A Ordinal Numbers

We give a short introduction to ordinal numbers. We largely follow the approach presented in [1]. Firstly, we have to define the notion of *well-ordering*.

Definition 28. A binary relation $<$ of a set A is called a well-ordering if the following hold:

1. $a \not< a$ for all $a \in A$
2. $a < b \;\&\; b < c \;\Rightarrow\; a < c$
3. $a < b$ or $a = b$ or $b < a$ for all $a, b \in A$
4. Every nonempty subset of A has a least element.

Definition 29. A set A is transitive if every element of A is a also a subset of A.

Definition 30. A set α is called an ordinal number if α is transitive and $\langle \alpha, \in \rangle$ is a well-ordering.

Example 2. Before we proceed we should observe some important ordinal numbers.

1. The empty set $\emptyset$ is an ordinal number. It is also denoted by 0.
2. The sets given by $1 := \{0\}$, $2 := \{0, 1\}$, $3 := \{0, 1, 2\}$,... are ordinal numbers.
3. If α is an ordinal number, then $\alpha + 1 := \alpha \cup \{\alpha\}$ is also an ordinal number. $\alpha + 1$ is called the *successor ordinal of* α. An ordinal that is not a successor is called a *limit ordinal*.
4. The *Axiom of Infinity*[1] ensures the existence of *the least inductive set* ω given by the set $\bigcap\{Y;\ 0 \in Y \;\&\; \forall x(x \in Y \Rightarrow x \cup \{x\} \in Y)\}$. ω is an ordinal number and the elements of ω are called *natural numbers* (i.e., ω is the *set of natural numbers*).
5. An ordinal α is a *cardinal number* if there is no $\beta \in \alpha$ such that there is a one-to-one mapping of β onto α. All natural numbers are cardinal numbers and the set of natural numbers is a cardinal number. In the context of cardinal numbers ω is usually denoted by $\aleph_0$.

Let $Y_1, ..., Y_n$ be sets and let $\phi(X, Y_1, ..., Y_n)$ be a set-theoretical property. The class of all sets X satisfying the property $\phi(X, Y_1, ..., Y_n)$ is denoted by $\{X;\ \phi(X, Y_1, ..., Y_n)\}$ and depends on the parameters Y_i. Two classes are equal iff they contain the same elements. If we assume that the axioms of set theory are consistent, then there are classes, such as the *class of all sets* V and the *class of all ordinals* Ω, that are no sets. We call classes that are sets *comprehension terms* (i.e., sets that are given by a set-theoretical property). For example, $A \cap B = \{x;\ x \in A \;\&\; x \in B\}$ is a set for all sets A and B. One can prove that the class $\{\alpha;\ \alpha \in \Omega$ such that there is an one-to-one mapping of α into $\omega\}$ is a cardinal number. This class is usually denoted by $\aleph_1$ and there is no cardinal κ such that $\aleph_0 \in \kappa \in \aleph_1$. Before we proceed we will give some facts about the class Ω. Notice that it is common to use $<$ instead of $\in$.

Lemma 10 (Properties of Ω). *Notice that it is common to use $<$ instead of $\in$.*

1. *$\langle \Omega, < \rangle$ satisfies statement 1., 2., and 3. of Definition 28.*
2. *For all ordinals α the equation $\alpha = \{\beta;\ \beta < \alpha\}$ holds.*
3. *Every nonempty class C of ordinals has a least element $\alpha \in C$. The equation $\bigcap C = \alpha$ holds.*
4. *If X is a set of ordinals, then $\bigcup X$ is also an ordinal. Moreover, $\bigcup X$ is the least upper bound of X.*
5. *The successor $\alpha + 1$ of an ordinal α is the least ordinal of the class $\{\beta \in \Omega;\ \alpha < \beta\}$.*

Now we are able to introduce two important methods used in this paper. On the one hand *Transfinite Induction* and on the other hand *Transfinite Recursion*.

Theorem 8 (Transfinite Induction). *Let C be an arbitrary class of ordinals such that for all ordinals α the following properties hold:*

1. $0 \in C$
2. $\alpha \in C \;\Rightarrow\; \alpha + 1 \in C$

[1] An axiom of Zermelo-Fraenkel.

3. $0 \in \alpha$ *limit ordinal* & $\forall \beta \in \alpha : \beta \in C \quad \Rightarrow \quad \alpha \in C$

Then C *is equal to the class of all ordinals* Ω*.*

Theorem 9 (Transfinite Recursion). *Let* $Y_1, .., Y_n$ *be arbitrary but fixed sets. Moreover, let* $t(X, Y_1, ..., Y_n)$ *be a comprehension term for any set* X*. Then there is a unique class*[2] F*, such that* F *is a function on* Ω *and* $F_\alpha = t((F_\beta)_{\beta \in \alpha}, Y_1, ..., Y_n)$ *for every ordinal* α*.*

Remark 6. The proof of the theorem provides a formula $\phi(X, Y_1, ..., Y_n)$ such that the class F given by $\{X;\ \phi(X, Y_1, ..., Y_n)\}$ is a function on Ω and $F_\alpha = t((F_\beta)_{\beta \in \alpha}, Y_1, ..., Y_n)$ for every ordinal α. Then, using Transfinite Induction, one can prove that F is unique.

Example 3 (Addition). The function $\alpha + \cdot : \Omega \to \Omega$ is defined to be the unique (class) function satisfying the following equation for every $\beta \in \Omega$:

$$\alpha + \beta = \begin{cases} \alpha, & \text{if } \beta = 0 \\ (\alpha + \gamma) + 1, & \text{if } \beta = \gamma + 1 \ \&\ \gamma \in \Omega \\ \bigcup_{\gamma < \beta} (\alpha + \gamma), & \text{if } \beta \text{ limit ordinal } > 0 \end{cases}$$

Notice that the above case distinction can be described by a comprehension term of the form $t(\,(\alpha + \zeta)_{\zeta \in \beta},\ \alpha)$.

Example 4 (Multiplication). The function $\alpha \bullet \cdot : \Omega \to \Omega$ is defined to be the unique (class) function satisfying the following equation for every $\beta \in \Omega$:

$$\alpha \bullet \beta = \begin{cases} 0, & \text{if } \beta = 0 \\ (\alpha \bullet \gamma) + \alpha, & \text{if } \beta = \gamma + 1 \ \&\ \gamma \in \Omega \\ \bigcup_{\gamma < \beta} (\alpha \bullet \gamma), & \text{if } \beta \text{ limit ordinal } > 0 \end{cases}$$

B Omitted Proof

Lemma 5 *Under the same conditions as in Theorem 3 for every formula* $\phi \in$ Form *and every assignment* h *the following holds:*

$$[\![\phi]\!]_h^{I_\infty} = \mathcal{T}_\alpha \quad \Rightarrow \quad [\![\phi]\!]_h^{I_i} = \mathcal{T}_\alpha \textit{ for some } i \in \aleph_1$$

Proof. We prove this by induction on the structure of ϕ. We use $I_H(\psi)$ as an abbreviation of

$$[\![\psi]\!]_h^{I_\infty} = \mathcal{T}_\alpha \quad \Rightarrow \quad [\![\psi]\!]_h^{I_i} = \mathcal{T}_\alpha \text{ for some } i \in \aleph_1.$$

Case 1: ϕ is an atomic formula. Then, using Definition 15 and Definition 20, the statement of the lemma is obviously true.
Case 2: $\phi = \psi_1 \vee \psi_2$ and both $I_H(\psi_1)$ and $I_H(\psi_2)$ holds. Let us assume that $[\![F]\!]_h^{I_\infty} = \mathcal{T}_\alpha$ holds. Moreover, we assume, without loss of generality, that $[\![\psi_2]\!]_h^{I_\infty} \leq [\![\psi_1]\!]_h^{I_\infty}$ holds. This implies $[\![\psi_1]\!]_h^{I_\infty} = \mathcal{T}_\alpha$ and $[\![\psi_2]\!]_h^{I_\infty} \leq \mathcal{T}_\alpha$. Then, using $I_H(\psi_1)$, there is an $i_0 \in \aleph_1$ such that $[\![\psi_1]\!]_h^{I_{i_0}} = \mathcal{T}_\alpha$. This implies $[\![\psi_2]\!]_h^{I_{i_0}} \leq \mathcal{T}_\alpha$. (Since otherwise, using Lemma 4 and Theorem 1, we would get $\mathcal{T}_\alpha < [\![\psi_2]\!]_h^{I_\infty}$ and this contradicts the assumption.) Hence we get that $[\![\phi]\!]_h^{I_{i_0}} = \mathcal{T}_\alpha$ holds.
Case 3: $\phi = \neg(\psi)$. Let us assume that $[\![F]\!]_h^{I_\infty} = \mathcal{T}_\alpha$ holds. This obviously implies $[\![\psi]\!]_h^{I_\infty} = F_{\alpha-1}$. Hence, using Theorem 1, we get that $[\![\psi]\!]_h^{I} = \mathcal{F}_{\alpha-1}$ and $[\![\phi]\!]_h^{I} = \mathcal{T}_\alpha$ hold.
Case 4: $\phi = \psi_1 \wedge \psi_2$ and both $I_H(\psi_1)$ and $I_H(\psi_2)$ holds. Let us assume that $[\![\phi]\!]_h^{I_\infty} = \mathcal{T}_\alpha$ holds. Moreover, we assume, without loss of generality, that $[\![\psi_1]\!]_h^{I_\infty} \leq [\![\psi_2]\!]_h^{I_\infty}$ holds. This implies $[\![\psi_1]\!]_h^{I_\infty} = \mathcal{T}_\alpha$ and $\mathcal{T}_\alpha \leq [\![\psi_2]\!]_h^{I_\infty}$. Then, using $I_H(\psi_1)$, there is an $i_0 \in \aleph_1$ such that $[\![\psi_1]\!]_h^{I_{i_0}} = \mathcal{T}_\alpha$.

[2] The class depends on the sets $Y_1, ..., Y_n$.

Firstly, we assume that $\mathcal{T}_\alpha = [\![\psi_2]\!]_h^{I_\infty}$ holds. Hence, using $I_H(\psi_2)$, we get that there is a $j_0 \in \aleph_1$ such that $\mathcal{T}_\alpha = [\![\psi_2]\!]_h^{I_{j_0}}$. This implies, together with Lemma 4 and Theorem 1, that $[\![\psi_1]\!]_h^{I_{max(i_0,j_0)}} = \mathcal{T}_\alpha = [\![\psi_1]\!]_h^{I_{max(i_0,j_0)}}$, and hence $[\![\phi]\!]_h^{I_{max(i_0,j_0)}} = \mathcal{T}_\alpha$. Secondly, let us consider the case $\mathcal{T}_\alpha < [\![\psi_2]\!]_h^{I_\infty}$. Then, using Lemma 4 and Theorem 1, we get that $\mathcal{T}_\alpha < [\![\psi_2]\!]_h^{I_{i_0}}$. This implies that $[\![\phi]\!]_h^{I_{i_0}} = \mathcal{T}_\alpha$ holds.
Case 5: $\phi = \forall v(\psi)$ and $I_H(\psi)$. Let us assume that $[\![\phi]\!]_h^{I_\infty} = \mathcal{T}_\alpha$. This obviously implies that

$$inf\left\{[\![\psi_1]\!]_{h[v\mapsto u]}^{I_\infty};\ u \in HU\right\} = \mathcal{T}_\alpha. \tag{15}$$

Hence there is a partition $H_U = H_{U_1} \dot{\cup} H_{U_2}$ such that $H_{U_1} = \left\{u \in H_U;\ \mathcal{T}_\alpha < [\![\psi]\!]_{h[v\mapsto u]}^{I_\infty}\right\}$ and $H_{U_2} = \left\{u \in H_U;\ \mathcal{T}_\alpha = [\![\psi]\!]_{h[v\mapsto u]}^{I_\infty}\right\}$. Then, using Lemma 4 and Theorem 1, we get that

$$\forall u \in H_{U_1} : \forall i < \aleph_1 : [\![\psi]\!]_{h[v\mapsto u]}^{I_\infty} = [\![\psi]\!]_{h[v\mapsto u]}^{I_i}. \tag{16}$$

$I_H(\psi)$ implies that for all $u \in H_{U_2}$ there is an $i \in \aleph_1$ such that $\mathcal{T}_\alpha = [\![\psi]\!]_{h[v\mapsto u]}^{I_i}$. This justifies the following definition:

$$\zeta : H_{U_2} \to \aleph_1 : u \mapsto \min\{i \in \aleph_1; [\![\psi]\!]_{h[v\mapsto u]}^{I_i} = \mathcal{T}_\alpha\}$$

Theorem 2 implies that the countable image $\zeta(H_{U_2})$ cannot be cofinal in $\aleph_1$. Hence, there is an $i_0 \in \aleph_1$ such that for all $u \in H_{U_2}$ the property $\zeta(u) \leq i_0$ holds. Using again Lemma 4 and Theorem 1 we get that

$$\forall u \in H_{U_2} : [\![\psi]\!]_{h[v\mapsto u]}^{I_{i_0}} = \mathcal{T}_\alpha = [\![\psi]\!]_{h[v\mapsto u]}^{I_\infty}.$$

Hence, using (15) and (16), we get that

$$[\![\phi]\!]_h^{I_{i_0}} = \inf\{[\![\psi]\!]_{h[v\mapsto u]}^{I_{i_0}};\ u \in H_U\} = \inf\{[\![\psi]\!]_{h[v\mapsto u]}^{I_\infty};\ u \in H_U\} = \mathcal{T}_\alpha.$$

Case 6: $\phi = \exists v(\psi)$ and $I_H(\psi)$ holds. Let us assume that $[\![\phi]\!]_h^{I_\infty} = \mathcal{T}_\alpha$. This implies the following:

$$\sup\left\{[\![\psi]\!]_{h[v\mapsto u]}^{I_\infty};\ u \in H_U\right\} = \mathcal{T}_\alpha \tag{17}$$

Hence, using Lemma 1, we get that there is an $u_0 \in H_U$ such that $[\![\psi]\!]_{h[v\mapsto u_0]}^{I_\infty} = \mathcal{T}_\alpha$. $I_H(\psi)$ implies that there is an $i_0 < \aleph_1$ such that $[\![\psi]\!]_{h[v\mapsto u_0]}^{I_{i_0}} = \mathcal{T}_\alpha$. Finally, using Lemma 4, Theorem 1, and (17), we get that $\forall u \in H_U : [\![\psi]\!]_{h[v\mapsto u]}^{I_{i_0}} \leq \mathcal{T}_\alpha = [\![\psi]\!]_{h[v\mapsto u_0]}^{I_{i_0}}$, and thus $[\![\phi]\!]_h^{I_{i_0}} = \mathcal{T}_\alpha$ holds. □

B.1 Example

In this subsection we will consider a formula-based logic program P_A from the area of arithmetic. The underlaying language contains a constant symbol $\dot{\mathbf{0}}$, a function symbol **S** of arity 1, a predicate symbol **add** of arity 3, a predicate symbol **multiple** of arity 2, a predicate symbol **smaller** of arity 2, a predicate symbol **prime** of arity 1, and a predicate symbol **primesucc** of arity 2. For each natural number n we use $\dot{\mathbf{n}}$ as an abbreviation of $\mathbf{S^n(\dot{0})}$. Moreover, $(\phi \Rightarrow \psi)$ is an abbreviation of the formula $(\neg\phi \vee \psi)$. The program P_A is given by the following rules:

(R1) $\mathbf{add(x_0,\dot{0},x_0)} \leftarrow$
(R2) $\mathbf{add(x_0,S(x_1),S(x_2))} \leftarrow \mathbf{add(x_0,x_1,x_2)}$
(R3) $\mathbf{multiple(\dot{0},x_0)} \leftarrow$
(R4) $\mathbf{multiple(x_2,x_0)} \leftarrow \exists \mathbf{x_1}(\mathbf{multiple(x_1,x_0)} \wedge \mathbf{add(x_1,x_0,x_2)})$
(R5) $\mathbf{smaller(x_0,S(x_0))} \leftarrow$
(R6) $\mathbf{smaller(x_0,S(x_1))} \leftarrow \mathbf{smaller(x_0,x_1)}$
(R7) $\mathbf{prime(x_0)} \leftarrow \mathbf{smaller(S(\dot{0}),x_0)} \wedge$
$\wedge \forall \mathbf{x_1}((\mathbf{smaller(S(\dot{0}),x_1)} \wedge \mathbf{smaller(x_1,x_0)}) \Rightarrow \neg\mathbf{multiple(x_0,x_1)})$

(R8) $\mathbf{primesucc(x_1,x_0)} \leftarrow \mathbf{smaller(x_0,x_1)} \wedge \mathbf{prime(x_0)} \wedge \mathbf{prime(x_1)} \wedge$
$\wedge \forall \mathbf{x_2}((\mathbf{smaller(x_0,x_2)} \wedge \mathbf{smaller(x_2,x_1)}) \Rightarrow \neg \mathbf{prime(x_2)})$

Remark 7. The predicates of the program can be understood as follows:

- $\mathbf{add}(x,y,z)$ "$x + y = z$"
- $\mathbf{multiple}(x,y)$ "x is a multiple of y"
- $\mathbf{smaller}(x,y)$ "$x < y$"
- $\mathbf{prime}(x)$ "x is a prime number"
- $\mathbf{primesucc}(y,x)$ "y is the prime successor to the prime number x"

Proposition 7. *The first approximant M_0 of the program P_A is as described in Figure 1.*

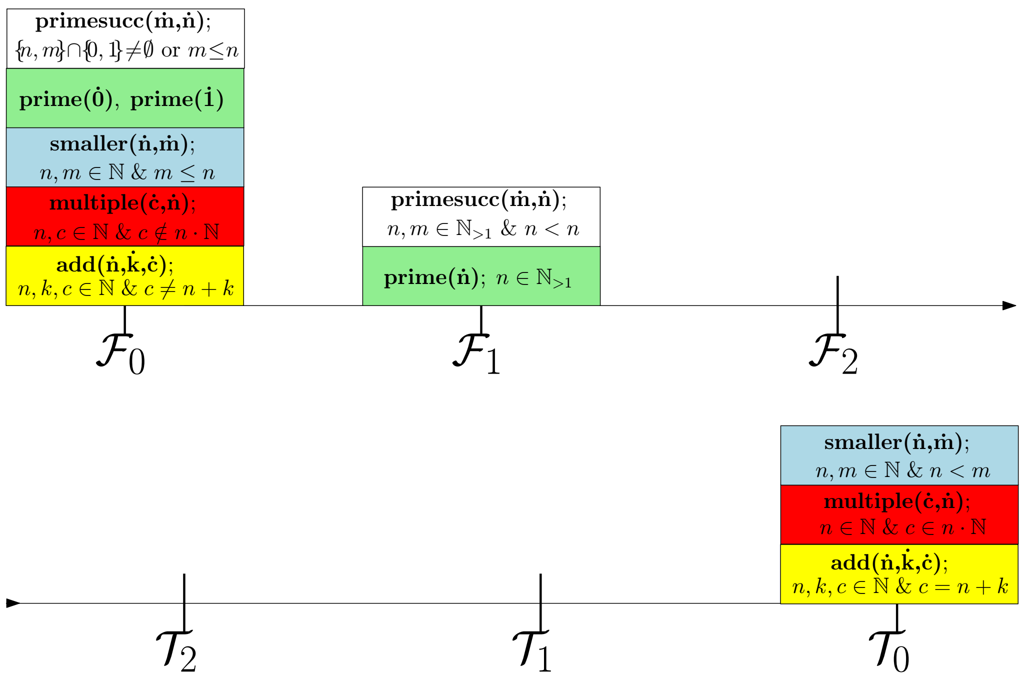

Fig. 1. The first approximant M_0

Proof. As a first approximation we calculate the approximant M_0 from the interpretation $\emptyset$ that maps everything to the least value $\mathcal{F}_0$. By induction on the natural number $m \in \mathbb{N}$ it is easy to prove that the following statements must hold true:

$$n \in \mathbb{N} \,\&\, 0 \leq k < m \;\Rightarrow\; \mathbf{add}(\dot{\mathbf{n}},\dot{\mathbf{k}},\mathbf{n} \dotplus \mathbf{k}) \in T^m_{P_A}(\emptyset)\|\mathcal{T}_0 \tag{18}$$

$$(a,b,c) \in \mathbb{N}^3 \,\&\, (m \leq b \text{ or } c \neq a+b) \;\Rightarrow\; \mathbf{add}(\dot{\mathbf{a}},\dot{\mathbf{b}},\dot{\mathbf{c}}) \in T^m_{P_A}(\emptyset)\|\mathcal{F}_0 \tag{19}$$

$$n \in \mathbb{N} \,\&\, 0 \leq k < m \;\Rightarrow\; \mathbf{multiple}(\dot{\mathbf{kn}},\dot{\mathbf{n}}) \in T^m_{P_A}(\emptyset)\|\mathcal{T}_0 \tag{20}$$

$$(b,a) \in \mathbb{N}^2 \,\&\, \; b \neq ka \; (\forall k : 0 \leq k < m) \Rightarrow \mathbf{multiple}(\dot{\mathbf{b}},\dot{\mathbf{a}}) \in T^m_{P_A}(\emptyset)\|\mathcal{F}_0 \tag{21}$$

$$n \in \mathbb{N} \,\&\, 0 < k \leq m \;\Rightarrow \mathbf{smaller}(\dot{\mathbf{n}},\mathbf{n} \dotplus \mathbf{k}) \in T^m_{P_A}(\emptyset)\|\mathcal{T}_0 \tag{22}$$

$$(a,b) \in \mathbb{N}^2 \,\&\, b \neq a+k \; (\forall k : 0 < k \leq m) \Rightarrow \mathbf{smaller}(\dot{\mathbf{a}},\dot{\mathbf{b}}) \in T^m_{P_A}(\emptyset)\|\mathcal{F}_0 \tag{23}$$

The above statements (18) and (19), together with Definition 20, imply that the following must hold:

$$n, k \in \mathbb{N} \;\Rightarrow\; \mathbf{add(\dot{n},\dot{k},\dot{\overline{n + k}})} \in T^{\omega}_{P_A,0}(\emptyset)\|\mathcal{T}_0 \tag{24}$$

$$(a, b, c) \in \mathbb{N}^3 \;\&\; c \neq a + b \;\Rightarrow\; \mathbf{add(\dot{a},\dot{b},\dot{c})} \in T^{\omega}_{P_A,0}(\emptyset)\|\mathcal{F}_0 \tag{25}$$

The above statements (20) and (21), together with Definition 20, imply that the following must hold:

$$n, k \in \mathbb{N} \;\Rightarrow\; \mathbf{multiple(\dot{\overline{kn}},\dot{n})} \in T^{\omega}_{P_A,0}(\emptyset)\|\mathcal{T}_0 \tag{26}$$

$$(b, a) \in \mathbb{N}^2 \;\&\; b \notin \mathbb{N}a \;\Rightarrow \mathbf{multiple(\dot{b},\dot{a})} \in T^{\omega}_{P_A,0}(\emptyset)\|\mathcal{F}_0 \tag{27}$$

The above statements (22) and (23), together with Definition 20, imply that the following must hold:

$$n \in \mathbb{N} \;\&\; 0 < k \;\Rightarrow \mathbf{smaller(\dot{n}, \dot{\overline{n + k}})} \in T^{\omega}_{P_A,0}(\emptyset)\|\mathcal{T}_0 \tag{28}$$

$$(a, b) \in \mathbb{N}^2 \;\&\; b \leq a \;\Rightarrow\; \mathbf{smaller(\dot{a},\dot{b})} \in T^{\omega}_{P_A,0}(\emptyset)\|\mathcal{F}_0 \tag{29}$$

It is easy to prove that $T^{\omega}_{P_A,0}(\emptyset)$ is a fixed point with respect to the ground atoms of the form $\mathbf{add}(\,\cdot\,,\cdot\,,\cdot\,)$, $\mathbf{multiple}(\,\cdot\,,\cdot\,)$, and $\mathbf{smaller}(\,\cdot\,,\cdot\,)$. Hence, using again Definition 20 and Definition 23, we get that the following hold, where $n, k, l \in \mathbb{N}$:

$$M_0(\,\mathbf{add(\dot{n},\dot{k},\dot{l})}\,) = \begin{cases} \mathcal{T}_0, & \text{if } n + k = l \\ \mathcal{F}_0, & \text{otherwise} \end{cases} \tag{30}$$

$$M_0(\,\mathbf{multiple(\dot{n},\dot{k})}\,) = \begin{cases} \mathcal{T}_0, & \text{if } n \in \mathbb{N}k \\ \mathcal{F}_0, & \text{otherwise} \end{cases} \tag{31}$$

$$M_0(\,\mathbf{smaller(\dot{n},\dot{k})}\,) = \begin{cases} \mathcal{T}_0, & \text{if } n < k \\ \mathcal{F}_0, & \text{otherwise} \end{cases} \tag{32}$$

The above statement (32) implies that

$$\mathbf{smaller(\dot{1},\dot{0})},\ \mathbf{smaller(\dot{1},\dot{1})} \in T^{\alpha}_{P_A,0}(\emptyset)\|\mathcal{F}_0$$

for each ordinal $\alpha \in \aleph_1$. Hence, using rule (R7), we get that

$$\mathbf{prime(\dot{0})},\ \mathbf{prime(\dot{1})} \in T^{\alpha}_{P_A,0}(\emptyset)\|\mathcal{F}_0$$

for every ordinal $\alpha \in \aleph_1$. This implies the following statement:

$$\mathbf{prime(\dot{0})},\ \mathbf{prime(\dot{1})} \in M_0\|\mathcal{F}_0 \tag{33}$$

Let us assume that n is a natural number such that $1 < n$. Definition 15 implies that for all interpretations I the value

$$[\![\forall \mathbf{x_1}((\mathbf{smaller(S(\dot{0}),x_1)} \wedge \mathbf{smaller(x_1,\dot{n})}) \Rightarrow \neg\mathbf{multiple(\dot{n},x_1)})]\!]^I$$

is an element of $[\mathcal{F}_1, \mathcal{T}_1]$. Thus, using (R7), we get that (for all $\alpha < \aleph_1$)

$$\mathbf{prime(\dot{n})} \notin T^{\alpha}_{P_A,0}(\emptyset)\|\mathcal{T}_0.$$

Statement (22) implies that $\mathbf{smaller(\dot{1},\dot{n})} \in T^{n-1}_{P_A,0}(\emptyset)\|\mathcal{T}_0$ and hence we get that $\mathbf{prime(\dot{n})} \notin T^{n}_{P_A,0}(\emptyset)\|\mathcal{F}_0$. This implies, together with Definition 20 and Definition 23, that

$$\text{for all } n \in \mathbb{N}_{>1}\text{: } \mathbf{prime(\dot{n})} \notin M_0\|\mathcal{F}_0 \;\&\; \mathbf{prime(\dot{n})} \notin M_0\|\mathcal{T}_0. \tag{34}$$

To complete the construction of the first approximant M_0 we have to consider the predicate **primesucc**. Let m, n be natural numbers such that $m \leq n$. Hence, using statement (32), we get that $\mathbf{smaller}(\dot{n}, \dot{m}) \in T^{\alpha}_{P_A,0}(\emptyset)\|\mathcal{F}_0$ for all $\alpha \in \aleph_1$. This implies for all $n, m \in \mathbb{N}$:

$$\text{if } m \leq n, \text{ then } \mathbf{primesucc(\dot{m},\dot{n})} \in M_0\|\mathcal{F}_0 \tag{35}$$

Statement (33) implies the following (for all $n, m \in \mathbb{N}$):

$$\text{if } \{n, m\} \cap \{0, 1\} \neq \emptyset, \text{ then } \mathbf{primesucc(\dot{m},\dot{n})} \in M_0\|\mathcal{F}_0 \tag{36}$$

Let $n, m \in \mathbb{N}$ be natural numbers such that $1 < n < m$. This, together with the statements (32) and (34) and Lemma 4, implies that there is an $\alpha \in \aleph_1$ such that the following four properties hold (for all ordinals $k \geq \alpha$):

$$\mathbf{smaller(\dot{n},\dot{m})} \notin T^{k}_{P_A,0}(\emptyset)\|\mathcal{F}_0$$

$$\mathbf{prime(\dot{n})} \notin T^{k}_{P_A,0}(\emptyset)\|\mathcal{F}_0$$

$$\mathbf{prime(\dot{m})} \notin T^{k}_{P_A,0}(\emptyset)\|\mathcal{F}_0$$

$$\mathcal{F}_0 < [\![\forall \mathbf{x_2}((\mathbf{smaller(\dot{n},x_2)} \wedge \mathbf{smaller(x_2,\dot{m})}) \Rightarrow \neg\mathbf{prime(x_2)})]\!]^{T^{k}_{P_A,0}(\emptyset)} < \mathcal{T}_0$$

These properties imply, together with (R8), that for all $n, m \in \mathbb{N}$ the following must hold:

$$1 < n < m \;\Rightarrow\; \mathbf{primesucc(\dot{m},\dot{n})} \notin M_0\|\mathcal{F}_0 \cup M_0\|\mathcal{T}_0 \tag{37}$$

The above results completely describe the first approximant M_0 of P_A.

Proposition 8. *The second approximant M_1 of the program P_A is as described in Figure 2.*

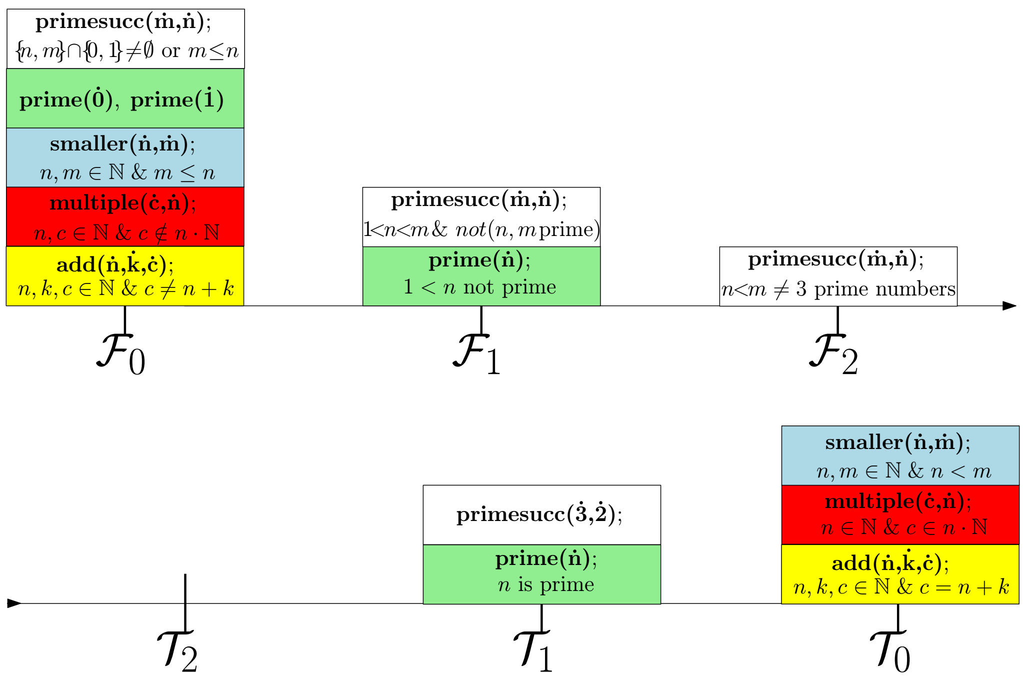

Fig. 2. The second approximant M_1

Proof. As a second approximation we calculate the approximant M_1 from the approximant M_0. The results (31), (32) and the body of the rule (R7) imply that for all natural numbers $n > 1$ and all $0 \leq \alpha \in \aleph_1$ the following holds:

$$\text{If } n \text{ is a prime number, then } \mathbf{prime(\dot{n})} \in T_{P_A}(M_0)\|\mathcal{T}_1.$$
$$\text{If } n \text{ is not prime number, then } \mathbf{prime(\dot{n})} \in T^{\alpha}_{P_A,1}(M_0)\|\mathcal{F}_1.$$

And this obviously implies

$$M_1(\,\mathbf{prime(\dot{n})}\,) = \begin{cases} \mathcal{T}_1, & \text{if } n \text{ is prime} \\ \mathcal{F}_1, & \text{if } 1 < n \text{ is not prime .} \\ \mathcal{F}_0, & \text{otherwise} \end{cases} \tag{38}$$

Let us assume that $1 < n < m$ are natural numbers such that n or m is not a prime number. Hence, using Statement (38) and rule (R8), we get that for all $\alpha \in \aleph_1$ the ground atom

$$\mathbf{primesucc(\dot{m}, \dot{n})} \text{ is an element of } T^{\alpha}_{P_A,1}(M_0)\|\mathcal{F}_1. \tag{39}$$

Let us assume that $n < m$ are prime numbers such that $n + 1 = m$. Obviously, this implies $n = 2$ and $m = 3$. The above statement implies that

$$\mathbf{prime}(\dot{2}), \mathbf{prime}(\dot{3}) \in T_{P_A}(M_0)\|\mathcal{T}_1.$$

Moreover, it is easy to prove that

$$[\![\forall z \left(\left(\mathbf{smaller}(\dot{2}, z) \wedge \mathbf{smaller}(z, \dot{3})\right) \Rightarrow \neg \mathbf{prime}(z)\right)]\!]_{T_{P_A}(M_0)} = \mathcal{T}_1.$$

This implies that

$$\mathbf{primesucc}(\dot{2}, \dot{3}) \in M_1\|\mathcal{T}_1. \tag{40}$$

Let us now assume that $n < m$ are prime numbers such that $n + 1 < m$. The above statement implies that

$$\mathbf{prime}(\dot{n}), \mathbf{prime}(\dot{m}) \in T^{1}_{P_A,1}(M_0)\|\mathcal{T}_1.$$

Moreover, it is easy to prove that

$$[\![\forall z \left(\left(\mathbf{smaller}(\dot{n}, z) \wedge \mathbf{smaller}(z, \dot{m})\right) \Rightarrow \neg \mathbf{prime}(z)\right)]\!]_{T^{\alpha}_{P_A,1}(M_0)} \in [\mathcal{F}_2, \mathcal{T}_2]$$

for all $\alpha \in \aleph_1$. This implies that both $\mathbf{primesucc}(\dot{n}, \dot{m}) \notin T^{2}_{P_A,1}(M_0)\|\mathcal{F}_1$ and $\mathbf{primesucc}(\dot{n}, \dot{m}) \notin T^{\alpha}_{P_A,1}(M_0)\|\mathcal{T}_1$ (for all $\alpha \in \aleph_1$) must hold true. Hence, using the above statements (39) and (40), we get that the following holds:

$$M_1(\,\mathbf{primesucc}(\dot{n}, \dot{m})\,) = \begin{cases} \mathcal{F}_1, & \text{if } 1 < n < m \text{ and } (n \text{ or } m \text{ not prime}) \\ \mathcal{T}_1, & n = 2 \text{ and } m = 3 \\ \mathcal{F}_2, & \text{if } m \neq 3 \text{ and } n < m \text{ prime} \end{cases}$$

The above results completely describe the second approximant M_1.

Proposition 9. *The least infinite-valued model $M^{\infty}_{P_A}$ of the program P_A is as described in Figure 3.*

Proof. As a third approximation we calculate the approximant M_2 from the approximant M_1. Statement (32), Statement (38), and the rule (R8) obviously imply the following (similar to the above argumentation):

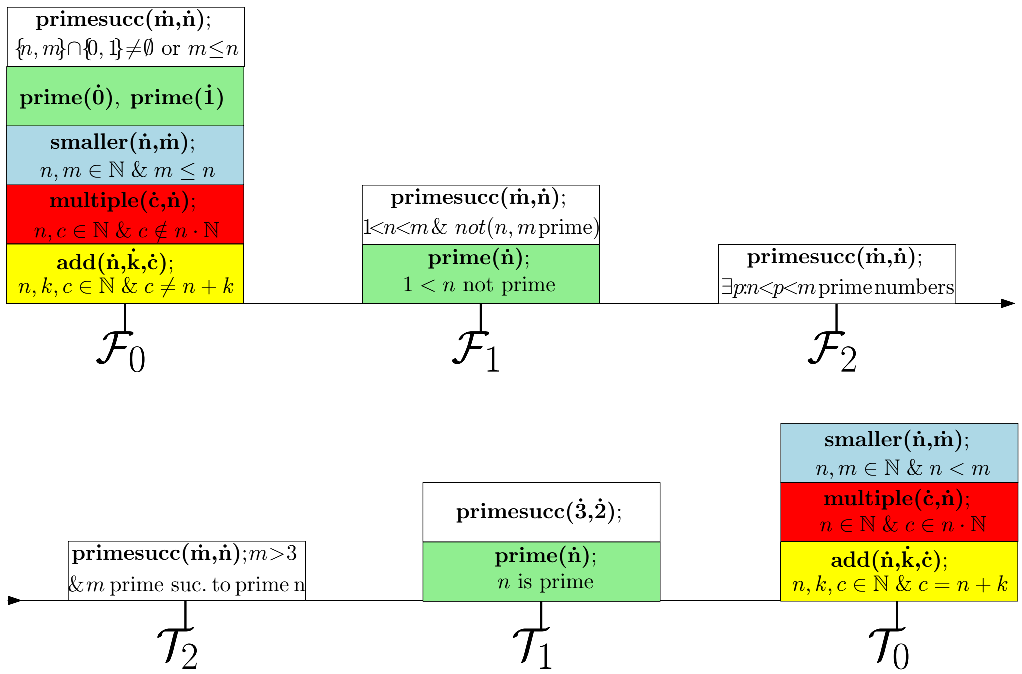

Fig. 3. The least infinite-valued model $M_{P_A}^{\infty}$

If $n < p < m$ prime numbers, then $\mathbf{primesucc(\dot{m},\dot{n})} \in M_2 \| \mathcal{F}_2$.
If $n < m \neq 3$ & m prime successor to n, then $\mathbf{primesucc(\dot{m},\dot{n})} \in M_2 \| \mathcal{T}_2$.

The above statement completely describes the third approximant M_2. Moreover, all ground atoms receive a value within $[\mathcal{F}_0, \mathcal{F}_2] \cup [\mathcal{T}_2, \mathcal{T}_0]$ (with respect to M_2). Hence we get that M_2 is equal to the least infinite-valued model $M_{P_A}^{\infty}$.

Corollary 2. *The depth δ_{P_A} of the program P_A is equal to 3. Moreover, the collapsed Model M_{P_A} is a classical 2-valued interpretation and the following statements hold true ($n, m, l \in \mathbb{N}$):*

1. $M_{P_A}(\,\mathbf{add(\dot{n},\dot{m},\dot{l})}\,) = \mathcal{T} \quad \Leftrightarrow n + m = l$
2. $M_{P_A}(\,\mathbf{multiple(\dot{n},\dot{m})}\,) = \mathcal{T} \quad \Leftrightarrow n \in \mathbb{N}m$
3. $M_{P_A}(\,\mathbf{smaller(\dot{n},\dot{m})}\,) = \mathcal{T} \quad \Leftrightarrow n < m$
4. $M_{P_A}(\,\mathbf{prime(\dot{n})}\,) = \mathcal{T} \quad \Leftrightarrow$ n is a prime number
5. $M_{P_A}(\,\mathbf{primesucc(\dot{n},\dot{m})}\,) = \mathcal{T} \Leftrightarrow$ n is the prime successor to the prime m