A Note on Kripkean Countermodels for Intuitionistically Unprovable Sequents

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Abstract. We present a slight modification to an ordinary multi-succedent sequent calculus for intutionistic propositional logic which allows straightforward extraction of Kripkean countermodels from failing derivation constructions.

For classical propositional logic there exists an immediate relation between failing sequent calculus derivations for a given propositional sequent and Boolean counterexamples for this sequent: any completed branch constructed during an attempt to derive a sequent which does not end in an axiom may be directly read as a countermodel for this sequent. For intuitionistic logic the situation is more complicated. First of all sequent calculi for intuitionistic logic have to contain noninvertible rules. Therefore nonderivability of a given sequent is not witnessed by a single non closed branch but by a tree of sequents all of whose branches are non closed. Considering for instance a well known multisuccedent calculus LJ for intuitionistic propositional logic with axioms $M, a \Rightarrow a, N$ and rules

$$(\mathrm{E}\wedge)\frac{M, a, b, a \wedge b \Rightarrow N}{M, a \wedge b \Rightarrow N} \qquad \frac{M \Rightarrow a, a \wedge b, N \qquad M \Rightarrow b, a \wedge b, N}{M \Rightarrow a \wedge b, N}(\mathrm{I}\wedge)$$

$$(\mathrm{E}\vee)\frac{M, a, a \vee b \Rightarrow N \qquad M, b, a \vee b \Rightarrow N}{M, a \vee b \Rightarrow N} \qquad \frac{M \Rightarrow a, b, a \vee b, N}{M \Rightarrow a \vee b, N}(\mathrm{I}\vee)$$

$$(\mathrm{E}\rightarrow)\frac{M, a \rightarrow b \Rightarrow a, N \qquad M, b, a \rightarrow b \Rightarrow N}{M, a \rightarrow b \Rightarrow N} \qquad \frac{M, a \Rightarrow b}{M \Rightarrow a \rightarrow b, N}(\mathrm{I}\rightarrow)$$

(cf. [3]) nonderivability of the sequent $\Rightarrow a \rightarrow b, b \rightarrow a$ is witnessed by the tree of sequents

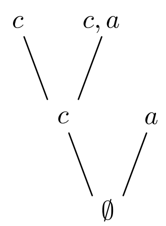
$$\frac{a \Rightarrow b \qquad b \Rightarrow a}{\Rightarrow a \rightarrow b, b \rightarrow a}$$

indicating the fact that both possible choices of the principal formula lead to nonclosed completed branches.

Moreover the trees witnessing nonderivability of a sequent do not in general correspond to Kripkean counterexamples for this sequent (cf. [1]) in a straightforward manner. For instance the sequent $(c \rightarrow a) \rightarrow a, (a \rightarrow b) \rightarrow a \Rightarrow a$ has an (infinite) tree witnessing its nonderivability which starts as follows:

$$
\frac{\dfrac{\dfrac{\dfrac{\begin{array}{c}\vdots\\ M,c\Rightarrow a\end{array} \qquad M,c,a\Rightarrow b}{M,c\Rightarrow a,c\to a,a\to b}}{M,c\Rightarrow a,c\to a}}{M,c\Rightarrow a} \qquad M,a\Rightarrow b}{\dfrac{M\Rightarrow a,c\to a,a\to b}{\dfrac{M\Rightarrow a,c\to a}{M\Rightarrow a}}}
$$

(where M consists of the two formulas $(c \to a) \to a$ and $(a \to b) \to a$.) Here the branches ending in $M, a \Rightarrow b$ and $M, c, a \Rightarrow b$ are finite because the right premisses of both possible E→-inferences equal the conclusion; the leftmost branch, however, contains infinitely many copies of the sequent $M, c \Rightarrow a$. So if we consider a straightforward Kripkean model of the form



obtained from this sequent tree, then the formula $(a \to b) \to a$ is false in every world on the leftmost branch and thus the sequent $(c \to a) \to a, (a \to b) \to a \Rightarrow a$ is evaluated to true in the bottom world.

Now we consider a slight modification LS of the calculus LJ. LS has the same rules as LJ except for I→: the I→-rule of LS is a two-premiss rule of the form

$$
\frac{M, a \Rightarrow b, N \qquad M, a \Rightarrow b}{M \Rightarrow a \to b, N}
$$

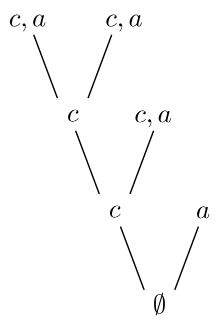
The new two-premiss I→-rule is obviously admissible for the calculus LJ; on the other hand if the new calculus LS derives a sequent $M, a \Rightarrow b$, then it also derives $M, a \Rightarrow b, N$, thus the one-premiss I→ is admissible for LS and the two calculi are equivalent.

For LS we shall show that sequent trees witnessing nonderivability of a given sequent may be directly read as Kripkean counterexamples for this sequent. Considering e.g. the previous example we have the following failing deduction tree:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{M,c,c,a \Rightarrow b}{(*)\quad M,c,c \Rightarrow a,a,a \to b} \qquad M,c,a \Rightarrow b}{M,c \Rightarrow a,c \to a,a \to b}}{M,c \Rightarrow a,c \to a}}{M,c \Rightarrow a} \qquad M,a \Rightarrow b}{M \Rightarrow a,c \to a,a \to b}}{M \Rightarrow a,c \to a}}{M \Rightarrow a}$$

(Here the sequent marked with an asterisk has been obtained as the left premiss of an application of our new I→-rule.)

Now this tree immediately yields the following obvious counterexample for our sequent:



(Note that this countermodel is a more elaborate form of the well known countermodel for the so called Peircean formula $((a \to b) \to a) \to a$; the countermodel constructed in this way is, therefore, in general not minimal.)

In order to show that model extraction based on our calculus LS succeeds for all nonderivable sequents, we start by defining the notion of a *refutable* sequent:

Definition. A sequent s is refutable, iff

a) s is not an axiom of LS and moreover

i) for all formulas $a \wedge b$ occurring on the left hand side of s and all formulas $a \vee b$ occurring on the right hand side of s both formulas a and b occur on the left hand side resp. right hand side and

ii) for all formulas $a \wedge b$ occurring on the right hand side of s and all formulas $a \vee b$ occurring on the left hand side of s either the formula a or the formula b occurs on the right hand side resp. left hand side and

iii) for all formulas $a \to b$ occurring on the left hand side of s either the formula a occurs on the right hand side of s or the formula b occurs on the left hand side of s and

iv) for all formulas $a \to b$ occurring on the right hand side of s the formula a occurs on the left hand side and the formula b occurs on the right hand side of s or

b) s is not an axiom of LS and moreover
i) condition *i*) of part a) does not hold and the sequent obtained from s by adding the subformulas a and b of the offending formula $a \wedge b$ resp. $a \vee b$ to the left hand side resp. right hand side of s is refutable or
ii) condition *ii*) of part a) does not hold and one of the sequents obtained from s by adding either the subformula a or the subformula b of the offending formula $a \wedge b$ resp. $a \vee b$ to the right hand side resp. left hand side of s is refutable or
iii) condition *iii*) of part a) does not hold and one of the sequents obtained from s by adding either the subformula a of the offending formula $a \to b$ to the right hand side of s or the subformula b of $a \to b$ to its left hand side is refutable or
iv) condition *iv*) of part a) does not hold and the sequent obtained from s by adding the subformula a of the offending formula $a \to b$ to the left hand side of s and the subformula b to its right hand side is refutable or

c) none of the previous conditions holds and s is not an axiom of LS and $a_1 \to b_1, \ldots, a_n \to b_n$ are all implications on the right hand side of s for which either a_i is not on the left hand side of s or b_i is not on its right hand side and all n sequents obtained from s by adding a_i to its left hand side and replacing its right hand side by b_i are refutable.

Using this definition we may associate with any refutable sequent s its *refutation tree* as follows:

Definition. a) If s is refutable according to condition a), then the refutation tree of s consists of s alone,

b) if refutability of s is established from refutability of a sequent s' by one of the conditions b), then the refutation tree of s is the same as the refutation tree of s',

c) if refutability of s is established from refutability of the sequents $s_1, \ldots, s_n$ according to condition c), then the refutation tree of s consists of the n refutation trees for the s_i made into a new tree by adding s as the new root.

We show that the refutable sequents are exactly the sequents not derivable by LS:

Proposition 1. *No refutable sequent is derivable.*

Proof. If a sequent s is refutable according to condition a), then it is not an axiom and for all possible LS-inferences leading to s one of the premisses is contained in s; thus s is not derivable by LS.

If refutability of s is established according to condition b) from refutability of another sequent s', then s' is not derivable by the induction hypothesis and since s is contained in s', the sequent s is not derivable either.

If refutability of s is established according to condition c) from refutability of all sequents $s_1, \ldots, s_n$, then the inferences with principal formulas $a_1 \to b_1, \ldots, a_n \to b_n$ are the only inferences leading to s for which none of the premisses is contained in s. Therefore nonderivability of the sequents $s_1, \dots, s_n$ according to the induction hypothesis implies nonderivability of s. □

The converse inclusion is proved by considering the following calculus CRIP due to Dyckhoff and Pinto [2] which consists of axioms of the form $p_1 \to b_1, \dots, p_n \to b_n, M \Rightarrow N$, where all p_i are propositional variables, M and N are disjoint sets of propositional variables and none of the p_i occurs in M and rules

$$\frac{M, a, b \Rightarrow N}{M, a \wedge b \Rightarrow N}\ (1) \quad \frac{M \Rightarrow a, N}{M \Rightarrow a \wedge b, N}\ (2) \quad \frac{M \Rightarrow b, N}{M \Rightarrow a \wedge b, N}\ (3) \quad \frac{M, a \Rightarrow N}{M, a \vee b \Rightarrow N}\ (4)$$

$$\frac{M, b \Rightarrow N}{M, a \vee b \Rightarrow N}\ (5) \qquad \frac{M \Rightarrow a, b, N}{M \Rightarrow a \vee b, N}\ (6) \qquad \frac{M, p, b \Rightarrow N}{M, p, p \to b \Rightarrow N}\ (7)$$

$$\frac{M, b \Rightarrow N}{M, (c \to d) \to b \Rightarrow N}\ (8) \quad \frac{M, c \to d, d \to b \Rightarrow N}{M, (c \vee d) \to b \Rightarrow N}\ (9) \quad \frac{M, c \to (d \to b) \Rightarrow N}{M, (c \wedge d) \to b \Rightarrow N}\ (10)$$

$$\frac{M_1, c_1, d_1 \to b_1 \Rightarrow d_1 \quad \dots \quad M_m, c_m, d_m \to b_m \Rightarrow d_m \quad M, e_1 \Rightarrow f_1 \quad \dots \quad M, e_n \Rightarrow f_n}{M \Rightarrow e_1 \to f_1, \dots, e_n \to f_n, N}$$

where p in rule (7) is a propositional variable and where in the last rule, i.e. rule (11) M is of the form $(c_1 \to d_1) \to b_1, \dots, (c_m \to d_m) \to b_m, M'$, M_i is M with $(c_i \to d_i) \to b_i$ deleted, all formulas of M' are either propositional variables or formulas of the form $p \to b$, where p is a propositional variable not occurring in M' and N is a set of propositional variables disjoint from M'.

The derivable sequents of the calculus CRIP are exactly the nonderivable sequents of LJ, hence the same as the nonderivable sequents of LS. Thus in order to show that all nonderivable sequents of LS are refutable we have to show

Proposition. *All sequents derivable by CRIP are refutable.*

Proof. If a sequent s is an axiom of CRIP of the form $p_1 \to b_1, \dots, p_n \to b_n, M \Rightarrow N$, where M consists of atomic formulas, then $p_1 \to b_1, \dots, p_n \to b_n, M \Rightarrow N, p_1, \dots, p_n$ is refutable and the sequent s results from this sequent by at most n applications of condition b)*iii*).

For the cases corresponding to applications of the rules (1) to (8) we use the following

Lemma 1. *a) If a sequent $M, a, b \Rightarrow N$ is refutable, then so is the sequent $M, a, b, a \wedge b \Rightarrow N$.*

b) If a sequent $M, a \Rightarrow N$ or a sequent $M, b \Rightarrow N$ is refutable, then so is the sequent $M, a, a \vee b \Rightarrow N$ resp. $M, b, a \vee b \Rightarrow N$.

c) If a sequent $M \Rightarrow a, b, N$ is refutable, then so is the sequent $M \Rightarrow a, b, a \vee b, N$.

d) If a sequent $M \Rightarrow a, N$ or a sequent $M \Rightarrow b, N$ is refutable, then so is the sequent $M \Rightarrow a, a \wedge bN$ resp. $M \Rightarrow b, a \wedge b$.

e) If a sequent $M, b \Rightarrow N$ is refutable, then so is the sequent $M, b, a \to b \Rightarrow N$.

f) If a sequent $M, a \Rightarrow b, N$ is refutable, then so is the sequent $M, a \Rightarrow b, a \to b, N$.

Proof. All claims are proved by straightforward inductions on the definition of refutability. In all cases the only relevant clauses are conditions a): If the original sequents satisfied condition a), then the transformed sequents satisfy a), too. Conditions b) and c) may be directly reduced to the relevant induction hypotheses. □

Now if a premiss of one of the rules (1) to (6) is refutable, then the sequent obtained from it by introducing the corresponding principal formula is refutable, too and an application of condition b)*i*) or b)*ii*) shows that the conclusion is refutable, too.

If, however, a premiss of a rule (7) or (8) is refutable, then the principal formula $p \to b$ resp. $(c \to d) \to b$ may be introduced and an application of condition b)*iii*) yields the refutability of the conclusion.

The cases corresponding to applications of the rules (9) and (10) are covered by parts a) and b) of the following

Lemma 2. *a) If a sequent $M, c \to b, d \to b \Rightarrow N$ is refutable, then so is the sequent $M, (c \vee d) \to b \Rightarrow N$.*

b) If a sequent $M, c \to (d \to b) \Rightarrow N$ is refutable, then so is the sequent $M, (c \wedge d) \to b \Rightarrow N$.

c) If a sequent $M, c, d \to b$ is refutable, then so is the sequent $M, (c \to d) \to b$.

Proof. a) If $M, c \to b, d \to b \Rightarrow N$ is refutable according to condition a) and M contains b, then $M, (c \vee d) \to b \Rightarrow N$ is refutable according to the same condition. If, however, N contains both c and d, then $M, c \to b, d \to b \Rightarrow c \vee d, N$ is refutable according to a) and so is $M, (c \vee d) \to b \Rightarrow c \vee d, N$, but from the latter sequent the required sequent $M, (c \vee d) \to b \Rightarrow N$ results by an application of condition b)*iii*).

For the induction step we use the obvious

Lemma 3. *If a sequent $M, a \to b \Rightarrow N$ is refutable and the sequent $M, a \to b, b \Rightarrow N$ is not refutable, then the sequent $M, a \to b \Rightarrow a, N$ is refutable. Moreover the refutation of the latter sequent is not longer than the refutation of the former one.*

Thus if $M, c \to b, d \to b \Rightarrow N$ is refutable according to condition b), and $M, c \to b, d \to b, b \Rightarrow N$ is not refutable, then $M, c \to b, d \to b \Rightarrow c, d, N$ is refutable and the induction hypothesis is applicable to this sequent, resulting in a refutable sequent $M, (c \vee d) \to b \Rightarrow c, d, N$. From this sequent we obtain the refutable sequent $M, (c \vee d) \to b \Rightarrow c, d, c \vee d, N$ by an application of lemma 1 and from this sequent we obtain the required sequent $M, (c \vee d) \to b \Rightarrow N$ by applications of b)*ii*) and b)*iii*). If, however $M, c \to b, d \to b, b \Rightarrow N$ is refutable, then by the induction hypothesis the sequent $M, (c \vee d) \rightarrow b, b \Rightarrow N$ is refutable and thus the required sequent $M, (c \vee d) \rightarrow b, b \Rightarrow N$ is refutable by an application of b)ii).

If refutability of the sequent $M, c \rightarrow b, d \rightarrow b \Rightarrow N$ results by an application of condition c), then either M contains the formula b or N contains both formulas c and d. In the former case the induction hypothesis applies to all refutable sequents $M, c \rightarrow b, d \rightarrow b, a_i \Rightarrow b_i$ resulting in refutable sequents $M, (c \vee d) \rightarrow b, a_i \Rightarrow b_i$ and from these sequents we obtain the required sequent by an application of condition c). In the latter case the formulas c and d disappear from the refutable sequents $M, c \rightarrow b, d \rightarrow b \Rightarrow a_i$ and applying condition c) again to the transformed sequents $M, (c \vee d) \rightarrow b, a_i \Rightarrow b_i$ we may introduce besides N also the formula $c \vee d$. From the resulting sequent we may then obtain the required sequent $M, (c \vee d) \rightarrow b \Rightarrow N$ by an application of b)iii).

b) If $M, c \rightarrow (d \rightarrow b) \Rightarrow N$ is refutable by condition a) and N contains c, then $M, c \rightarrow (d \rightarrow b) \Rightarrow c \wedge d, N$ is refutable and therefore both $M, (c \wedge d) \rightarrow b \Rightarrow c \wedge d, N$ and $M, (c \wedge d) \rightarrow b \Rightarrow N$ are refutable. If, however, M contains $d \rightarrow b$, then either N contains d or M contains b. The former case is symmetric to the case just mentioned, while in the latter case the required sequent $M, (c \wedge d) \rightarrow b \Rightarrow N$ is itself refutable by condition a).

If $M, c \rightarrow (d \rightarrow b) \Rightarrow N$ is refutable by condition b), then either $M, c \rightarrow (d \rightarrow b), d \rightarrow b \Rightarrow N$ or $M, c \rightarrow (d \rightarrow b) \Rightarrow c, N$ is refutable. In the latter case refutations of $M, (c \wedge d) \rightarrow b \Rightarrow c, N$ and $M, (c \wedge d) \rightarrow b \Rightarrow c, c \wedge d, N$ and $M, (c \wedge d) \rightarrow b \Rightarrow c \wedge d, N$ and $M, (c \wedge d) \rightarrow b \Rightarrow N$ result from applications of the induction hypothesis, lemma 1, condition b)ii) and condition b)iii) respectively. In the former case either $M, c \rightarrow (d \rightarrow b), d \rightarrow b, b \Rightarrow N$ is refutable and therefore $M, (c \wedge d) \rightarrow b), d \rightarrow b, b \Rightarrow N$ is refutable by the induction hypothesis and $M, (c \wedge d) \rightarrow b) \Rightarrow N$ is refutable by condition b)iii) or $M, c \rightarrow (d \rightarrow b), d \rightarrow b \Rightarrow d, N$ is refutable. Then from the sequent $M, c \rightarrow (d \rightarrow b) \Rightarrow d, N$ as before by applications of the induction hypothesis, lemma 1, condition b)ii) and condition b)iii) we obtain the required sequent $M, (c \wedge d) \rightarrow b \Rightarrow N$. Finally if our sequent results from an application of condition c), then we may apply the induction hypothesis to a sequent $M, c \rightarrow (d \rightarrow b), b, a_i \Rightarrow b_i$ resp. one of the sequents $M, c \rightarrow (d \rightarrow b), a_i \Rightarrow b_i$. In the former case the sequent $M, (c \wedge d) \rightarrow b) \Rightarrow N$ results by the same application of condition c). In the latter case the formula $c \wedge d$ may be added to the formulas of N and the formula c resp. d in the sequent obtained by applying condition c). This sequent, then, is the required sequent $M, (c \wedge d) \rightarrow b) \Rightarrow N$.

c) If $M, c, d \rightarrow b \Rightarrow N$ is refutable by condition a), then either M contains b and $M, c, (c \rightarrow d) \rightarrow b \Rightarrow N$ is refutable by condition a) or N contains d and $M, c, d \rightarrow b \Rightarrow c \rightarrow d, N$ and $M, c, (c \rightarrow d) \rightarrow b \Rightarrow c \rightarrow d, N$ are refutable by condition a) and from the latter sequent the required sequent $M, c, (c \rightarrow d) \rightarrow b \Rightarrow N$ results by an application of b)iii).

If $M, c, d \rightarrow b \Rightarrow N$ results by an application of condition b)iii) from a refutable sequent $M, c, d \rightarrow b \Rightarrow d, N$, then we may apply the induction hypothesis to this sequent and according to lemma 1 we may add the formula

$c \to d$ to the right hand side of the resulting sequent obtaining the sequent $M, c, (c \to d) \to b \Rightarrow d, c \to d, N$ and from this sequent by an application of condition b)*iv*) and an application of condition b)*iii*) we arrive at the required sequent $M, c, (c \to d) \to b \Rightarrow N$.

If $M, c, d \to b \Rightarrow N$ results by an application of condition b)*iii*) from a refutable sequent $M, c, d \to b, b \Rightarrow N$, then we may apply the induction hypothesis to this sequent and from the resulting sequent we may obtain the sequent $M, c, (c \to d) \to b \Rightarrow N$ by an application of b)*iii*).

If our sequent $M, c, d \to b \Rightarrow N$ results from a sequent s' by a different application of condition b), then we may apply the induction hypothesis to s' and the same application of b) leads to the required transformed sequent.

If $M, c, d \to b \Rightarrow N$ results from n sequents $M, c, d \to b, a_i \Rightarrow b_i$ by an application of condition c), then N contains the formula d and we may apply the induction hypothesis to all these sequents and again apply the condition c) with the modification that we also introduce the formula $c \to d$ on the right hand side to obtain the sequent $M, c, (c \to d) \to b \Rightarrow c \to d, N$. (This is possible because the left hand side of this sequent contains the formula c and the right hand side contains d.) From this sequent by an application of b)*iii*) we obtain the required sequent $M, c, (c \to d) \to b \Rightarrow N$. □

Now the only remaining rule to consider is rule (11): If all sequents $d_i, c_i \to b_i, M_i \Rightarrow d_i$ and all sequents $M, e_j \Rightarrow f_j$ are refutable and $p_1, \ldots, p_t$ are all propositional variables for which M' contains a formula $p_k \to g$, then by the preceding lemma the sequents $d_i, (c_i \to d_i) \to b_i, M_i \Rightarrow d_i$ are refutable and from these sequents and the sequents $M, e_j \Rightarrow f_j$ by an application of condition c) we may arrive at the sequent $M \Rightarrow c_1 \to d_1, \ldots, c_m \to d_m, e_1 \to f_1, \ldots, e_n \to f_n, p_1, \ldots, p_t, N$, because none of the p_k occurs in M. From this sequent by $m+t$ applications of b)*iii*) we arrive at the required sequent $M \Rightarrow e_1 \to f_1, \ldots, e_n \to f_n, N$. □

Having thus established that all nonderivable sequents are refutable we may associate with every nonderivable sequent s a so called *canonical countermodel* $C(s)$ whose underlying tree is the same as the underlying tree of the refutation tree of s and whose assignment function stipulates a propositional variable to be true at a given node of this tree, iff the propositional variable occurs on the left hand side of the corresponding sequent in the refutation tree. Then we show

Theorem. *If a sequent s is not a theorem of intuitionistic propositional logic, then s is evaluated to false in the canonical countermodel $C(s)$ of s.*

Proof. s is refutable, so we may use induction on the definition of refutability:

If s is refutable according to condition a), then the theorem obviously holds. If it is refutable according to condition b) and by virtue of refutability of some other sequent s', then $C(s)$ equals $C(s')$ and therefore by the induction hypothesis s' is evaluated to false in $C(s)$, but s' contains s, hence s is evaluated to false, too. If, finally, s is found to be refutable according to condition c), then a straightforward induction on the complexity of formulas of s shows that all formulas on the left hand side of s are evaluated to true and all formulas on the right hand side of s are evaluated to false. The only significant case we have to consider is the case of implications on the right hand side of s: If $a \to b$ occurs on the right hand side of s and a occurs on its left hand side and b on its right hand side, then by the induction hypothesis a is evaluated to true and b is evaluated to false and hence $a \to b$ is evaluated to false. If, however, either a does not occur on the left hand side or b does not occur on its right hand side, then we have a refutable sequent s' obtained from s by adding a to the left hand side and replacing the right hand side by b. Then by the induction hypothesis a is true and b is false in the submodel $C(s')$ of $C(s)$ and therefore $a \to b$ is false in $C(s)$. □

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