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# Definitional Reflection and Basic Logic

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ABSTRACT

In their *Basic Logic*, Sambin, Battilotti and Faggian give a foundation of logical inference rules by reference to certain reflection principles. We investigate the relationship between these principles and the principle of *Definitional Reflection* proposed by Hallnäs and Schroeder-Heister.



## 1. Introduction

*Basic Logic* (in short: BL) is an approach to embed various systems of propositional and quantifier logic into a framework of inferential definitions using principles of 'reflection' to generate logical rules. *Definitional Reflection* (in short: DR) is an approach to generate inference principles from clausal definitions. Both approaches not only share the term 'reflection', but show a similarity the precise kind of which will be discussed in this paper. For BL we rely on Sambin et al. [9] and Sambin [8]. DR is explained, e.g., in Hallnäs [6], Hallnäs and Schroeder-Heister [7] and Schroeder-Heister [11].[2] We confine ourselves to propositional logic in order to make the conceptual points clear.

Both BL and DR belong to the framework of proof-theoretic semantics according to which the meaning of logical constants is explained in terms of certain inference principles governing them (for an overview see [13]). Proof-theoretic semantics is normally viewed as a foundational enterprise, based on a critique of certain presuppositions of denotational semantics. However, in addition to this foundational stance, proof-theoretic semantics often claims that it is able to integrate different logics, given as deductive systems, within a single framework, which then allows to reach certain results in a uniform way. This is both true for BL and DR. For example, in the framework of BL, cut elimination is available for a variety of systems by a single method, and in the framework of DR, the rules governing a variety of logical (and also nonlogical) operators are treated uniformly in a general way. In this sense, both approaches are significant for the taxonomy

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[2] A survey of definitional reflection is subject of current work by Hallnäs and the author.

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of logical systems. In the context discussed here, we are specifically dealing with the sequent calculus and the symmetries and asymmetries inherent in this framework. A special role is played by substructural features of logical rules, especially the distinction between the additive and multiplicative association of reasoning contexts as discussed in linear logic.

## 2. Definitional reflection

The idea of definitional reflection is related to the program of developing general elimination rules for logical constants as presented in Schroeder-Heister [10]. Given $m$ introduction rules for an $n$-ary constant $c$ of propositional logic

$$\frac{A_1(p_1,\ldots,p_n)}{c(p_1,\ldots,p_n)} \quad \ldots \quad \frac{A_m(p_1,\ldots,p_n)}{c(p_1,\ldots,p_n)}$$

where the $A_i(p_1,\ldots,p_n)$ are premisses structured in a certain way, the elimination rules are

$$\frac{c(p_1,\ldots,p_n) \qquad \begin{array}{c}[A_1(p_1,\ldots,p_n)]\\ C\end{array} \quad \ldots \quad \begin{array}{c}[A_m(p_1,\ldots,p_n)]\\ C\end{array}}{C}$$

where the brackets indicate the possibility of discharging the assumption structures mentioned. In the case of conjunction and implication these rules run as follows:

$$\frac{p_1 \qquad p_2}{p_1 \wedge p_2} \qquad \frac{p_1 \wedge p_2 \qquad \begin{array}{c}[p_1 \quad p_2]\\ C\end{array}}{C} \qquad \frac{\begin{array}{c}[p_1]\\ p_2\end{array}}{p_1 \rightarrow p_2} \qquad \frac{p_1 \rightarrow p_2 \qquad \begin{array}{c}[p_1 \Rightarrow p_2]\\ C\end{array}}{C}$$

In the E-rules the premiss structures occur in assumption position. There the fact that premisses may depend on assumptions is represented by 'rules of higher levels', i.e., by using some sort of 'structural' implication '$\Rightarrow$'.[3] The idea behind this approach is that the I-rules represent a kind of 'definition' of $c$, and the E-rule says that everything that follows from each defining condition $A_i$ of $c$ follows from $c$ itself (for simplicity, we omit the arguments $p_1,\ldots,p_n$). The E-rule is called a rule of 'definitional reflection', since it expresses an act of 'reflection' on the inferential definition given by the I-rules.

More generally, the idea of definitional reflection was proposed by Hallnäs for a definitional system of clauses of the form

$$a \Leftarrow B$$

in the style of an inductive definition or of a logic program. Here $a$ is an atom and $B$ a potentially complex condition formulated in some logical or structural system. In the simplest case $B$ may just be a list of atoms, in more complicated cases one may consider expressions in first-order or even higher-order logic or some fragment thereof. The relationship to logic programming (which includes resolution-based evaluation mechanisms) has been investigated by Hallnäs and Schroeder-Heister [7], and is also the background of certain extensions of logic programming. Formulated in a sequent-style framework, a definition of $a$ consisting of clauses

$$\begin{cases} a \Leftarrow B_1 \\ \quad\vdots \\ a \Leftarrow B_n \end{cases}$$

leads to right and left introduction rules of the following form:

$$\frac{\Gamma \vdash B_i}{\Gamma \vdash a}\,(\vdash a) \qquad \frac{\Gamma, B_1 \vdash C \quad \ldots \quad \Gamma, B_n \vdash C}{\Gamma, a \vdash C}\,(a\vdash) \tag{1}$$

where the left-rules are called rules of 'definitional closure' and the right-rule is called 'definitional reflection'. Whereas the closure rules are local rules representing reasoning along the clauses of the given definition, the reflection rule is of a more global kind: It reflects upon the definition as a whole, saying that everything that can be obtained from each defining condition of $a$, can be obtained from $a$ itself. If variables are present in $a$, these rules becomes more complicated, with certain provisos to be respected in their application. Also, there is no restriction in the given definition to be monotone or well-founded, so, as in logic programming, the programmer has full freedom to formulate a definition. In this sense DR is much more general than generalized elimination rules for logical constants, which are a special application of it. However, the philosophical idea is much the same: Taking all definientia of a definition together, then the definiendum expresses what these definitientia have in common, i.e., their 'common content'. Definitional reflection has at its target this common content.

[3] We do not discuss here ways of avoiding higher-level rules, which in standard logic is possible, though not in the general case. See [14] for details.

### 3. Definitional reflection for denial and generalized definitional reflection

The idea of definitional reflection can be naturally extended to the case where a denial operator is available (see [12]). Suppose $\sim a$ denotes the denial of an atom $a$ (the denial operator '$\sim$' being an outermost operator expressing a form of judgment) and $\sim B$ denotes the denial of a defining condition $B$ in some sense (we leave open how exactly this $\sim$-operation is defined for non-atomic conditions $B$). Then, given the following defining clauses for $a$:

$$\begin{cases} a \Leftarrow B_1 \\ \quad\vdots \\ a \Leftarrow B_n \end{cases} \tag{2}$$

the following assertion and denial rules can be defined:

$$\frac{B_i}{a}\,(a) \qquad \frac{\sim B_1 \quad \ldots \quad \sim B_n}{\sim a}\,(\sim a)$$

Here again $(a)$ is a closure rule with respect to the definitional clauses, whereas $(\sim a)$ is a rule of definitional reflection telling that we can deny $a$ if we have denied all possible defining conditions of $a$. This approach is particularly powerful, if, in addition to assertion clauses of the form (2), we also use denial clauses (i.e., clauses with denials in their heads and bodies), which leads to a system which, on the logic programming side, is related to extended logic programming.

Quite generally, definitional reflection says that everything we can do with each defining condition $B_i$ of $a$ we can do with the definiendum $a$ itself. If we have defined an operation $^*$ that can be applied both to the definiendum $a$ and to defining conditions $B_i$, then DR with respect to $^*$ says that, if we are entitled to apply $^*$ to each $B_i$, we are entitled to apply it to $a$:

$$\frac{B_i}{a}\,(a) \qquad \frac{B_1^* \quad \ldots \quad B_n^*}{a^*}\,(a^*) \tag{3}$$

The rule $(a^*)$ is also called a rule *complementary* to the rules $(a)$. In fact, depending on the form of the premisses $B_i$ there may be more than one complementary rule. For example, in the case of denial, the definitional rule

$$\frac{a_1 \quad a_2}{a}$$

has as complementary rules

$$\frac{\sim a_1}{\sim a} \qquad \frac{\sim a_2}{\sim a}$$

Furthermore, $^*$ need not be unique, i.e., need not be an operation in the genuine sense, but may just express that there is some complementarity between $B$ and $B^*$. In that case we also speak of $B$ and $B^*$ as being *complementary* to one another.

A schema such as $(a^*)$ is called *generalized definitional reflection* (*with respect to* $^*$). Of course, $^*$ cannot be defined arbitrarily. In the case of denial, it has a natural meaning, whereas in other cases we have to carefully motivate it and to give adequacy conditions for it. Below we shall motivate and define such a notion in connection with sequent-style definitional clauses.

### 4. Basic logic

Unlike DR, where arbitrary clausal definitions are considered, BL is confined to logic, but in a very general sense. Out of certain definitional principles, the rules of logic are generated for various systems of logic, taking into account possible substructural distinctions. As a simple example, we consider the definition of (additive) disjunction $\vee$ in BL.[4] Here Sambin et al. start with 'definitional equations', by which they mean a pair of rules

$$\frac{\Gamma, p_1 \vdash \Delta \qquad \Gamma, p_2 \vdash \Delta}{\Gamma, p_1 \vee p_2 \vdash \Delta} \tag{4}$$

where the double line indicates that the rule can be read both top-down and bottom-up. (We here use Došen's [3] terminology and notation of 'double-line rules'.) In BL the top-down direction is called 'formation', the bottom-up direction 'implicit reflection'. "We say that a certain connective obeys the principle of reflection, or is defined by following the principle of reflection, if we can obtain its definition by solving an equation like this." (Sambin [8, p. 304]). This equation is 'solved' by first generating the axioms $p_1 \vdash p_1 \vee p_2$ and $p_2 \vdash p_1 \vee p_2$ by means of trivialization and implicit reflection, i.e., by applying implicit reflection to the initial sequent $p_1 \vee p_2 \vdash p_1 \vee p_2$, and then, by means of cut, the rules

$$\frac{\Gamma \vdash p_1, \Delta}{\Gamma \vdash p_1 \vee p_2, \Delta} \qquad \frac{\Gamma \vdash p_2, \Delta}{\Gamma \vdash p_1 \vee p_2, \Delta} \tag{5}$$

[4] We use $\wedge$ and $\vee$ for additive conjunction and disjunction, and use Girard's notation $\otimes$ and $\parr$ for the corresponding multiplicatives.

If we add the formation rule (the top-down direction of (4)), we obtain the $\vee$-rules of the classical sequent calculus. So the general idea of BL is to start with a double-line pair of formation/implicit-reflection rules, generate certain axiom sequents by means of trivial initial sequents and implicit reflection, and then obtain the inverse of formation by means of cut. Actually, this whole procedure can be inverted, yielding implicit reflection back from (explicit) reflection (this backwards procedure is called 'verification' in BL).

It should be noticed that cut is really necessary to establish the sequent calculus rules, i.e., that these rules cannot be shown to be admissible given double line rules only. More precisely, given the double line rule for a single logical constant, the sequent calculus rules are in fact admissible (without using cut). In our example of disjunction, the right introductions (5) are admissible with respect to the double-line rule (4), as the $p_i$ $(i = 1, 2)$ in the succedent of the premiss of (5) is either superfluous, i.e., already $\Gamma \vdash \Delta$ is derivable, or it occurs in the antecedent too, in which case we can obtain the conclusion of (5) by using $p_i \vdash p_1 \vee p_2$, which is derivable using (4). However, as soon as we have different logical constants with the double line rules not always operating on the same side of the turnstile, this argument is no longer valid. This situation cannot be avoided as double line rules for $\vee$ vs. $\wedge$, and also for additive vs. multiplicative logical constants operate on different sides of the turnstile. This is a crucial point for the comparison of BL with DR in the final section.

## 5. Definitional reflection for a sequent-style framework

In order to compare BL and DR, we have to show first how reasoning in the multiple-succedent sequent calculus including the substructural distinctions between additive and multiplicative constants, which is the target of BL, can be represented in the framework of DR. One approach would be to consider definitional clauses $a \Leftarrow B$ in which the defined atom $a$ is a sequent. Then the rules of definitional closure and reflection (1) lead to hypersequents, i.e., to sequents of sequents. This possibility of dealing with the symmetric relationship between right and left rules in the sequent calculus has been investigated by de Campos Sanz and Piecha [2]. At the meta-level they use the single-succedent sequent framework of definitional reflection to investigate derivability in the object-level sequent calculus (see footnote 5). Here we propose a different approach, which uses generalized definitional reflection (3) with an appropriate notion * of complementation. We assume that certain defining rules for operators are given in a sequent-style framework, corresponding to, but more general than the formation rules in BL. These rules are then complemented in a certain way, thus generating the reflection rules.

We take it that a definitional rule ('formation rule' in Sambin et al.'s terminology) *defines* a connective $c$. For simplicity, we skip the reference to propositional arguments (such as $p_1, \ldots, p_n$) in our notation. In the conclusion of the defining rule, this $c$ may occur either on the left or on the right of the turnstile '$\vdash$', and is embedded into a left and right context, which as usual is denoted by capital Greek letters such as $\Delta, \Gamma$, etc. So the conclusion of a defining rule for $c$ has either the form $\Gamma \vdash c, \Delta$ or $\Gamma, c \vdash \Delta$. The right and left sides of sequents are always supposed to be multisets.

The connective $c$ is assumed to be defined in terms of certain conditions $A_1, \ldots, B_1, \ldots,$ which themselves are embedded in contexts. In the elementary propositional case considered here we assume that these conditions are (possibly empty) lists of propositional constants (lists always understood as expressing multisets). In more complex cases they may be (lists of) complex propositions. To indicate their list structure, we also write $\overrightarrow{A}$ instead of $A$. We furthermore assume that, when passing from the premisses to the conclusion of a defining rule, contexts may be associated either in an additive or in a multiplicative way. Based on these assumptions, a defining rule for a connective $c$ has the following general form:

$$\frac{\begin{array}{ccc} \Gamma_1, \overrightarrow{A_{11}} \vdash \overrightarrow{B_{11}}, \Delta_1 & \ldots & \Gamma_1, \overrightarrow{A_{1k_1}} \vdash \overrightarrow{B_{1k_1}}, \Delta_1 \\ & \vdots & \\ \Gamma_n, \overrightarrow{A_{n1}} \vdash \overrightarrow{B_{n1}}, \Delta_n & \ldots & \Gamma_n, \overrightarrow{A_{nk_n}} \vdash \overrightarrow{B_{nk_n}}, \Delta_n \end{array}}{\Gamma_1, \ldots, \Gamma_n \vdash c, \Delta_1, \ldots, \Delta_n}$$

or, with the same premisses, but $c$ occurring on the left side in the conclusion

$$\frac{\begin{array}{ccc} \Gamma_1, \overrightarrow{A_{11}} \vdash \overrightarrow{B_{11}}, \Delta_1 & \ldots & \Gamma_1, \overrightarrow{A_{1k_1}} \vdash \overrightarrow{B_{1k_1}}, \Delta_1 \\ & \vdots & \\ \Gamma_n, \overrightarrow{A_{n1}} \vdash \overrightarrow{B_{n1}}, \Delta_n & \ldots & \Gamma_n, \overrightarrow{A_{nk_n}} \vdash \overrightarrow{B_{nk_n}}, \Delta_n \end{array}}{\Gamma_1, \ldots, \Gamma_n, c \vdash \Delta_1, \ldots, \Delta_n}$$

Here every line $i$ represents a list of premisses whose contexts $\Gamma_i$ and $\Delta_i$ are additively associated (within line $i$), whereas the contexts of the $n$ premiss lines are multiplicatively associated when passing over to the conclusion. Suppressing contexts, these rules can be formulated as

$$\frac{\langle\overrightarrow{A_{11}} \vdash \overrightarrow{B_{11}} \quad \ldots \quad \overrightarrow{A_{1k_1}} \vdash \overrightarrow{B_{1k_1}}\rangle \quad \ldots \quad \langle\overrightarrow{A_{n1}} \vdash \overrightarrow{B_{n1}} \quad \ldots \quad \overrightarrow{A_{nk_n}} \vdash \overrightarrow{B_{nk_n}}\rangle}{\vdash c} \tag{6}$$

and

$$\frac{\langle\overrightarrow{A_{11}} \vdash \overrightarrow{B_{11}} \quad \ldots \quad \overrightarrow{A_{1k_1}} \vdash \overrightarrow{B_{1k_1}}\rangle \quad \ldots \quad \langle\overrightarrow{A_{n1}} \vdash \overrightarrow{B_{n1}} \quad \ldots \quad \overrightarrow{A_{nk_n}} \vdash \overrightarrow{B_{nk_n}}\rangle}{c \vdash} \tag{7}$$

where the angle brackets in the premisses indicate additive association (within each pair of brackets, respectively). If $n$ is 1, we have a purely additive association, if $k_1 = \cdots = k_n = 1$, we have a purely multiplicative association, and if $n > 1$ and $k_i > 1$ for some $i$, we have a mixture of additive and multiplicative associations. One might think of even more general forms of rules, which we do not discuss here.

Now a definition is a list of rules of the form (6) that defines $c$ in succedent position, or a list of rules of the form (7) that defines $c$ in antecedent position. Then the defining rules for the well-known additive and multiplicative logical constants all fall under this schema, where for each of them there is a definition on the right side and one on the left side. That there is a certain equilibrium between both aspects, so that the left defining rules can be generated from the right defining rules, and vice versa, is a result of what follows.

In order to apply definitional reflection in the form (3), we must define a $^*$-operation which associates with each premiss and with each conclusion of a defining rule certain expressions which generate the reflection schema. More precisely, we define complementary pairs of sequents or sets of sequents $\Sigma$ and $\Sigma^*$, as we do not have a unique $^*$-operation. We always consider sequents which focus on certain formulas, i.e. formulas, which are not contextual. The formulas focused on, also called *designated formulas*, are denoted by Latin letters, whereas Greek letters denote contexts. If we have a sequent with a single designated formula $A$ such as $\Gamma_1, A \vdash \Delta_1$, then a complementary sequent is one with $A$ occurring on the opposite side of the turnstile, i.e. $\Gamma_2 \vdash A, \Delta_2$. A natural way of expressing this is that by applying cut to a sequent $S$ and a sequent $S^*$ complementary to it, the formula $A$ is eliminated. For a set of sequents $\Sigma$ a complementary set of sequents $\Sigma^*$ would be one such that from $\Sigma \cup \Sigma^*$ by applying cut with respect to designated formulas a sequent without any designated formula can be obtained.

More precisely, if

$$\Sigma = \left\{\Gamma_1, \overrightarrow{A_1} \vdash \overrightarrow{B_1}, \Delta_1\ ; \quad \ldots \quad ; \ \Gamma_n, \overrightarrow{A_n} \vdash \overrightarrow{B_n}, \Delta_n\right\}$$

$$\Sigma^* = \left\{\Gamma_1', \overrightarrow{A_1'} \vdash \overrightarrow{B_1'}, \Delta_1'\ ; \quad \ldots \quad ; \ \Gamma_n', \overrightarrow{A_n'} \vdash \overrightarrow{B_n'}, \Delta_n'\right\}$$

then $\Sigma$ and $\Sigma^*$ are a complementary pair (in short: are complementary), if from $\Sigma \cup \Sigma^*$ we can derive $\Gamma_1, \ldots, \Gamma_n, \Gamma_1', \ldots, \Gamma_n' \vdash \Delta_1, \ldots, \Delta_n, \Delta_1', \ldots, \Delta_n'$ by means of cut only. Here cut is, as usual, understood in its multiplicative version

$$\frac{\Gamma_1 \vdash A, \Delta_1 \quad \Gamma_2, A \vdash \Delta_2}{\Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2}$$

in which all contexts are joined together in the conclusion. In our notation, in which contexts are suppressed, complementarity means complementarity for any assignment of contexts. So for the fact that

$\Sigma$ is complementary to $\Sigma^*$

we also write:

$$\left\{\overrightarrow{A_1} \vdash \overrightarrow{B_1}\ ; \quad \ldots \quad ; \ \overrightarrow{A_n} \vdash \overrightarrow{B_n}\right\} \quad \text{is complementary to} \quad \left\{\overrightarrow{A_1'} \vdash \overrightarrow{B_1'}\ ; \quad \ldots \quad ; \ \overrightarrow{A_n'} \vdash \overrightarrow{B_n'}\right\}$$

We then can use the notation with angle brackets to indicate that equal contexts have to be assigned to all sequents within such brackets. For example,

$$\left\{\langle A_1 \vdash B_1\ ;\ A_2 \vdash B_2\rangle\ ;\ A_3 \vdash B_3\ ;\ A_4 \vdash B_4\right\}$$

is understood as standing for

$$\{\Gamma_1, A_1 \vdash B_1, \Delta_1\ ;\ \Gamma_1, A_2 \vdash B_2, \Delta_1\ ;\ \Gamma_3, A_3 \vdash B_3, \Delta_3\ ;\ \Gamma_4, A_4 \vdash B_4, \Delta_4\}$$

i.e., the first two sequents must have the same context, which corresponds to additive association when used as a premiss. Using this notation we obtain the following table of complementary pairs, where each cell lists a set of sequents (with suppressed contexts), such that each (right or left) neighboring cell lists a set complementary to it.

<table>
<tr><td>$\vdash A$</td><td>$A \vdash$</td></tr>
<tr><td>$\vdash A_1, A_2$</td><td>$A_1 \vdash \quad A_2 \vdash$</td></tr>
<tr><td>$A_1, A_2 \vdash$</td><td>$\vdash A_1 \quad \vdash A_2$</td></tr>
<tr><td rowspan="2">$\langle \vdash A_1 \quad \vdash A_2 \rangle$</td><td>$A_1 \vdash$</td></tr>
<tr><td>$A_2 \vdash$</td></tr>
<tr><td rowspan="2">$\langle A_1 \vdash \quad A_2 \vdash \rangle$</td><td>$\vdash A_1$</td></tr>
<tr><td>$\vdash A_2$</td></tr>
<tr><td>$\vdash$</td><td></td></tr>
<tr><td></td><td>$\vdash$</td></tr>
</table>

(8)

The first case is needed for negation, the four cases below for the standard binary constants, and the last two for the interpretation of the nullary constants. If we take $A_1, A_2$ and $A_3$ to be propositional variables $p_1, p_2, p_3$, and if we put each item in the left column on top of an inference line with conclusion $\vdash c$, and each corresponding item in the right column on top of an inference line with conclusion $c \vdash$, where $c$ is the constant to be defined, we obtain the standard right- and left-introduction rules of (additive and multiplicative) propositional logic. If we exchange $\vdash c$ and $c \vdash$, we obtain the rules for constants with the negated meaning. For example, for the case of multiplicative disjunction $\parr$, the rules are

$$\frac{\vdash p_1, p_2}{\vdash p_1 \parr p_2} \qquad \frac{p_1 \vdash \quad p_2 \vdash}{p_1 \parr p_2 \vdash} \quad \text{(see second line of the table)}$$

which according to our notational conventions means the same as

$$\frac{\Gamma \vdash p_1, p_2, \Delta}{\Gamma \vdash p_1 \parr p_2, \Delta} \qquad \frac{\Gamma_1, p_1 \vdash \Delta_1 \quad \Gamma_2, p_2 \vdash \Delta_2}{\Gamma_1, \Gamma_2, p_1 \parr p_2 \vdash \Delta_1, \Delta_2}$$

Applying cut to the conclusions of these two rules yields

$$\frac{\Gamma \vdash p_1 \parr p_2, \Delta \quad \Gamma_1, \Gamma_2, p_1 \parr p_2 \vdash \Delta_1, \Delta_2}{\Gamma, \Gamma_1, \Gamma_2 \vdash \Delta, \Delta_1, \Delta_2}$$

which can also be obtained by applying cut (in this case twice) to the premisses of the respective rules:

$$\frac{\dfrac{\Gamma \vdash p_1, p_2, \Delta \qquad \Gamma_1, p_1 \vdash \Delta_1}{\Gamma, \Gamma_1 \vdash p_2, \Delta, \Delta_1} \qquad \Gamma_2, p_2 \vdash \Delta_2}{\Gamma, \Gamma_1, \Gamma_2 \vdash \Delta, \Delta_1, \Delta_2}$$

The general criterion that we extract from this characterization is that we associate with a set of definitional rules

$$\frac{\Sigma_1}{S} \quad \ldots \quad \frac{\Sigma_n}{S}$$

one or more complementary inference rules of the form

$$\frac{\Sigma_1^* \quad \ldots \quad \Sigma_n^*}{S^*}$$

such that $S$ and $S^*$ as well as, for every $i$, $\Sigma_i$ and $\Sigma_i^*$ are complementary pairs. Therefore the notion of complementarity can also be described as saying that cut with the conclusions $S$ and $S^*$ can be reduced to cuts within the premisses $\Sigma_i \cup \Sigma_i^*$. Thus we call our criterion the *criterion of cut reduction*.

This does not mean that we assume cut as a primitive rule. We only use it as a local adequacy condition in the sense that cut with the conclusions of right- and left-rules can be reduced to cut(s) with their premisses: *If* we have premiss-cut *then* we have conclusion-cut. Cut does not need to hold globally for the whole system, i.e., it need not be admissible. For example, the reduction of conclusion-cut to premiss-cut is also defined for circular definitional rules. If we define $c$ in terms of $\neg c$ by means of the rules

$$\frac{\vdash \neg c}{\vdash c} \qquad \frac{\neg c \vdash}{c \vdash}$$

then our adequacy condition holds (a cut with $c$ is reducible to a cut with $\neg c$) although cut is not admissible globally in such a system. In proof-theoretic terminology, we only require that the main reduction step for cut is defined, not that cut is admissible.

However, the criterion of cut reduction is not sufficient. For example,

| | |
|---|---|
| $\vdash A \quad B, C \vdash D$ | $\vdash B \quad A \vdash C \quad D \vdash$ |

represents a complementary pair. If we use it to define a 4-place propositional operator $c_1$

$$\frac{\vdash p_1 \quad p_2, p_3 \vdash p_4}{\vdash c_1(p_1, p_2, p_3, p_4)} \qquad \frac{\vdash p_2 \quad p_1 \vdash p_3 \quad p_4 \vdash}{c_1(p_1, p_2, p_3, p_4) \vdash}$$

then the $c_1$-left rule is too weak as it does not give us everything back which (intuitively) is contained in the $c_1$-right rule. A simpler example is

$$\begin{array}{|l|l|}\hline \langle \vdash A \qquad \vdash B\rangle & A \vdash \\ \hline \end{array} \tag{9}$$

which leads to a constant with the full right rule but with only one of the left rules for (additive) conjunction.

These examples suggest to introduce as a second adequacy criterion what we call *uniqueness*, which guarantees that rules are not only *not too strong* but also *strong enough*, i.e., that from the defined expression on one side of the turnstile we obtain an expression of the same strength on the other side. Suppose rules of the form (6) or (7) are given, and suppose a copy of (6) or (7) is given with the only difference that the conclusions of the rules have $c^+$ instead of $c$ for a new constant $c^+$. Then we expect the complementary rules to be strong enough that from both systems of rules $c \vdash c^+$ and therefore (for symmetry reasons)

$$c \dashv\vdash c^+$$

is derivable. The criterion of uniqueness in addition to cut reduction roughly corresponds to $\eta$-reduction in addition to $\beta$-reduction in the typed $\lambda$-calculus.

Thus our adequacy conditions are *cut-reduction* and *uniqueness*. These conditions are related to the adequacy conditions of conservativeness and uniqueness Belnap [1] has proposed, with the only (but crucial) difference that unlike conservativeness, cut-reduction is not a *global* criterion. (For further remarks see Došen and Schroeder-Heister [4,5].)

In general, it is not possible to define for every set of sequents a set of complementary sequents in such a way that the resulting complementary pair defines a constant uniquely. In the above example, there is no set of left-rules for $c_1$ which would give $c_1$ the meaning of $p_1 \otimes (p_2^\perp \parr p_3^\perp \parr p_4)$, which should be the case, if

$$\frac{\vdash p_1 \quad p_2, p_3 \vdash p_4}{\vdash c_1(p_1, p_2, p_3, p_4)}$$

were the right-rules for $c_1$. This means that in the general case of an arbitrary $n$-ary constant defined by (6) or (7) corresponding complementary rules cannot always be defined directly, at least not with the means of expression introduced here. So the problem of non-appropriate complementary rules is not always just due to some rule being lacking as in the case of (9). If we wanted to define complementarity directly for every set of defining conditions, we would have to introduce certain structural means of expression such as a structural implication '$\Rightarrow$' to represent the turnstile within sequents, which would lead to some sort of hypersequents. Instead we define complementarity for a limited case and then express the remaining rules in terms of an operator already definable. This limited case is still more general than what is needed for the standard logical constants.

We consider a constant defined by the single definitional rule

$$\frac{\langle \overrightarrow{A_1} \vdash \overrightarrow{B_1} \quad \dots \quad \overrightarrow{A_k} \vdash \overrightarrow{B_k}\rangle}{\vdash g} \tag{10}$$

To define complementarity, we use the following notation: If $\vec{A}$ is $A_1, \dots, A_n$, then $\overrightarrow{\vdash A}$ denotes the list of sequents $\vdash A_1; \dots; \vdash A_n$, and $\overrightarrow{A \vdash}$ denotes the list of sequents $A_1 \vdash; \dots; \ A_n \vdash$, where the semicolon is used to separate sequents rather than items in the antecedent or succedent of a sequent. Then we have as complementary pairs

$$\begin{array}{|c|c|}\hline & \overrightarrow{\vdash A_1} \quad \overrightarrow{B_1 \vdash} \\ \cline{2-2} \langle \overrightarrow{A_1} \vdash \overrightarrow{B_1} \quad \dots \quad \overrightarrow{A_n} \vdash \overrightarrow{B_n}\rangle & \vdots \\ \cline{2-2} & \overrightarrow{\vdash A_n} \quad \overrightarrow{B_n \vdash} \\ \hline \end{array} \tag{11}$$

which include

$$\begin{array}{|l|l|}\hline A_1, \dots, A_k \vdash B_n, \dots, B_\ell & \vdash A_1 \quad \dots \quad \vdash A_k \quad B_1 \vdash \quad \dots \quad B_\ell \vdash \\ \hline \end{array}$$

as a special case. (11) covers all cases in (8) and therefore all the standard logical constants. As left inferences complementing (10) we obtain

$$\frac{\overrightarrow{\vdash A_1} \quad \overrightarrow{B_1 \vdash}}{g \vdash} \quad \dots \quad \frac{\overrightarrow{\vdash A_k} \quad \overrightarrow{B_k \vdash}}{g \vdash} \tag{12}$$

It is easy to see that both adequacy conditions are met, i.e., that not only cut-reduction but also uniqueness holds. For that it is important to notice that, whereas the premisses of (10) are additively understood, the premisses of each rule in (12) are multiplicatively understood. If the defining rule for $g$ has $g\vdash$ in the conclusion, i.e., defines $g$ on the left side of the turnstile:

$$\frac{\langle \overrightarrow{A_1} \vdash \overrightarrow{B_1} \quad \ldots \quad \overrightarrow{A_k} \vdash \overrightarrow{B_k} \rangle}{g \vdash}$$

then the conclusions of (12) are inverted as well:

$$\frac{\vdash \overrightarrow{A_1} \quad \overrightarrow{B_1} \vdash}{\vdash g} \quad \ldots \quad \frac{\vdash \overrightarrow{A_k} \quad \overrightarrow{B_k} \vdash}{\vdash g} \tag{13}$$

Based on the reflection rules (12) and (13), we can easily show that the inverse to (10) is admissible, i.e. that the double-line rule

$$\frac{\Gamma, \overrightarrow{A_1} \vdash \overrightarrow{B_1}, \Delta \qquad \ldots \qquad \Gamma, \overrightarrow{A_k} \vdash \overrightarrow{B_k}, \Delta}{\Gamma, \vdash g, \Delta}$$

is admissible. This means that the general rule (6) can be formulated as

$$\frac{\vdash g_1 \quad \ldots \quad \vdash g_n}{\vdash c} \tag{14}$$

A list of $m$ rules of this form

$$\frac{\vdash g_{11} \quad \ldots \quad \vdash g_{1k_1}}{\vdash c} \quad \ldots \quad \frac{\vdash g_{m1} \quad \ldots \quad \vdash g_{mk_m}}{\vdash c}$$

can then, by means of (11), be complemented by

$$\frac{g_{11}, \ldots, g_{1k_1} \vdash \quad \ldots \quad g_{m1}, \ldots, g_{mk_m} \vdash}{c \vdash}$$

and analogously, if $c$ is defined on the left side. In this way, $c$ is defined via the intermediary definitions of certain $g$'s which allows us to write the definition of $c$ in the special form (14).

A final example for the case in which we can define $c$ only via some intermediary $g$ and not directly, is the following. Each set $\{\langle \vdash A\ ; \vdash B\rangle\}, \{\vdash C\}$ is complementary to each set $\{\langle A \vdash\ ;\ C \vdash\rangle\}, \{\langle B \vdash;\ C \vdash\rangle\}$. Therefore it is tempting to characterize a ternary constant $c_2$ by the following right and left rules:

$$\frac{\langle \vdash p_1 \quad \vdash p_2 \rangle}{\vdash c_2(p_1, p_2, p_3)}, \quad \frac{\vdash p_3}{\vdash c_2(p_1, p_2, p_3)} \qquad \frac{\langle p_1 \vdash \quad p_3 \vdash \rangle}{c_2(p_1, p_2, p_3) \vdash} \qquad \frac{\langle p_2 \vdash \quad p_3 \vdash \rangle}{c_2(p_1, p_2, p_3) \vdash}$$

However, the criterion of uniqueness is not satisfied, which is due to the fact that the right-introduction rules give $c_2$ the meaning of $(p_1 \wedge p_2) \vee p_3$, whereas the left-introduction rules give it the meaning of $(p_1 \vee p_3) \wedge (p_2 \vee p_3)$, which is not the same (if we do not assume particular structural inference rules). $(p_1 \wedge p_2) \vee p_3$ does not have a direct structural characterization.

## 6. Comparison of BL with DR

In order to compare BL with DR, we fix some terminology. Using Sambin et al.'s term, the top-down direction of a double-line rule in BL, or a definitional rule in DR, is called a *formation rule*. A formation rule is always an introduction rule, as for example $\vee$-introduction on the left side of the turnstile (4). In BL there can only be a single formation rule for a defined constant, because otherwise it could not be inverted. In DR, there can be more than one formation rule. This is, for example, the case if $\vee$-introduction on the right (5) is used as the definition of disjunction. The bottom-up direction of a double-line rule is called an *elimination rule*, as it eliminates a constant either from the right or from the left side. For example, the elimination rule contained in (4) eliminates $\vee$ from the left side of the turnstile. There can be several elimination rules for a constant, if the corresponding formation rule has several premisses. Elimination corresponds to what is called 'implicit reflection' in BL. The rules which introduce a constant on the opposite side of the turnstile as compared to the formation rules, i.e., those rules, which are justified by means of a reflection procedure, are called *reflection rules*. Such rules are, for example, the right-introduction rules for $\vee$ (5) with respect to the left $\vee$-introduction rule as a formation rule. The following table exemplifies our terminology for the case of additive and multiplicative conjunction.

| | formation | elimination | reflection |
|---|---|---|---|
| $\wedge$ (via right-formation) | $\dfrac{\langle \vdash p_1 \quad \vdash p_2 \rangle}{\vdash p_1 \wedge p_2}$ | $\dfrac{\vdash p_1 \wedge p_2}{\vdash p_1}$ $\dfrac{\vdash p_1 \wedge p_2}{\vdash p_2}$ | $\dfrac{p_1 \vdash}{p_1 \wedge p_2 \vdash}$ $\dfrac{p_2 \vdash}{p_1 \wedge p_2 \vdash}$ |
| $\wedge$ (via left-formation), DR only | $\dfrac{p_1 \vdash}{p_1 \wedge p_2 \vdash}$ $\dfrac{p_2 \vdash}{p_1 \wedge p_2 \vdash}$ | | $\dfrac{\langle \vdash p_1 \quad \vdash p_2 \rangle}{\vdash p_1 \wedge p_2}$ |
| $\otimes$ (via left-formation) | $\dfrac{p_1, p_2 \vdash}{p_1 \otimes p_2 \vdash}$ | $\dfrac{p_1 \otimes p_2 \vdash}{p_1, p_2 \vdash}$ | $\dfrac{\vdash p_1 \qquad \vdash p_2}{\vdash p_1 \otimes p_2}$ |
| $\otimes$ (via right-formation), DR only | $\dfrac{\vdash p_1 \qquad \vdash p_2}{\vdash p_1 \otimes p_2}$ | | $\dfrac{p_1, p_2 \vdash}{p_1 \otimes p_2 \vdash}$ |

Using this terminology, the main difference between BL and DR is the following: By means of the notion of complementarity, DR establishes a direct duality between formation and reflection rules. Given certain formation rules, the reflection rules can be generated by the procedure described above. Conversely, by the same procedure, the formation rules are obtained back from the reflection rules. In the above table, this is expressed by the fact that with respect to formation and reflection, the second line is the converse of the first one, and the fourth the converse of the third one. This works generally in the case of arbitrary $n$-ary propositional connectives. Contrary to that, BL generates reflection from formation only *via elimination*. In fact elimination is the immediate converse of formation, so there is some plausibility in considering (formation + elimination) as a sort of 'equation', which is then 'solved' by passing to reflection. However, the way via elimination first imposes certain restrictions on the formation inferences, and secondly requires cut as a means of passing from elimination to reflection.

Using (formation + elimination) as the starting point presupposes that formation must be a single rule, since otherwise there would be no elimination.[5] This again means that (formation + elimination) acts for certain constants on the left side and for others on the right side of the turnstile. In the above example, the defining rules (formation + elimination) for additive conjunction define $\wedge$ on the right side of $\vdash$, whereas those for multiplicative conjunction define $\otimes$ on the left side of $\vdash$. So there is a certain lack of uniformity in BL, which is avoided in DR where we have full symmetry between formation and reflection, i.e. formation and reflection can be interchanged. This is essentially due to the fact that in DR defining rules can be non-deterministic, i.e. there can be more than one rule to define a constant. DR has taken this inspiration from inductive definitions and logic programs. From that point of view, BL is an approach which is based on explicit rather than inductive definitions, for which the non-uniformity with respect to the right vs. left side of a sequent is taken into account. In the DR framework, the definition of logical constants is a special case within a framework of completed clausal (inductive) definitions.

Arbitrary clausal definitions in the sense of DR could be framed in a BL-style framework as well: Non-deterministic (i.e. multiple) clauses defining an expression $a$ on the right side of the turnstile can be translated into BL by an appropriate single formation rule defining $a$ on its left side, from which the original right-side definition can be obtained by equation-solving in the BL sense. It is, however, not in the spirit of inductively defining a domain of reasoning to start with left-side rules and generate right-side ones. On the other hand, since BL is mainly concerned with the foundational purpose of giving a taxonomy of logical constants, and here especially with using the power of substructural distinctions, it might appear unfair to apply criteria from a different area (inductive definitions and logic programming). However, the full symmetry in DR between formation and reflection achieved by non-deterministic formation is something that to our mind has strong appeal particularly for foundational purposes.

The second essential difference between BL and DR concerns the role of cut in the justification of inferences. In order to pass from elimination to reflection, BL first extracts from elimination one or more axioms by means of trivialization, i.e. by using an initial sequent as a premiss. For additive conjunction these axioms are $p_1 \wedge p_2 \vdash p_1$ and $p_1 \wedge p_2 \vdash p_2$, for multiplicative conjunction it is $p_1, p_2 \vdash p_1 \otimes p_2$, for multiplicative disjunction it is $p_1 \parr p_2 \vdash p_1, p_2$ etc. In fact, the elimination inferences are used exclusively to generate these axioms. From these axioms and the premisses of reflection,

[5] Instead of focusing on double-line rules, which implies that there is a single formation and possibly several elimination rules, one might consider a framework which allows for more than one formation rule, but has a single hypersequential elimination rule. This is the way taken by de Campos Sanz and Piecha [2], where this hypersequential elimination rule is generated by some form of definitional reflection at the meta-level.

the conclusion of reflection is obtained by means of cut. For example, in the case of multiplicative conjunction, we have the following derivation (suppressing contexts):

$$\dfrac{\vdash p_2 \qquad \dfrac{\vdash p_1 \qquad \dfrac{}{p_1, p_2 \vdash p_1 \otimes p_2}\,\text{Axiom}}{p_2 \vdash p_1 \otimes p_2}\,\text{Cut}}{\vdash p_1 \otimes p_2}\,\text{Cut}$$

Although Sambin et al. are fully aware of the problem of using cut at this place, they present no real solution to it. The final system of BL does enjoy cut-elimination — the fact that it allows for a general cut-elimination proof is one of its outstanding features. However, this later proof of cut elimination is nothing one can rely upon for the initial justification of the system, quite independently of any questions of vicious circles.[6] In the initial justification of reflection, cut is used with respect to a different system, namely one based on formation and elimination, or, what amounts to the same, one based on formation and certain axioms derived from elimination. In contradistinction to the final system based on formation and reflection, in the initial system cut is not admissible and has therefore to be laid down explicitly as a fundamental principle.

In DR, we only use cut reduction as an adequacy criterion for reflection rules, not the validity of cut itself. In fact, the example in the previous paragraph shows that Sambin et al. only use very limited applications of cut, namely applications of cut to axioms and to results of applications of cut to axioms. From the viewpoint of DR, this may be interpreted as reducing a cut with $p_1 \otimes p_2$ to a cut with $p_1$ and $p_2$. However, as shown in Section 5, such a reduction does not require cut as a principle, not even in its limited application to axioms. Complementarity of defining conditions, on which cut reduction is based, can be formulated directly, without assuming the validity of cut.

The fact that certain formation rules do not have an immediate reflection counterpart but proceed via an internal codification step (the definition of $g$ in Section 5) is nothing that divides BL from DR. That, for example, the ternary constant $c_3$ with the meaning of $(p_1 \wedge p_2) \otimes p_3$, which has the formation rule

$$\dfrac{\langle \vdash p_1 \quad \vdash p_2 \rangle \quad \vdash p_3}{\vdash c_3(p_1, p_2, p_3)}$$

has no direct reflection rule in DR, but has first to be formulated by coding the premiss $\langle \vdash p_1 \ \vdash p_2 \rangle$ into a single sequent $\vdash g(p_1, p_2)$ (where $g$ has the intuitive meaning $p_1 \wedge p_2$), has its counterpart in BL in the fact that there is no elimination rule directly corresponding to this formation rule.[7]

The adequacy criterion of uniqueness which in DR is crucial to ensure that the reflection rules are not only valid, but also sufficiently strong with respect to the formation rules, have their counterpart in BL in what Sambin et al. call 'verification'. In BL, verification is a procedure that generates the elimination rules from the reflection rules by means of cut. According to Sambin et al. it corresponds to verifying that the solution of an equation is in fact a solution. However, this way of presenting it does not give verification its proper standing. Verifying that a solution of an equation obtained by a certain algorithm is in fact a solution, is a second check which mathematically is not relevant if the algorithm to solve the equation has been applied properly in the first place. What must be made sure in the present case is not that the 'solutions' to a double line equation (= rule) are solutions, but that they represent *all* solutions. Exactly this can be established by 'verification': We can check whether we obtain *all* elimination rules from the reflection rules. Understood in this way, verification is essential to establish that the reflection rules are as strong as they are intended to be. This could have been stated more explicitly in the presentation of BL.

Taking these arguments together, for foundational purposes BL has the appeal of being based on (formation + elimination) as rules, which are immediate inverses of one another, at the cost of an asymmetry between right and left sides of the turnstile, and at the cost of having to assume cut as a fundamental principle of reasoning. DR, on the contrary, exhibits a full duality of formation and reflection, i.e. of right- and left-rules of the sequent calculus, based on elementary notions of complementarity and cut reduction.

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[6] Apparently, Sambin et al. envisage some sort of hermeneutics according to which, in the justification of the system, cut is anticipated or expected to hold, while this anticipation is fulfilled by the later proof of cut elimination. See [9], p. 994.

[7] We cannot use $c_2$ as considered at the end of Section 5 as an example, since $c_2$ has two formation rules, which is not allowed in BL.

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