}, \tag{25}$$

where $f$ is a quantity (here, density or velocity), and $F^f := fu$ is its flux. We split the task of updating the quantity $f$ in time into two steps: an *advection step*, in which the left hand side of the equation is applied, and the *sources* or *forces step*, in which the external forces or sources are applied. The continuity equation, having no sources, only requires the advection step; the momentum equation, in the other hand, requires both steps. This process is called *operator splitting*. In each of the steps, the finite differences method is applied, and the fluxes must be carefully computed (the principle that we use for flux calculation is called the *first order upwind method*, which we will not discuss here).

### 3.2.4 Time step

The selection of the time step $\Delta t$ is not completely arbitrary, but it should satisfy the *CFL-criterion* (after Courant, Friedrich and Levy): the time step should be small enough such that no perturbation should propagate further than $C\Delta r$, where $\Delta r$ is the width of a cell and $C$ has a value lower

than 0.75. The propagation speed of a perturbation in each cell is given by $c_s + |u|$, where $c_s$ was defined in eq. 24. Then,

$$\Delta t \leq C \min_i \left( \frac{\Delta r}{c_{si} + |u_i|} \right), \tag{26}$$

where the operator $\min_i$ returns the minimum value of the quantity among all the grid cells.

### 3.2.5 Boundary conditions

The boundary conditions we will use are as follows. At the extremes of the grid, we use closed boundaries, that is, matter cannot enter or leave the domain, which implies that the velocity must be zero (and so, the flux is also zero), at both extremes. For the density and pressure, we shall use reflective boundaries (the radial derivatives vanish), since we assume spherical symmetry. The boundary conditions are summarized as

$$u = 0 \quad \text{for} \quad r = 0 \quad \text{and} \quad r = r_{\max} \tag{27}$$

$$\frac{\partial \rho}{\partial r} = 0 \quad \text{for} \quad r = 0 \quad \text{and} \quad r = r_{\max}. \tag{28}$$

These boundary conditions are conveniently implemented through the use of ghost cells.

### 3.2.6 Initial conditions

The initial density will be filled with the equilibrium configuration we obtained in the first part of the exercise, inside the star (the central density is 1 in code units). Outside the star, a small value shall be used for the density, for example, $\rho_{\min} = 10^{-6}$ (density floor). The density should not be set equal to zero, because several terms in the equations we are solving contain terms inversely proportional to the density. The initial velocity should be set initially to zero, but later, we also introduce perturbations by setting an initial value for the velocity.

### 3.2.7 Units

The hydrodynamics equations, 20, 22 and 23 can be adimensionalized by substituting the relations

$$\begin{cases} \rho = \rho_o \tilde{\rho} & u = u_0 \tilde{u} \\ p = p_o \tilde{p} & t = t_o \tilde{t} \\ r = r_o \tilde{r}, & \end{cases}$$

where the symbols with a tilde are adimensional, and then forcing some terms to become 1. The velocity is built as

$$u = \frac{r_o}{t_0} \tilde{u}. \tag{29}$$

The following relations are then obtained:

$$t_0 = 1/\sqrt{G\rho_0} \tag{30}$$

$$r_0^2 := \frac{1}{A^2} = \frac{n+1}{4\pi} \frac{K}{G} \rho_0^{1/n-1} \tag{31}$$

$$p_0 = \rho_0 u_0^2 = \frac{n+1}{4\pi} K \rho_0^{1+1/n} \tag{32}$$

and so, the units are set by the choice of the central density $\rho_0$ and the polytropic index $n$. If we set

$$K = \frac{4\pi}{n+1} \tag{33}$$

our scale lengths simplify, and the the equation of state, eq. 23, is transformed to

$$\tilde{p} = \frac{4\pi}{n+1}\tilde{\rho}^{1+1/n} \tag{34}$$

# 4 Bonus application: the Chandrasekhar limit

Another, unrelated application of the Lane-Emden equation is obtaining the Chandrasekhar limit, that is, the upper mass limit for a stable white dwarf. Stars with cores more massive than the Chandrasekhar limit form either neutron stars or black holes.

The equation of state in this case is computed for a degenerated Fermi gas of electrons. The process can be summarized this way: first, the distribution function of the degenerated particles in the phase space is found, which defines the Fermi momentum (so that the particles are arranged with the lowest energy possible). The number density and degeneracy pressure are computed as a function of the Fermi momentum with the help of the distribution function, and two limiting cases arise: the non-relativistic case, and the ultra-relativistic case. We take the latter case for computing the Chandrasekhar limit. After combining the pressure and the number density, one obtains an equation of state of the form

$$p = K\rho^{4/3}, \tag{35}$$

where $K$ is found in terms of fundamental constants:

$$K = \frac{3^{1/3}}{8\pi^{1/3}} \frac{hc}{m_H^{4/3}\mu_e^{4/3}}, \tag{36}$$

with $\mu_e = 1$ for a hydrogen star, and $\mu_e = 2$ for a white dwarf. For the non-relativistic case, the equation of state is found to be $p_{\mathrm{nr}} = K_{\mathrm{nr}}\rho_{\mathrm{nr}}^{5/3}$, which is the adiabatic case, and with $K_{\mathrm{nr}}$ of the order of $10^7$ when expressed in SI units.

The total mass of the star can be calculated as

$$M = \int_0^R 4\pi r^2 \rho dr = 4\pi(1/A)^3\rho_c \int_0^{z_1} z^2 w^n \, dz, \tag{37}$$

where $z_1$ is the adimensional radius of the star, that is, the point where the adimensional density $w$ goes to zero. For $n = 3$ ($\gamma = 4/3$), the mass of the star is independent of the central density, as it can be proven if $A$ is substituted in eq. 37:

$$M \propto (\rho_c^{1-1/3})^{-3/2}\rho_c = 1.$$

Now, if we multiply eq. 11 both sides by $z^2$, and integrate between 0 and $z_1$, we see that

$$\int_0^{z_1} z^2 w^n dz = -z_1^2 \left(\frac{dw}{dz}\right)_{z=z_1} = -z_1^2\tilde{\varsigma}(z_1). \tag{38}$$

Using eq. 41 in eq. 37, and substituting all of the values for the constants, one can find the Chandrasekhar limit

$$M_{\mathrm{Ch}} = \frac{\sqrt{6}}{32\pi}\left(\frac{hc}{G}\right)^{3/2}\left(\frac{2}{\mu_e}\right)^2 \frac{|z_1^2\tilde{\varsigma}(z_1)|}{m_H^2}. \tag{39}$$

# 5 Procedure

## 5.1 Requirements

The following packages should be installed in the computer that will be used for the exercise:

1. GFortran
2. Python 3.x
3. Numpy
4. Scipy
5. Matplotlib

Notes:

- A Linux machine is preferred, however, if this is not possible, there are several platform-dependent options to get the configuration working:
  - MacOS users: install *Homebrew* and then, the GNU compilers, that include Fortran. Install Python 3.x from Homebrew, and then the rest of Python packages using PIP (e.g., `pip install numpy`). Tip: check that when you run `python` and `pip` from your terminal, you are running the Homebrew-version and not the default Python included with MacOS!
  - Windows users: try installing the Windows Subsystem for Linux, and then, install GFortran and Python via the command line (e.g.,`sudo apt install gfortran`). Install the Python packages via PIP.
  - If nothing seems to work, then, download VirtualBox (`https://www.virtualbox.org`) and a version of Ubuntu, and install it in a virtual machine. Warning: this takes some space (and time to download), but after it is done, it should be easy to follow instructions.
- It is possible to use other programming languages or software packages for plotting and solving the Lane-Emden equation, but it may not be possible for the tutor to give technical support.

## 5.2 Equilibrium configurations

1. Write a program (preferably in Python) that solves the system of equations 17 subject to the boundary conditions in eq. 18. For this step, the following options are available:
   - Implement your own code directly by following a numerical method for an initial value problem, such as Runge-Kutta of 2nd order.
   - Use Scipy, for example with the function `scipy.integrate.solve_ivp`.
   - If you have no previous experience in solving numerically such initial value problems, then, let the tutor know.
2. Solve the Lane-Emden equations for the following values of the adiabatic exponent:
   - $\gamma = 3$
   - $\gamma = 5/3$
   - $\gamma = 7/5$
3. Plot the results of the density as a function of the radial coordinate, in code units.
4. Repeat the plots of the previous point but this time, use the following units: for $r$, solar radii, and for $\rho$, the mean density of the Sun ($\bar{\rho} = M_\odot / (\frac{4}{3}\pi R_\odot^3)$). Choose the central density for a Sun-like star and $n = 3$. Discuss the results and compare them with values from elsewhere. (Hint: see the Appendix for help calculating the polytropic constant $K$)

## 5.3 Stellar oscillations

For this section, a Fortran code will be distributed among the students during the experiment. No prior knowledge of Fortran (95+) is needed.

1. Read the code: make an outline of modules there, and the subroutines in each module. Which steps are called in the "main" `program`? Where are the initial conditions of the problem set? Which size is the grid?

2. Run the code by setting $u_i = 0$ and the analytical solution of the Lane-Emden equation for $n = 1$ (eq. 15). Plot the results with the plotting script provided. The equilibrium configurations should remain in equilibrium.

3. Introduce a perturbation in the velocity so that

$$u(r) = \frac{1}{10} c_s(r) \sin(\pi r / R_\star).$$

   Run the code, plot and discuss the results.

4. Incorporate to the program made in the the first part of the experiment an interpolation of the results in such a way that they fit into the grid of the hydrodynamics code. That is: the results obtained in the first part of the exercise had a different $\Delta r$ than what is needed for the second part of the exercise. The way to fix this is through interpolation into the grid used in the hydrodynamics code. The use of the function `scipy.interpolate.interp1d` is recommended. If you have no experience with interpolation, ask the tutor. The output (text) file of your program has to be in the following form: the first line outputs the star radius in code units, and the rest of the lines output the density vector (vertically).

5. Feed the output of your program for different $n$ into the Fortran hydrodynamics program.

6. Run the hydrodynamics program and plot the results for $u_i = 0$. Your equilibrium configuration should remain in equilibrium.

7. Introduce again a perturbation in the velocity so that

$$u(r) = \frac{1}{10} c_s(r) \sin(\pi r / R_\star).$$

   Run the code, plot and discuss the results.

8. Try other perturbations freely. Choose one to add it to your report.

## 5.4 Optional: Chandrasekhar limit

1. Run the program you created in the first part of the experiment, but for $n = 3$ (i.e., $\gamma = 4/3$). Output this time the value of the radius of the star $z_1$ and $\xi = dw/dz$ evaluated at that point.

2. Introduce those values in eq. 39. You should get the famous value of $M \approx 1.4 M_\odot$.

## References


- Choudhuri, A. (2010). *Astrophysics for Physicists*, Cambridge University Press. §5.2
- Kippenhahn, R. & Weigert, A. (1990) *Stellar Structure and Evolution*, Springer Verlag.
- Schwarz, H. R (1993) *Numerische Mathematik*, Teubner, Stuttgart.
- Stone, J. M. & Norman, M. L. (1992) ApJS 80, 753.

# A Fixing the polytropic constant for a Sun-like star

For a Sun-like star, the polytropic constant $K$ is not fixed. We can fix it with knowledge of the mass of the star and the radius of the star.

The total mass of the star can be calculated as

$$M = \int_0^R 4\pi r^2 \rho dr = 4\pi (1/A)^3 \rho_c \int_0^{z_1} z^2 w^n \, dz, \tag{40}$$

where $z_1$ is the adimensional radius of the star, that is, the point where the adimensional density $w$ goes to zero. Now, if we multiply eq. 11 both sides by $z^2$, and integrate between 0 and $z_1$, we see that

$$\int_0^{z_1} z^2 w^n dz = -z_1^2 \left(\frac{dw}{dz}\right)_{z=z_1} = -z_1^2 \xi(z_1). \tag{41}$$

We obtain

$$M = 4\pi \rho_c \left(\frac{z_1}{A}\right)^3 \left|\frac{1}{z_1}\xi(z_1)\right| = 4\pi \rho_c R_\star^3 \left|\frac{1}{z_1}\xi(z_1)\right| \tag{42}$$

dividing the previous equation over 3, we can easily obtain the central density in terms of the average density $\bar{\rho} = 3M/(4\pi R^3)$:

$$\frac{\bar{\rho}}{\rho_c} = \left|\frac{1}{z_1}\xi(z_1)\right|. \tag{43}$$

Now, at $r = R_\star$, we know that $A = z_1/R$. Once $A$ is determined in this way, we can use eq. 9 to calculate $K$.