Combinatorial Principles in Finite Variable Logic

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Abstract

For any $n > 0$ and a suitable $m > n$ we expose a simple three-variable equality-free combinatorial sentence $\mathfrak{F}_{m,n}$ of 1-order logic, which is provable in the corresponding $(n+3)$-variable equality-free Hilbert-Bernays formalism, but not in the corresponding $(n+2)$-variable equality-free Hilbert-Bernays formalism. The crucial negative part of the proof is based on a suitable nested finite-variable variant of the familiar Herbrand Lemma. The canonical translation of $\mathfrak{F}_{m,n}$ into the language of relation algebras yields the identity-free algebraic equation $\mathfrak{T}_{m,n} = 1$ having the same proof theoretical feature.

1 Introduction

There are two familiar *modus-ponens* axiomatizations of 1-order logic without function symbols, which we refer to as Hilbert-Bernays and Henkin-Tarski formalisms, respectively (cf. [8, 9, 12, 21] et al). Both formalisms are complete, and hence proof theoretically equivalent, in full 1-order logic with an infinite supply of individual variables. However, this is not true of the corresponding finite-variable domains, which are known to be incomplete (cf. e.g. [21]). To illuminate the formal distinction with respect to finite variable logic, note that Hilbert-Bernays formalisms include the *law of substitution* axiom/schema $\forall x F \to F[x/y]$ where $[x/y]$ denotes a chosen variable-substitution operator, for all F, x and y available, whereas

Henkin-Tarski formalisms include its identical instance $\forall x F \to F$ only. On the other hand, Henkin-Tarski formalisms include the equality symbol $=$, which enables to derive $\forall x F \to F[x/y]$ from other axioms, provided that the *general Leibniz law* axiom/schema $x = y \to (F \to F[x/y])$ is also included, for all F, x and y available. Moreover, in the presence of the general Leibniz law, $[x/y]$ can be simulated by setting $F[x/y] := \forall x\,(x = y \to F)$. However, the general Leibniz law is on the one hand proof theoretically stronger than the law of substitution, and on the other hand meaningless in the language without equality, whereas the basic Hilbert-Bernays 1-order language does not include equality. Moreover, in the language with equality, Hilbert-Bernays formalisms with the *simple Leibniz law* (being an instance of the general Leibniz law for atomic F only) are conservative extensions of the correlated equality-free formalisms, although the simple Leibniz law is proof theoretically weaker than the general Leibniz law. Another difference is related to Gentzen-style cutfree proof theoretical *sequent calculi* which explicitly use $[x/y]$ and the corresponding substitution rule(s) (see e.g. [2, 3, 4, 5, 6, 22]), thus being more suitable to Hilbert-Bernays, rather than Henkin-Tarski, formalisms. Summing up, it seems that basic Hilbert-Bernays formalisms provide us with the adequate formal framework for 1-order finite variable logic without equality. *In fact, I conjecture that Henkin and Tarski finite-variable formalisms are both conservative extensions of the correlated Hilbert-Bernays formalisms without equality* (see [8, 21] and [4] for precise definitions and loose comparisons), in that every equality-free sentence provable in either Henkin or Tarski formalism is also provable in the correlated Hilbert-Bernays equality-free formalism (the other direction is easy — see e.g. [4]); see also Conjecture 37 below.

It is known that the whole 1-order logic (with or without equality) is embeddable (mod polynomial translation) into both Henkin-Tarski and Hilbert-Bernays n-variable logic for each $n > 3$ (see [2, 16, 21]). A more careful analysis (see [4]) shows that every valid m-variable sentence is already provable in the correlated $(m+2)$-variable Hilbert-Bernays formalism extended by Tarski *conjugated quasi-projections*, and hence also in the analogous extensions of Henkin-Tarski formalisms.[1] It turns out that the conjugated quasi-projections are essential for the latter results, and the sentences $\mathfrak{F}_{m,n}$ (see Abstract above and Definition 1 below) provide us with a complete collection of simple three-variable counterexamples for the basic Hilbert-Bernays formalisms.

Moreover, these sentences can be referred to as simple combinatorial both 1-order and relation-algebra solutions to the Hilbert-Bernays analogues of [11, Problem 2.12] (mod canonical back-and-forth interpretations into the language of cylindric algebras).[2] The combinatorial contents of $\mathfrak{F}_{m,n}$ are relevant to the Greenwood-Gleason theorem (an instance of the familiar Ramsey theorem) which, in turn, was extensively

[1] The analogous claim about $(m+1)$-variable Tarski formalism posed in [21] is probably wrong — there is a fatal error in the attached proof of Theorem 4.8 (xvi).

[2] See also Remark 41 below. In a recent private communication, Hirsch, Hodkinson and Maddux [10] claimed the analogous solutions with regard to Tarski formalisms. Their proof uses entirely different techniques, and their three-variable sentences are relevant to the ones exposed in [18], where a weaker proof theoretical result was claimed. Sentences of this kind are more involved than ours.

used in [17] where a weaker proof theoretical result was claimed with regard to not explicitly shown three-variable sentences with equality, which arise by translating the defining equations of certain complicated relation algebras into the 1-order language. It should be noted that $\mathfrak{F}_{m,n}$ include $m + n + 2$ different predicate symbols, and hence they cannot a priori serve as Hilbert-Bernays-style solutions to the analogous problem posed in [21, p. 93] with regard to the underlying one-generated language of relation algebras. One can overcome this trouble via the relation algebra translation intended in [17], since the underlying one-generated relation algebras of [17] are generated by a single relation. Hence by rewriting every predicate occurring in $\mathfrak{F}_{m,n}$ to its intended one-generated expansion one can pass from $\mathfrak{F}_{m,n}$ to the required one-generated solutions of the Hilbert-Bernays analogue of the problem [21, p. 93]. However, the resulting one-predicate sentences will be less transparent than $\mathfrak{F}_{m,n}$ and the corresponding negative proofs will be more involved, accordingly. On the contrary, in the present paper, we look for transparent three-variable equality-free sentences of any given finite-variable proof theoretical strength, which have as simple as possible syntactical structure, so that their provability a/o improvability in Hilbert-Bernays formalisms with a given number of distinct variables can be analyzed by constructive proof theoretical means. The sentences $\mathfrak{F}_{m,n}$ provide us with handy examples.[3]

These sentences are obtained by the direct 1-order translation of the appropriate nontrivial interpretation of the Greenwood-Gleason theorem (see Proposition 8 in the text). The proofs can be loosely characterized by the appropriate game-theoretical combinatorial propositions which are also relevant to the theorem of Greenwood and Gleason (see Propositions 10, 11 below). On the other hand, the formalization of the crucial negative part of the proof is based on the appropriate finite-variable nested analogue of the familiar Herbrand lemma, which, in turn, is proved using the appropriate finite-variable Herbrand-style nested cutfree analogue of the underlying Hilbert-Bernays modus-ponens formalism. The whole approach can be also used for the sake of finite-variable proof theoretical analysis of other sentences of simple syntactical shape. The finite-variable nested cutfree proof systems under consideration also provide us with the hierarchy of 1-order automated nested theorem provers which are entirely different from the familiar tableaux a/o resolution based theorem provers. This hierarchy is parametric in the total number of different variables occurring in proofs, which has special advantages for relation algebras. In particular, the four-variable nested theorem prover is adequate to the canonical Tarski formalism of the relation algebras (see [20] for more exhaustive presentations).

In what follows we expose the main results and address the combinatorial background. In the second section we prove the combinatorial and game-theoretical propositions used. In the final, third section, we sketch the finite-variable nested cutfree

[3]Previously publicized claims and proofs related to Problem 2.12 of [11] use methods of the algebraic logic (see [13, 14, 15, 17, 18, 19]). The attached proofs are extremely specific in the chosen algebraic formalisms, but too sketchy in the correlated a/o alleged 1-order finite-variable domains. The proof theoretical approach used in the present paper can help to specify a/o clarify some of the latter (see also Remark 36 below).

proof systems, prove the finite-variable nested analogue of the Herbrand lemma and infer the main theorems from the propositions and the nested Herbrand lemma in question.

1.1 Basic Definitions and Results

For any $m, n > 0$, consider the 1-order equality-free language $\mathcal{L}_{m,n}$ with unary predicate symbols $H_0, ..., H_m$ and binary predicate symbols $C_0, ..., C_n$. Let $*_{m,n}$ be the adjacent identity-free language of relation algebras with basic (binary) relations $X_0, ..., X_m, Y_0, ..., Y_n$, basic operations $+$ (boolean sum), $\cdot$ (boolean product), $^-$ (boolean negation), 1 (boolean unity), 0 (boolean zero), $\smile$ (relational converse), $\odot$ (relational product) and $\oplus$ (relational sum). Recall that, in $*_{m,n}$, $S+T = (S^- \cdot T^-)^-$, $0 = 1^-$ and $S \oplus T = (S^- \odot T^-)^-$. Now for any $2 < \lambda < \infty$, let $\mathcal{L}^{\lambda}_{m,n}$ denote the λ-variable restriction of $\mathcal{L}_{m,n}$. By $\mathbf{HBL}_{\lambda}$ we'll denote the canonical Hilbert-Bernays *modus-ponens* formalism of $\mathcal{L}^{\lambda}_{m,n}$; thus all axioms and rules of $\mathbf{HBL}_{\lambda}$ are just λ-variable restrictions of the Hilbert-Bernays formalism **HBL** specified to $\mathcal{L}_{m,n}$. For the sake of brevity, we expose the axioms of **HBL** in the basic syntax $\{\neg, \rightarrow, \forall\}$, as follows:

1. $(A \rightarrow B) \rightarrow ((B \rightarrow G) \rightarrow (A \rightarrow G))$
2. $(\neg A \rightarrow A) \rightarrow A$
3. $B \rightarrow (\neg B \rightarrow A)$
4. $\forall x A \rightarrow A[x/y]$
5. $\forall x(D \rightarrow A) \rightarrow (D \rightarrow \forall x A)$ [x not free in D]

where A, B, D, G and x, y respectively range over formulas and (individual) variables such that x does not occur free in D, while $A[x/y]$ denotes the canonical substitution of all free occurrences of x by y, in A, that simultaneously renames all bound occurrences of y by x (cf. e.g. [2]). By definition, a formula F is *provable* in **HBL** ($\mathbf{HBL}_{\lambda}$) (mod canonical abbreviations and/or translations) iff there exists a finite sequence $F_0, F_1, ..., F_p = F$ in $\mathcal{L}_{m,n}$ ($\mathcal{L}^{\lambda}_{m,n}$) such that for each $l \leqq p$, either F_l is an instance of one of the above axioms 1-5, or $F_l = \forall x F_i$ for some $i < l$ *(generalization rule)*, or else there are F_i and $F_j = F_i \rightarrow F_l$ where $i, j < l$ *(modus-ponens rule)*.

By ϕ we denote Greenwood-Gleason's function that is defined by the two recursive equations $\phi(0) = 2$, $\phi(n{+}1) = 1{+}(n{+}2){\cdot}\phi(n)$. The following sentence $\mathfrak{F}_{m,n}$ is obtained by direct three-variable translation of the appropriate interpretation of Greenwood-Gleason's combinatorial theorem (cf. [7]); the precise meaning will be explained in the next subsection (see Remark 7 below).

(1) Definition. Let $\mathfrak{F}_{m,n}$ denote the following $\mathcal{L}^{3}_{m,n}$-sentence

$$\bigvee\{\forall x \neg H_i(x) : i \leqq m\} \vee \exists x \exists y \bigwedge\{\neg C_j(x,y) : j \leqq n\} \vee$$
$$\bigvee\{\exists x \exists y \exists z \,[H_p(x) \wedge H_q(y) \wedge H_r(z) \wedge C_j(x,y) \wedge C_j(x,z) \wedge C_j(y,z)] : \\ j \leqq n \;\&\; p < q < r \leqq m\}$$

where $\bigvee\{F_{i_1,\dots,i_k} : \varphi(i_1,\dots,i_k)\}$ and $\bigwedge\{F_{i_1,\dots,i_k} : \varphi(i_1,\dots,i_k)\}$ respectively denote the disjunction and conjunction of all formulas $F_{i_1,\dots,i_k}$ whose indices $i_1,\dots,i_k$ satisfy a finite-domain condition φ.

(2) Theorem. *If $n > 0$ and $m \geqq \phi(n)$ then $\mathfrak{F}_{m,n}$ is provable in $\mathbf{HBL}_{n+3}$, but not in $\mathbf{HBL}_{n+2}$.*

(3) Definition. Let $\mathfrak{T}_{m,n}$ denote the following $\Lambda_{m,n}$-term

$$\sum\{0 \oplus X_i^- \oplus 0 : i \leqq m\} + 1 \odot \prod\{Y_j^- : j \leqq n\} \odot 1 +$$
$$\sum\{1 \odot \left[(X_p \oplus 0) \cdot (0 \oplus X_q^\smile) \cdot Y_j \cdot \left(Y_j \odot \left(Y_j^\smile \cdot (X_r \oplus 0)\right)\right)\right] \odot 1 : \\ j \leqq n \;\&\; p < q < r \leqq m\}$$

where $\sum\{T_{i_1,\dots,i_k} : \varphi(i_1,\dots,i_k)\}$ and $\prod\{T_{i_1,\dots,i_k} : \varphi(i_1,\dots,i_k)\}$ respectively denote the boolean sum and product of all terms $T_{i_1,\dots,i_k}$ whose indices $i_1,\dots,i_k$ satisfy a finite-domain condition φ.

(4) Theorem. *If $n > 1$ and $m \geqq \phi(n)$ then the canonical 1-order translation of the algebraic equation $\mathfrak{T}_{m,n} = 1$ is provable in $\mathbf{HBL}_{n+3}$, but not in $\mathbf{HBL}_{n+2}$.*

(5) Remark. $\mathfrak{T}_{m,n} = 1$ is the canonical algebraic interpretation of $\mathfrak{F}_{m,n}$ that preserves λ-variable Hilbert-Bernays provability for every $\lambda > 3$ under the intended interpretation $H_i(x) \Leftrightarrow \exists y x X_i y$, $C_j(x,y) \Leftrightarrow x Y_j y$. The statement $\mathfrak{F}_{m,n}$ and the proof of Theorem 2 are both closely related to the appropriate combinatorial propositions and games, which are discussed below.

By the above remark, Theorem 4 is an easy consequence of Theorem 2 (see Subsection 2.1 below). The proof of Theorem 2 is exposed in the rest of Section 2. The underlying proof theoretical techniques differ in every respect from the algebraic ones used in [10, 17, 18, 21].

1.2 Combinatorial Principles and Games.

In the sequel we always assume that $n > 0$ and $m \geqq \phi(n)$ where $\phi(0) = 2$ and $\phi(n+1) = 1 + (n+2) \cdot \phi(n)$. These assumptions are sufficient for the combinatorial Greenwood-Gleason's theorem, being in fact a particular case of the Ramsey theorem. For some n, the estimate $m \geqq \phi(n)$ can be improved, but this is less relevant to the contents below. The above recursive definition of ϕ is convenient for our game-theoretical variant of Greenwood-Gleason's proof (see below proof of Proposition 10).

(6) Theorem. `(Greenwood-Gleason [7]).` *Let G be any complete indirected graph with $m+1$ distinct vertices, whose edges are colored using at most $n+1$ distinct colors. Then G contains at least one monochromatic triangle (:= a 3-vertex subgraph whose all 3 edges have one and the same color).*

(7) Remark. The following proposition provides us with a suitable interpretation of Greenwood-Gleason's theorem, whose 3-variable translation results in our sentence $\mathfrak{F}_{m,n}$, where 'H_i' and 'C_j'are "holes" and "colors", except that for the sake of brevity we deal with oriented colorings of the threads. That is, in the definition of $\mathfrak{F}_{m,n}$, we assume that colors are assigned to the ordered pairs $\langle x, y\rangle$ such that $x \in H_i$ and $y \in H_j$ where $i < j \leqq m$.

(8) Proposition. *Suppose given $m+1$ distinct (golf-)holes, $n+1$ distinct colors and an unspecified collection of balls and colored threads (each thread having exactly one color). Moreover, suppose that every hole contains at least one ball, and suppose that every two balls are connected by at least one thread. Then there are at least three different balls sitting in three different holes such that every two of these balls are connected by threads having one and the same color.*

Proof. We choose exactly one ball in each hole and exactly one colored thread connecting each pair of the chosen balls. This provides us with a complete colored graph having $m + 1$ vertices and at most $n + 1$ colors. By Greenwood-Gleason's theorem, this graph has at least one monochromatic triangle. ■

(9) Remark. From the viewpoint of finite-variable formalization, this proof is bad, for it requires too many (namely, $m + 1$) distinct variables. This is due solely to the "too liberal" choice in question, since finite-variable logic allows us to use the same names for separated bound variable-occurrences. In fact, this proof can be formalized using only $n + 3$ distinct variables (see the next section).

(10) Proposition. *Consider the following two-player infinite game. Let $t \leqq m$ be fixed. There are given $m + 1$ distinct vertices $V_0, V_1, ..., V_m$ and $n + 1$ distinct colors. Player I chooses vertices, Player II assigns colors to the adjacent edges. The game runs as follows.*

- 1^{st} *move: Player I chooses $V_{i(0)}$, and Player II responds by coloring all edges $\langle V_{i(0)}, V_j\rangle$, $j \neq i(0)$.*
- 2^{nd} *move: Player I chooses another vertex $V_{i(1)}$, Player II responds by coloring all edges $\langle V_{i(1)}, V_j\rangle$, $j \neq i(0), i(1)$.*
- 3^{rd}*move, and so on, are analogous, except that the following condition is satisfied. If after the initial $k+1$ moves the number of all actually chosen vertices is $t+1$, then Player I at his $k+2^{nd}$ move must withdraw one of those previously chosen vertices, say $V_{i(p)}$, $p \leqq k$, (thus reducing their total number back to t)*

along with all previously colored adjacent vertices $\langle V_{i(p)}, V_j\rangle$, $j \neq i(p)$, before he is allowed to pick his $k + 2^{nd}$ vertex $V_{i(k+1)}$.

Clearly, after every initial $k + 1$ moves, both players produce a complete colored graph G_{k+1} consisting of all actually chosen vertices and colored edges; note that $|G_{k+1}| = \max(t + 1, k + 1)$. Player I wins iff some G_{k+1} contains a monochromatic triangle. Otherwise, Player II wins. The proposition's claim reads: Player I has a winning strategy for any $t > n+1$ (and, in fact, he already wins at his $n+3^{rd}$ move), but Player II has a winning strategy for any $t \leqq n + 1$.

Proof. Consider the given vertices $V_0, V_1, ..., V_m$ and colors $C_0, C_1, ..., C_n$.

Case 1. Suppose $t > n + 1$. Player I's winning strategy is as follows.

- 1^{st} move: He picks any vertex $V_{i(0)}$. She colors the set of at least $\phi(n)$ edges $\{\langle V_{i(0)}, V_j\rangle : j \neq i(0)\}$ using at most $n + 1$ given colors. Since $\phi(n) = 1 + (n + 1) \cdot \phi(n - 1)$, this coloring contains at least $\phi(n - 1) + 1$ edges having the same 'dominating' color, say, $C_{d(0)}$. Denote by S_0 the collection of the adjacent, at least $\phi(n - 1) + 1$, 'dominating' vertices.
- 2^{nd} move: He picks another vertex $V_{i(1)}$ where $i(1) = \min\{j \neq i(0) : V_j \in S_0\}$. She responds by coloring the adjacent set of the at least $\phi(n - 1)$ edges $\{\langle V_{i(1)}, V_j\rangle : j \in S_0\}$. If she colored one of the latter edges by $C_{d(0)}$ then he wins. Otherwise, her coloring uses at most n distinct colors, while $\phi(n - 1) = 1 + n \cdot \phi(n - 2)$, and hence this set of edges contains at least $\phi(n-2)+1$ edges having the same 'dominating' color, say, $C_{d(1)}$. Denote by S_1 the collection of the adjacent, at least $\phi(n - 2) + 1$, 'dominating' vertices.
- 3^{rd} move, and so on, are analogous.
- $n+2^{nd}$ move: After his $n+1^{st}$ move, she had only one remaining color, say, C_r, for at least $\phi(0) = 2$ edges, say $\langle V_{i(n)}, V_e\rangle$ and $\langle V_{i(n)}, V_f\rangle$, where $S_n = \{V_e, V_f\}$. He picks $V_{i(n+1)} = V_e$. She has no choice but to assign C_r to $\langle V_e, V_f\rangle$.
- $n + 3^{rd}$ move: After his $n + 2^{nd}$ move, the set of all chosen vertices has the cardinality $n + 2 < t + 1$. He picks his $n + 3^{rd}$ vertex $V_{i(n+2)} = V_f$ and wins the game, since $\{V_{i(n)}, V_e, V_f\}$ is the desired monochromatic triangle.

Case 2. Suppose $t \leqq n + 1$. Player II's winning strategy is as follows.

- $k + 1^{st}$ move, where $k \leqq t$: Regardless of his choice of the vertex $V_{i(k)}$, she colors every edge $\langle V_{i(k)}, V_j\rangle$ by C_k. So far, her chosen colors $C_0, ..., C_k$ were all pairwise different, and hence the correlated colored graphs did not contain monochromatic triangles.
- $k + 1^{st}$ move, where $k > t$: With his chosen vertex $V_{i(k)}$ she correlates his previously chosen vertex $V_{i(p)}, p < k$, that has been removed by the condition

rule. Then she assigns to every edge $\langle V_{i(k)}, V_j\rangle$ her previously chosen color of the removed edge $\langle V_{i(p)}, V_j\rangle$. Since she always uses different colors for the edges adjacent to his different vertices, the colored graph G_{k+1} thus obtained still has no monochromatic triangle. ∎

(11) Proposition. *Consider the following two-player infinite game. There are given $m+1$ distinct (golf-)holes $H_0, H_1, ..., H_m$ and $n+1$ distinct colors, as well as balls and threads — as in Proposition 8, except that all balls are labelled by natural numbers $\leqq n+1$, and all threads are initially blank. Moreover, it is assumed that every hole contains infinitely many distinct (labelled) balls. Player I chooses holes and balls inside them, Player II assigns colors to the connecting threads. The game runs as follows.*

- 1^{st} *move: Player I chooses a ball b_0 in a chosen hole $H_{\chi(0)}$, Player II has no response.*

- 2^{nd} *move: Player I chooses a ball b_1 in a chosen hole $H_{\chi(1)}$, Player II responds by coloring the thread $\langle b_0, b_1\rangle$, provided that $\chi(0) \neq \chi(1)$ and b_0 and b_1 have different labels (otherwise, $\langle b_0, b_1\rangle$ remains blank).*

- 3^{rd} *move, and so on, are analogous, where the coloring condition reads as follows. At her $k+1^{st}$ move, Player II is allowed to color only such threads $\langle b_i, b_k\rangle$, $i < k$, that $\chi(i) \neq \chi(k)$ and the label of b_i differs from the labels of all b_j, $i < j \leqq k$, for which $\langle b_i, b_j\rangle$ is not blank (thus, in particular, b_i and b_k have different labels).*

Clearly, after every initial $k+1$ moves, both players produce a structure S_{k+1} consisting of labelled balls connected by threads some of which are colored (the rest being blank). Player I wins iff some S_{k+1} contains a monochromatic triangle. Otherwise, wins Player II. The proposition's claim reads: Player II has a winning strategy.

Proof. Consider the given holes $H_0, H_1, ..., H_m$ and colors $C_0, C_1, ..., C_n$. Denote by $\ell(b)$ the label of a chosen ball b. Player II's winning strategy is as follows. At every $k+1^{st}$ move, she assigns $C_{\min(\ell(b_j),\ell(b_k))}$ to each thread $\langle b_j, b_k\rangle$, $j < k$, satisfying the coloring condition. It is readily seen that the coloring restriction guarantees that the same color can't be assigned to every thread $\langle b_i, b_j\rangle, \langle b_j, b_k\rangle, \langle b_i, b_k\rangle$ where $i < j < k$ (and $\chi(i) \neq \chi(j) \neq \chi(k) \neq \chi(i)$, which is irrelevant to the argument). Indeed, let $\alpha = \min\{\ell(b_i), \ell(b_j)\}, \beta = \min\{\ell(b_j), \ell(b_k)\}, \gamma = \min\{\ell(b_i), \ell(b_k)\}$. We have $\ell(b_i) \neq \ell(b_j) \neq \ell(b_k) \neq \ell(b_i)\}$. So if $\alpha = \beta$ then clearly $\alpha = \beta = \ell(b_j)$, and hence $\alpha = \beta = \gamma$ implies $\ell(b_j) = \ell(b_i)$ or $\ell(b_j) = \ell(b_k)$ — a contradiction. ∎

(12) Remark. Proposition 10 generalizes Greenwood-Gleason's theorem and gives hints to the positive part of the proof of Theorem 2. Proposition 11 refers more directly to the crucial negative part of the proof of Theorem 2.

2 Proof of Theorems

2.1 On Theorem 4

Proof. We deduce Theorem 4 from Theorem 2. As already mentioned (see Remark 5), the statement of Theorem 4 is the canonical algebraic translation of the statement of Theorem 2. In order to complete the proof, it will suffice to verify that $\mathbf{HBL}_4$ preserves this translation as equivalence — this is quite routine. On the other hand, one can deduce the same conclusion from the known results, as follows. Recall that the required passage is known to hold for Tarski's 3-variable formalism $\mathcal{L}^{\times}$ (cf. e.g. [21]) extending the familiar algebraic formalism **RA**. Note that $\mathcal{L}^{\times}$ has basic symbols for both the logic equality and algebraic identity and includes as axiom the general Leibniz law. Now our expressions $\mathfrak{F}_{m,n}$ and $\mathfrak{T}_{m,n}$ don't include equality and identity, respectively. Moreover, the identity-free reduct of **RA** is axiomatizable by adding a suitable axiom (see [1]) whose canonical translation is readily provable in $\mathbf{HBL}_4$, while the general 3-variable Leibniz law is also provable in $\mathbf{HBL}_4$ (see [4, 5]). Hence the desired equivalence also holds for $\mathbf{HBL}_4$, Q.E.D. ∎

2.2 Positive Part of Theorem 2

(13) Lemma. *For any $\lambda > 0$, all boolean tautologies and 1-4 below are provable in* $\mathbf{HBL}_\lambda$.

1. $\exists xA \wedge D \longleftrightarrow \exists x(A \wedge D)$ [x not free in D]
2. $A \longleftrightarrow A'$ [$A = A' \bmod$ bound renaming]
3. $\exists x(A \vee B) \longleftrightarrow \exists xA \vee \exists xB$
4. $\exists x(A \wedge B) \rightarrow \exists xA$

Proof. This is trivial (see e.g. [9] or [4]). ∎

(14) Claim. $\mathfrak{F}_{m,n}$ *is provable in* $\mathbf{HBL}_{n+3}$.

Proof. We fix the language $\mathcal{L}^{n+3}_{m,n}$ with unary predicate symbols $H_0, ..., H_m$, binary predicate symbols $C_0, ..., C_n$ and individual variables $v_0, ..., v_{n+2}$. Rewrite $\mathfrak{F}_{m,n}$ in positive implicative form $(P \wedge Q) \rightarrow R$:

$$\begin{aligned}&(\bigwedge\{\exists x H_i(x) : i \leqq m\} \wedge \forall x \forall y \bigvee\{C_j(x,y) : j \leqq n\}) \rightarrow \\ &\bigvee\{\exists x \exists y \exists z\, [H_p(x) \wedge H_q(y) \wedge H_r(z) \wedge C_j(x,y) \wedge C_j(x,z) \wedge C_j(y,z)] : \\ &\qquad\qquad j \leqq n \ \& \ \ p < q < r \leqq m\}\end{aligned}$$

whose two conjuncts and the conclusion we denote by P, Q and R, respectively. Now, in $\mathcal{L}^{n+3}_{m,n}$, denote by $\Delta_{i(0),\dots,i(k+1);d(0),\dots,d(k)}(v_0, ..., v_{k+1})$ the conjunction:

$$
\begin{aligned}
&H_{i(0)}(v_0) \wedge ... \wedge H_{i(k+1)}(v_{k+1})\\
&\wedge C_{d(0)}(v_0, v_1) \wedge ... \wedge C_{d(0)}(v_0, v_{k+1})\\
&\wedge C_{d(1)}(v_1, v_2) \wedge ... \wedge C_{d(1)}(v_1, v_{k+1})\\
&...\\
&\wedge C_{d(k)}(v_k, v_{k+1}),
\end{aligned}
$$

for any $k \leqq n+1$, any $i(0) < ... < i(k) \leqq m$ and any $d(0), ..., d(k) \leqq n$. By Lemma 13(4), $\exists v_0 ... \exists v_{n+2} \Delta_{i(0),...,i(n+2);d(0),...,d(n+1)}(v_0, ..., v_{n+2}) \to R$ holds in $\mathbf{HBL}_{n+3}$, since there are at least two equal indices among $d(0), ..., d(n+1) \leqq n$ (Dirichlet principle); then for any $j = d(\alpha) = d(\beta) \leqq n$ where $\alpha < \beta \leqq n+1$ we put $x := v_\alpha$, $y := v_\beta$, $z := v_{\beta+1}$, $p := i(\alpha)$, $q := i(\beta)$, $r := i(\beta+1)$ and observe that 1^{st}, α^{th} and β^{th} rows of $\exists v_0 ... \exists v_{n+2} \Delta_{i(0),...,i(n+2);d(0),...,d(n+1)}(v_0, ..., v_{n+2})$ provide us with the desired disjunct of R. In what follows we can safely put $i(0) = 0$. By the axiom 4 of $\mathbf{HBL}_{n+3}$ and Lemma 13(1,2), we first pass from $P \wedge Q$ to:

$$
\begin{aligned}
\exists v_0 \{H_0(v_0) &\wedge \exists v_1 [H_1(v_1) \wedge (C_0(v_0, v_1) \vee ... \vee C_n(v_0, v_1)]\\
&\wedge \exists v_1 [H_2(v_1) \wedge (C_0(v_0, v_1) \vee ... \vee C_n(v_0, v_1)]\\
&\quad ...\\
&\wedge \exists v_1 [H_m(v_1) \wedge (C_0(v_0, v_1) \vee ... \vee C_n(v_0, v_1)]\},
\end{aligned}
$$

from which, by boolean distributive law together with Lemma 16(3), we arrive at the disjunction of all possible conjunctions of the form:

$$
\begin{aligned}
\exists v_0 \{H_0(v_0) &\wedge \exists v_1 [H_1(v_1) \wedge C_{j(1)}(v_0, v_1)]\\
&\wedge \exists v_1 [H_2(v_1) \wedge C_{j(2)}(v_0, v_1)]\\
&\quad ...\\
&\wedge \exists v_1 [H_m(v_1) \wedge C_{j(m)}(v_0, v_1)]\}.
\end{aligned}
$$

By the basic combinatorial (Greenwood-Gleason's) argument used in the proof of Proposition 10, above, every conjunction contains a subconjunction of the form:

$$
\begin{aligned}
\exists v_0 \{H_0(v_0) &\wedge \exists v_1 [H_{j(1)}(v_1) \wedge C_{d(0)}(v_0, v_1)]\\
&\wedge \exists v_1 [H_{j(2)}(v_1) \wedge C_{d(0)}(v_0, v_1)]\\
&\quad ...\\
&\wedge \exists v_1 [H_{j(k+1)}(v_1) \wedge C_{d(0)}(v_0, v_1)]\},
\end{aligned}
$$

where $k \geqq \phi(n-1)$ and $0 < j(1) < ... < j(k+1) \leqq m$. We then pass to this subconjunction, by Lemma 13(4), and then let $i(1) = j(1)$ and rename v_1 by v_2 everywhere below the second line, by Lemma 13(2). As a result, we obtain, for every disjunction in question, a proof of:

$$\begin{aligned}\exists v_0 \exists v_1 \{\Delta_{0,i(1);d(0)}(v_0, v_1) & \wedge \exists v_2 \Delta_{0,j(2);d(0)}(v_0, v_2) \\ & \wedge \exists v_2 \Delta_{0,j(3);d(0)}(v_0, v_2) \\ & \quad ... \\ & \wedge \exists v_2 \Delta_{0,j(k+1);d(0)}(v_0, v_2)\}.\end{aligned}$$

By the same token, we further apply Q by inserting $\wedge \bigvee\{C_j(v_1, v_2) : j \leqq n\}$ in the scope of every $\exists v_2$ in the latter sentence, which, by Lemma 13(1,2) and boolean distributivity, allows us to transform every subconjunction in question into a disjunction of all possible conjunctions of the form:

$$\begin{aligned}\exists v_0 \exists v_1 \{\Delta_{0,i(1);d(0)}(v_0, v_1) & \wedge \exists v_2 [\Delta_{0,j(2);d(0)}(v_0, v_2) \wedge C_{d(1)}(v_1, v_2)] \\ & \wedge \exists v_2 [\Delta_{0,j(3);d(0)}(v_0, v_2) \wedge C_{d(1)}(v_1, v_2)] \\ & \quad ... \\ & \wedge \exists v_2 [\Delta_{0,j(k+1);d(0)}(v_0, v_2) \wedge C_{d(1)}(v_1, v_2)]\},\end{aligned}$$

where $k \geqq \phi(n-2)$ and $0 < i(1) < j(2) < j(3) < ... < j(k+1) \leqq m$. Having this, we argue as above, i.e., by setting $i(2) = j(2)$ and renaming v_2 by v_3, in order to pass to:

$$\begin{aligned}\exists v_0 \exists v_1 \exists v_2 \{\Delta_{0,i(1),i(2);d(0),d(1)}(v_0, v_1, v_2) & \wedge \exists v_3 \Delta_{0,i(1)j(3);d(0),d(1)}(v_0, v_1, v_3) \\ & \wedge \exists v_3 \Delta_{0,i(1),j(4);d(0),d(1)}(v_0, v_1, v_3) \\ & \quad ... \\ & \wedge \exists v_3 \Delta_{0,i(1),j(k+1);d(0),d(1)}(v_0, v_1, v_3)\}.\end{aligned}$$

We continue this procedure until every disjunction in question contains a conjunct of the desired form $\exists v_0 ... \exists v_{n+2} \Delta_{i(0),...,i(n+2);d(0),...,d(n+1)}(v_0, ..., v_{n+2})$, which yields the required conclusion R using the $n+3$ variables $v_0, ..., v_{n+2}$ only. ∎

2.3 Negative Part of Theorem 2: Proof Outline

We reduce the remaining negative part of Theorem 2 to Proposition 11 via a suitable *nested* variant of the familiar Herbrand-style result known as Herbrand lemma. Recall that the familiar "plain" Herbrand lemma, in the simplest form, states that a given 1-order formula F in positive normal form without universal quantifiers and function symbols is valid iff so is its propositional Herbrand expansion F^* that is obtained by successively rewriting, in F, every subformula $\exists x A$ for boolean A to the disjunction $\bigvee\{A[x/v_i] : v_i$ *occurs (free or bound) in* $F\}$. (Recall that $v_0, ..., v_i, ...$ is a fixed enumeration of all 1-order individual variables.) For example, if we take $F = \mathfrak{F}_{m,n}$ and let G be the corresponding $\forall$-free formula in $\mathcal{L}^{m+1}_{m,n}$ that arises from F by rewriting every subformula $\forall x \neg H_i(x)$ to $\neg H_i(v_i)$, then 1-order validity of F is equivalent to propositional validity of the corresponding quantifier-free $\mathcal{L}^{m+1}_{m,n}$-formula

G^*. Of course, this estimate is too loose for our considerations, since m is just too big as compared to n. Furthermore, what actually has to be evaluated is not the 1-order validity, but provability in $\mathbf{HBL}_{n+2}$ whose set of individual variables is just too small to eliminate all universal quantifiers. Therefore, we introduce a more sophisticated *nested* analogue of the Herbrand lemma (see Lemma 19 below). In the sequel we work in the *positive* λ-variable language $\mathcal{L}^\lambda$ with propositional connectives $\vee, \wedge$, quantifiers $\exists, \forall$, in which only atoms are in the domain of $\neg$. Actually, we are only interested in the particular case $\mathcal{L}^\lambda = \mathcal{L}^\lambda_{m,n}$, but all definitions and results hold true for any $\mathcal{L}^\lambda$ in question. We call *disjunctive equivalence* the disjunctive fragment of the ordinary boolean equality. To put it more precisely, two formulas are disjunctively equivalent iff they are equivalent modulo associativity, commutativity and contraction of disjunctions (but not conjunctions); it can also be viewed as the ordinary (hereditarily finite) set-theoretical equality on formulas, where disjunctions (possibly iterated) are regarded as sets of disjuncts. As usual, by L (possibly indexed) we denote arbitrary literals, i.e. either atomic formulas or their negations.

(15) Definition. Call '*purely existential*' any 1-order formula (or sentence) F of the shape $\exists x_1 ... \exists x_k \bigwedge \{L_i : i < r\}$ where $k, r > 0$, $x_1, ..., x_k$ any individual variables. Moreover, for any $\lambda > 0$ and any purely existential formula F, as above, we call the (uniquely determined) DNF $\bigvee \{\bigwedge \{L_i [x_0 / v_{l_1}] ... [x_k / v_{l_k}] : i < r\} : l_1, ..., l_k < \lambda\}$ the '*λ-variable boolean expansion of F*'.

(16) Definition. Let $s, \lambda > 0$, and let $\mathcal{L}(\overrightarrow{Ⓢ}_s)$ be the extension of the underlying 1-order language $\mathcal{L}$ without function symbols and equality by adding s distinct 0-ary special predicate symbols $Ⓢ_1, ..., Ⓢ_s$ not occurring in $\mathcal{L}$. As above, by $\mathcal{L}^\lambda$ and $\mathcal{L}^\lambda(\overrightarrow{Ⓢ}_s)$ we denote the corresponding λ-variable restrictions. Arguing in $\mathcal{L}(\overrightarrow{Ⓢ}_s)$ (in fact, in $\mathcal{L}^\lambda(\overrightarrow{Ⓢ}_s)$) we define $\textsc{up}(s, \lambda)$ (in words: *the class of universal s, λ-polynomials*) by recursion (mod disjunctive equivalence):

1. $\emptyset$ (the empty formula) is in $\textsc{up}(s, \lambda)$,
2. for any $0 < \sigma \leqq s$, $Ⓢ_\sigma$ is in $\textsc{up}(s, \lambda)$,
3. for any $l < \lambda$ and unary literal $L(v_l)$ of $\mathcal{L}$, if U is in $\textsc{up}(s, \lambda)$ then so is $\forall v_l (L(v_l) \vee U)$.

(17) Definition. For any $U, W \in \textsc{up}(s, \lambda)$, we let $U \sqsubseteq W$ (in words: *W extends U*) if either $U = W$ or else U can be obtained (mod disjunctive equivalence) from W by chain of successive rewriting of arbitrary subformulas $\forall x (L(x) \vee V \vee V')$ to $\forall x (L(x) \vee V) \vee V'$, where $V, V' \in \textsc{up}(s, \lambda)$ while $L(x)$ ranging over unary literals of $\mathcal{L}^\lambda$.

(18) Definition. For any $U \in \textsc{up}(s, \lambda)$ and $\mathcal{L}^\lambda$-formulas $G_1, ..., G_s$, denote by $U[Ⓢ_1 := G_1, ..., Ⓢ_s := G_s]$ the corresponding $\mathcal{L}^\lambda$-realization that arises by rewriting in U every occurrence of $Ⓢ_\sigma$ to G_σ $(0 < \sigma \leqq s)$. An $\mathcal{L}^\lambda$-realization is called '*purely existential*' iff so are all G_σ.

(19) Lemma. *For any $U \in \textsc{up}(s,\lambda)$, any purely existential $\mathcal{L}^{\lambda}$-sentences $S_1, ..., S_s$ and their λ-variable boolean expansions $S_1^0, ..., S_s^0$, respectively, the following holds. $U[Ⓢ_1 := S_1, ..., Ⓢ_s := S_s]$ is provable in the correlated λ-variable Hilbert-Bernays formalism $\mathbf{HBL}_\lambda$ iff there exists a $W \in \textsc{up}(s,\lambda)$ such that $U \sqsubseteq W$ and $W[Ⓢ_1 := S_1^0, ..., Ⓢ_s := S_s^0]$ is 1-order valid.*

(20) Example. Let $U = \forall x H(x) \vee Ⓢ$ and $S = \exists x \neg H(x)$ where $s = \lambda = 1$, $x = v_0$, $Ⓢ = Ⓢ_1$. Thus $U \in \textsc{up}(1,1)$. Note that $U[Ⓢ := S]$ is valid. Moreover $S^0 = \neg H(x)$. Now let $W = \forall x (H(x) \vee Ⓢ)$. Clearly $U \sqsubseteq W \in \textsc{up}(1,1)$. Furthermore, $W[Ⓢ := S^0] = \forall x (H(x) \vee \neg H(x))$ is valid, and hence, by the lemma, $U[Ⓢ := S]$ is provable in the correlated 1-variable formalism $\mathbf{HBL}_1$. On the other hand, $U[Ⓢ := S^0] = \forall x H(x) \vee \neg H(x)$ is obviously invalid, which shows that the condition $U \sqsubseteq W$ is essential for the conclusion.

(21) Example. Now let $\mathcal{L} = \mathcal{L}_{m,n}$ (see 1.1) and recall that our sentence $\mathfrak{F}_{m,n}$ is

$$\begin{aligned}&\bigvee\{\forall x \neg H_i(x) : i \leqq m\} \vee \exists x \exists y \bigwedge\{\neg C_j(x,y) : j \leqq n\} \vee \\ &\bigvee\{\exists x \exists y \exists z\, [H_p(x) \wedge H_q(y) \wedge H_r(z) \wedge C_j(x,y) \wedge C_j(x,z) \wedge C_j(y,z)] : \\ &\qquad\qquad j \leqq n \ \&\ p < q < r \leqq m\}.\end{aligned}$$

Let $S_1 := \exists x \exists y \bigwedge\{C_j(x,y) : j \leqq n\}$, and for any $j \leqq n$, $p < q < r \leqq m$, let

$$S_{\langle p,q,r,j\rangle} := \exists x \exists y \exists z\, [H_p(x) \wedge H_q(y) \wedge H_r(z) \wedge C_j(x,y) \wedge C_j(x,z) \wedge C_j(y,z)].$$

There are $\frac{n+1}{6} m(m^2-1)$ such $\langle p,q,r,j\rangle$, so let $s = 1 + \frac{n+1}{6} m(m^2-1)$, $\lambda = n+2$, $x = v_0$, $y = v_1$, $z = v_2$, and let $\kappa \mapsto \langle \kappa(0), \kappa(1), \kappa(2), \kappa(3)\rangle$, where $1 < \kappa \leqq s$, be a chosen bijective enumeration of all quadruples $\langle p,q,r,j\rangle$ in question.This yields

$$\begin{aligned}&\mathfrak{F}_{m,n} = \bigvee\{\forall v_\iota H_i(v_\iota) : i \leqq m \ \&\ \iota < \lambda\} \vee S_1 \vee ... \vee S_s, \text{ where for any } 1 < \kappa \leqq s, \\ &S_\kappa = \exists x \exists y \exists z [\neg H_{\kappa(0)}(x) \wedge \neg H_{\kappa(1)}(y) \wedge \neg H_{\kappa(2)}(z) \wedge \neg C_{\kappa(3)}(x,y) \wedge \\ &\qquad\qquad \neg C_{\kappa(3)}(x,z) \wedge \neg C_{\kappa(3)}(y,z)].\end{aligned}$$

For every $0 < \sigma \leqq s$, let S_σ^0 be the λ-variable boolean expansion of S_σ. Let

$$U = \bigvee\{\forall v_\iota H_i(v_\iota) : i \leqq m \ \&\ \iota < \lambda\} \vee Ⓢ_1 \vee ... \vee Ⓢ_s$$

be the correlated polynomial from $\textsc{up}(s,\lambda)$ in $\mathcal{L}(\overrightarrow{Ⓢ}_s)$. Now suppose $U[Ⓢ_1 := S_1, ..., Ⓢ_s := S_s]$ is provable in the corresponding λ-variable Hilbert-Bernays formalism $\mathbf{HBL}_\lambda$. Then, by the lemma, there exists a W from $\textsc{up}(s,\lambda)$ such that $U \sqsubseteq W$ and $W[Ⓢ_1 := S_1^0, ..., Ⓢ_s := S_s^0]$ is 1-order valid.

We postpone the proof of the lemma to the next subsection, since it is based on a special, 'nested' cutfree variant of the finite-variable Hilbert-Bernays formalism. We complete this subsection by deducing from the last example and Proposition 11 the desired

(22) Claim. *$\mathfrak{F}_{m,n}$ is not provable in $\mathbf{HBL}_{n+2}$.*

Proof. Let $\mathcal{L}, s, \lambda$ and $S_1, ..., S_s$ be as in Example 21, and let $U \in \mathrm{UP}(s, \lambda)$. For any $U' \in \mathrm{UP}(s, \lambda)$ define a natural number $\forall^\natural(U')$ (in words: the $\forall$-depth of U') by recursion: $\forall^\natural(\emptyset) = \forall^\natural(Ⓢ_\sigma) = 0, \forall^\natural(F \vee G) = \max\{\forall^\natural(F), \forall^\natural(G)\}, \forall^\natural(\forall v_\iota(H(v_\iota) \vee F)) = 1 + \forall^\natural(F)$. Note that $U' \sqsubseteq U''$ implies $\forall^\natural(U') \leqq \forall^\natural(U'')$. For any t, let $\mathrm{UP}(U, s, \lambda, t)$ be the finite (mod disjunctive equivalence) class of all $W \in \mathrm{UP}(s, \lambda)$ such that $U \sqsubseteq W$ and $t = \forall^\natural(W)$. We observe that every $\mathrm{UP}(U, s, \lambda, t)$ contains the (uniquely determined) 'maximal' polynomial U_t that is defined recursively as follows.

- $U_0 = \emptyset$
- $U_{r+1} = \bigvee\{\forall v_l(H_i(v_l) \vee Ⓢ_1 \vee ... \vee Ⓢ_s \vee U_r) : l < \lambda \ \&\ i \leqq m\}$

It is readily seen that if $W[Ⓢ_1 := S_1^0, ..., Ⓢ_s := S_s^0]$ is 1-order valid for some $W \in \mathrm{UP}(U, s, \lambda, t)$ then $U_t[Ⓢ_1 := S_1^0, ..., Ⓢ_s := S_s^0]$ is also 1-order valid; in the sequel we denote the latter realization by R_t. On the other hand, if $\mathfrak{F}_{m,n}$ is provable in $\mathbf{HBL}_{n+2} = \mathbf{HBL}_\lambda$ then so is $U[Ⓢ_1 := S_1, ..., Ⓢ_s := S_s]$, since this realization is an isomorphic syntactical (mod bound renaming) variant of $\mathfrak{F}_{m,n}$ that arises by interchanging every H_i with $\neg H_i$ and every C_j with $\neg C_j$. Hence, by Example 24, in order to prove the claim, *ad contrario*, it will suffice to show that every R_t is in fact 1-order invalid, where we can safely assume $t > 1$. To this end, we first observe that dropping all $\forall v_l$, $l \leqq n + 1$, will reduce R_t to a propositional formula. Furthermore, R_t is 1-order valid iff dropping all $\forall v_l$ while simultaneously renaming v_l by chosen, pairwise distinct variables from the new, infinite variable domain $b_0, ..., b_\iota, ...$ ($v_l \neq b_\iota$) will reduce R_t to a propositional tautology — call it P_t. This propositional formula P_t, can just as well be referred to as the ordinary *qf-expansion* of R_t according to a chosen separation of the universal variables occurring in R_t. Now consider the shape of P_t. Clearly, P_t is a DNF whose disjuncts are either $H_i(b_\iota)$ or $C_{\varepsilon,\zeta} := \bigwedge\{C_j(b_\varepsilon, b_\zeta) : j \leqq n\}$, or else $M_{\kappa,\xi,\varepsilon,\zeta} := \neg H_{\kappa(0)}(b_\xi) \wedge \neg H_{\kappa(1)}(b_\varepsilon) \wedge \neg H_{\kappa(2)}(b_\zeta) \wedge \neg C_{\kappa(3)}(b_\xi, b_\varepsilon) \wedge \neg C_{\kappa(3)}(b_\xi, b_\zeta) \wedge \neg C_{\kappa(3)}(b_\varepsilon, b_\zeta)$, for any $i \leqq m$, $1 < \kappa \leqq s$, $\iota \leqq t^*$ and some $\xi, \varepsilon, \zeta \leqq t^*$, where $t^* < |U_t|$ is fixed. Note that $C_{\varepsilon,\zeta}$ and $M_{\kappa,\xi,\varepsilon,\zeta}$ are the resulting renaming of S_1^0 and S_κ^0, respectively. Moreover, it is readily seen that P_t is a tautology iff for every choice-function $f : \langle \varepsilon, \zeta \rangle \mapsto f(\varepsilon, \zeta) \leqq n$, the resulting structure $\mathfrak{S}_t^f = \langle \{H_i(b_\iota) : i \leqq m \ \&\ \iota \leqq w^*\}, \{C_{f(\varepsilon,\zeta)}(b_\varepsilon, b_\zeta) : \varepsilon, \zeta \leqq t^* : C_{\varepsilon,\zeta} \text{ occurs in } P_t\} \rangle$ has a monochromatic triangle. Indeed, by the distributive law and basic (resolution or tableau) characterization of boolean validity, it is necessary and sufficient that, for every f in question, P_t contains at least one $M_{\kappa,\xi,\varepsilon,\zeta}$ whose all 6 conjuncts's atoms occur in $\mathfrak{S}_t^f$ — which obviously yields the result. Having noticed this, we wish to refine the necessity condition. To this end, we first refine U_t by dropping $Ⓢ_\kappa$ for all $\kappa > 1$; denote the resulting restricted polynomial, its boolean realization and the correlated propositional formula, as above, by U_t^-, R_t^- and P_t^-, respectively. Now take P_t^- and consider the following 'choice-generator'. Pick exactly one (quantified) disjunct D_0^{R} of the disjunction R_t^-, and fix the adjacent disjunct D_0^{P} of P_t^-. Note that D_0^{P} is one-one uniquely determined by the atom $H_{\chi(0)}(b_{\theta(0)})$ corresponding to the chosen prefix $\forall v_\iota(H_{\chi(0)}(v_\iota)$ of R_t^-. Next pick exactly one quantified disjunct D_1^{R} of the disjunction D_0^{R}, and fix the adjacent disjunct D_1^{P} of P_t^-. Note that D_1^{P} is one-one uniquely determined by the atom $H_{\chi(1)}(b_{\theta(1)})$ corresponding to the chosen prefix $\forall v_l(H_{\chi(1)}(v_l)$ of D_1^{R}. In D_1^{P}, pick exactly one conjunct $C_q(b_{\xi(0)}, b_{\xi(1)})$ from every conjunction $C_{\xi(0),\xi(1)}$ that arises from the corresponding boolean disjunct of D_1^{R}, where by definition $\langle \xi(0), \xi(1)\rangle := \langle \theta(0), \theta(1)\rangle$ *if* $\chi(0) < \chi(1)$ *and* $\langle \xi(0), \xi(1)\rangle := \langle \theta(1), \theta(0)\rangle$ *if* $\chi(1) < \chi(0)$, *else undefined*. Next pick exactly one quantified disjunct D_2^{R} of the disjunction D_1^{R}, and so on, such that the subsequently chosen conjuncts $C_q(b_{\theta(i)}, b_{\theta(k)})$ provide us with partial choice-functions f which are defined for all $C_{\theta(i),\theta(k)}$ for which $H_{\chi(i)}(b_{\theta(i)}), H_{\chi(k)}(b_{\theta(k)})$, where $i \neq k$ and $b_{\theta(i)} \neq b_{\theta(k)}$, are the previously chosen prefix-atoms such that $\chi(i) < \chi(k)$. Obviously, $\forall^\natural(D_i^{\mathrm{R}}) > \forall^\natural(D_{i+1}^{\mathrm{R}})$ holds for every admissible i, which yields $i < t$, and hence this generating procedure actually runs by induction on $t = \forall^\natural(U_t)$. Moreover, arguing by induction on t, one can easily prove that if P_t is a tautology then there is at least one such partial choice-functions f for which the resulting partial structure

$$\widetilde{\mathfrak{S}}_t^f = \left\langle \{H_{\chi(j)}(b_{\theta(j)}) : j \leqq t\}, \{C_{f(\xi(i),\xi(k))}(b_{\xi(i)}, b_{\xi(k)}) : i \neq k \leqq t\}\right\rangle$$

already contains a monochromatic triangle. In order to complete the proof, it will suffice to interpret the latter conclusion in game-theoretical terms of Proposition 11. Obviously, without loss of generality we can rename the new variables such that $\theta(\iota) = \iota$, in order to rewrite the sequence of chosen prefix-atoms in the form $H_{\chi(0)}(b_0)$, $H_{\chi(1)}(b_1), ..., H_{\chi(t)}(b_t)$. Furthermore, we observe that separating universal variables in R_t^- provides us with the 'label-assignment' $\ell : b_\iota \mapsto \ell(b_\iota) \leqq n+1$, $\iota \leqq t$, where by definition $\ell(b_\iota) := \ell(\iota) = l$ iff b_ι is the renamed universal variable v_l from R_t^-. Moreover, we observe that $C_{i,k}$ occurs in P_t iff the occurrence Ⓢ$_1$ in U_t corresponding to the underlying boolean expansion G_1^0 occurs in the scope of every $\forall v_{\ell(j)}$ where $i \leqq j \leqq k$ or $k < j \leqq i$. From this we infer that if $i < k$ and both $\ell(i) \neq \ell(j)$ and $\chi(i) \neq \chi(j)$ hold for all $i < j \leqq k$, then the unordered pair $\langle b_i, b_k\rangle$ must be in the domain of f. But this is exactly the coloring condition of Proposition 11. Summing up, from boolean validity of P_t we infer that Player I has a winning strategy in the game of Proposition 11 — a contradiction. Hence P_t^s is invalid for all t, and hence so is R_t^s, Q.E.D. ■

2.4 Proof of the Nested Herbrand Lemma

In this subsection we prove a suitable generalization of Lemma 19 — Lemma 34 below — to be referred to as *the* nested Herbrand lemma. This generalization deals with *uni-*

versal multinomials and *semiexistential* formulas in $\mathcal{L}^\lambda(\overrightarrow{\circledS}_s)$ and $\mathcal{L}^\lambda$, respectively. The crux of the proof is to replace the *modus-ponens* λ-variable formalism $\mathbf{HBL}_\lambda$ by the appropriate, equivalent Herbrand-style *nested cutfree sequent calculus* NSC_λ (being in turn a suitable *nested* generalization of the familiar Gentzen-Schütte-Rasiowa-Tait *cutfree sequent calculus* which, in the original "plain" form, is too weak in the finite variable domain). We adopt basic notations of the previous subsection. Now by definition NSC_λ is a formula-rewriting system in the language $\mathcal{L}^\lambda$ (in particular, $\mathcal{L}^\lambda_{m,n}$) whose rewrite rules are:

1. $A \vee B \hookrightarrow B \vee A$
2. $C \vee (A \vee B) \hookrightarrow (C \vee A) \vee B$
3. $C \vee (A \wedge B) \hookrightarrow (C \vee A) \wedge (C \vee B)$
4. $A \hookrightarrow A \vee A$
5. $\top \vee A \hookrightarrow \top$
6. $\top \wedge \top \hookrightarrow \top$
7. $L \vee \neg L \hookrightarrow \top$
8. $\forall x D \hookrightarrow D$ $\quad$ `[`x `not free in` D`]`
9. $\forall x A \vee D \hookrightarrow \forall x (A \vee D)$ $\quad$ `[`x `not free in` D`]`
10. $A[x/y] \hookrightarrow \forall x A$
11. $\exists x A \hookrightarrow A[x/y]$

where $A, B, C, D \mid L \mid x, y \mid \top$ are arbitrary formulas | literals | variables | the truth value in $\mathcal{L}^\lambda$, respectively, such that x is not free in D. By definition, a formula F is *derivable* in NSC_λ (mod canonical abbreviations and/or translations) iff there exists a finite sequence $F = F_0, F_1, ..., F_p = \top$ in $\mathcal{L}^\lambda$ such that for each $l < p$, F_{l+1} arises by rewriting, in F_l, a chosen positive (i.e. non-negated) subformula Q to Q', which we abbreviate $F_{l+1} = F_l[Q \hookrightarrow Q']$, provided that $Q \hookrightarrow Q'$ is a rewrite rule from the above list 1–11. Generally, for any rewrite rule $Q \hookrightarrow Q'$ of NSC_λ, we call the analogous relation $F' = F[Q \hookrightarrow Q']$ a *reduction*, in NSC_λ, corresponding to $Q \hookrightarrow Q'$; this relation we abbreviate by $F \rightsquigarrow F'$. Furthermore, by $F \rightsquigarrow^* F^*$ we denote the corresponding transitive reduction $F = F_0 \rightsquigarrow F_1 \rightsquigarrow ... \rightsquigarrow F_p = F^*$, which we also call a *reduction chain* from F to F^*, in NSC_λ. Thus, in particular, F is derivable in NSC_λ iff there exists a reduction chain from F to $\top$ in NSC_λ. Note that all reductions are valid inverse implications, i.e. for every reduction $F \rightsquigarrow F'$, the implication $F' \rightarrow F$ is 1-order valid. In what follows we often drop associative dis/con/junctive brackets and use without proof the admissibility in NSC_λ of the associative and commutative dis/con/junctive laws and bound renaming (cf. [6]). The corresponding *Nested Hauptsatz* reads:

(23) Theorem. *Any $\mathcal{L}^\lambda$-formula F is provable in* $\mathbf{HBL}_\lambda$ *iff it is derivable in* NSC_λ.

Proof. See [2, 3] for an outline and correction with regard to a slightly different proof system, called RPC_λ. An exhaustive proof of the theorem as stated above is exposed in [6]. ∎

Working in $\mathcal{L}^\lambda(\overrightarrow{Ⓢ}_s)$, let $\mathsf{NSC}_\lambda(\overrightarrow{Ⓢ}_s)$, be the analogous proof system that extends NSC_λ by adding, for purely technical reasons, to the basic rules 1–11 the following new rewrite rules 4^+, 9^+, while assuming that all special symbols $Ⓢ_\sigma$ are 0-ary predicate symbols thus having no free variables.

4^+. $Ⓢ_\sigma \hookrightarrow Ⓢ_\sigma \vee ... \vee Ⓢ_\sigma$ `[of any positive length]`

9^+. $Ⓢ_\sigma \vee D \hookrightarrow \forall x(Ⓢ_\sigma \vee ... \vee Ⓢ_\sigma \vee D)$ `[of any positive length;` x `not free in` D`]`

Note that 4^+ and 9^+ are easily derivable by 4 plus trivial repetition $A \hookrightarrow A$ and 4 plus 10, respectively. Furthermore, in the proofs below, we also introduce *complex special symbols* $Ⓢ_{\sigma,i,j}$ which are treated as ordinary literals which include (free) variables; these complex special symbols by definition don't obey the new rules $4^+, 9^+$.

(24) Definition. Let $F = \exists x_1 ... \exists x_k \bigwedge\{L_i : i < r\}$ be any purely existential formula in $\mathcal{L}^\lambda$, $k, r > 0$. We say G is '*generated*' by F if there exists a reduction chain from F to G or a disjunct of G, in NSC_λ. Moreover, we call such G '*semiexistential*' if it is not a disjunction. For any semiexistential G, we denote by $G[\exp]$ the correlated '*semiboolean*' expansion of G that arises by successively replacing, in G, every subformula $\exists xB$ by $\bigvee\{B[x/v_l] : l < \lambda\}$. Thus semiboolean formulas don't contain existential quantifiers.

(25) Definition. Working in $\mathcal{L}^\lambda(\overrightarrow{Ⓢ}_s)$, we define $\textsc{um}(s, \lambda)$ (in words: *the class of universal s, λ-multinomials*; abbr.: U, V, W, possibly indexed) by recursion (mod disjunctive equivalence):

1. $\emptyset$ (the empty formula) is in $\textsc{um}(s, \lambda)$,
2. any $\mathcal{L}^\lambda$-literal is in $\textsc{um}(s, \lambda)$,
3. any (finite) $\mathcal{L}^\lambda(\overrightarrow{Ⓢ}_s)$-conjunction $Ⓢ_{\sigma(1)} \wedge ... \wedge Ⓢ_{\sigma(r)}$ is in $\textsc{um}(s, \lambda)$, $r > 0$,
4. $\textsc{um}(s, \lambda)$ is closed under $\vee$ and $\forall v_l$, $l < \lambda$,
5. if $U \vee (V \wedge W)$ is in $\textsc{um}(s, \lambda)$ then so is $(U \vee V) \wedge (U \vee W)$.

(26) Definition. For any $U \in \textsc{um}(s, \lambda)$, let $\overrightarrow{O} = O_1, ..., O_t = Ⓢ_{\sigma(1)}, ..., Ⓢ_{\sigma(t)}$ be the list of all of occurrences in U of the special symbols from $\overrightarrow{Ⓢ}_s$. Let $\overrightarrow{F} = F_{\sigma(1)}, ..., F_{\sigma(t)}$

and $\overrightarrow{G} = G_1, ..., G_t$ be two adjacent lists of respectively purely existential and semiexistential formulas, in $\mathcal{L}^\lambda$, such that every G_r is generated by $F_{\sigma(r)}$ ($0 < r \leqq t$, $0 < \sigma(r) \leqq s$). We say that $\overrightarrow{G}$ is correlated with $\overrightarrow{O}, \overrightarrow{F}$ if for any $0 < i \neq j \leqq t$ it holds that $\sigma(i) = \sigma(j)$ implies $F_{\sigma(i)} = F_{\sigma(j)}$. That is to say, with equal special symbols Ⓢ we correlate equal purely existential formulas F, although their generated semiexistential formulas G might be different.

(27) Definition. Let $U, \overrightarrow{O}, \overrightarrow{F}, \overrightarrow{G}$ be as above, i.e. $\overrightarrow{G}$ is correlated with $\overrightarrow{O}, \overrightarrow{F}$. Denote by $U[\overrightarrow{O} := \overrightarrow{G}]$ the adjacent $\mathcal{L}^\lambda$-realization that arises by rewriting in U every occurrence O_r to G_r ($0 < r \leqq t$). In the same context, we denote by $U[\overrightarrow{Ⓢ} := \overrightarrow{F}]$ the analogous realization with respect to the "original" purely existential list $\overrightarrow{F}$. Now let $\overrightarrow{F}[\exp] = F_{\sigma(1)}[\exp], ..., F_{\sigma(t)}[\exp]$ and $\overrightarrow{G}[\exp] = G_1[\exp], ..., G_t[\exp]$ be the correlated lists of boolean and semiboolean expansions, respectively. Then by $U[\overrightarrow{Ⓢ} := \overrightarrow{F}[\exp]]$ and $U[\overrightarrow{O} := \overrightarrow{G}[\exp]]$ we denote the corresponding boolean and semiboolean realizations, respectively. In the sequel we denote arbitrary realizations by X, Y, Z (possibly indexed). Thus X, Y, Z in fact range over arbitrary $\mathcal{L}^\lambda$-formulas. By $U[O_r := V]$ and $X[G_r := Y]$ we denote the corresponding composite multinomial and realization which arise by rewriting in U and $X = U[\overrightarrow{O} := \overrightarrow{G}]$ a chosen O_r and G_r to V and Y, respectively.

(28) Definition. For any $U \in \text{UM}(s, \lambda)$, a $\mathcal{L}^\lambda$-realization $X = U[\overrightarrow{O} := \overrightarrow{G}]$ is called '*normal*' if the following two conditions hold. First, every maximal existential subformula of X occurs in the list $\overrightarrow{G}$. That is, if $\exists yY$ is an occurrence which is not a proper subformula of any occurrence $\exists zZ$, in X, then $\exists yY = G_r$ for some $0 < r \leqq t$. Second, every conjunction occurring in X is multinomial. That is, if $Y \wedge Z$ occurs in X then U has an occurrence $V \wedge W$ whose subrealization induced by X coincides with $Y \wedge Z$.

(29) Definition. Let $F_\sigma = \exists x_1 ... \exists x_k \bigwedge\{L_{\sigma,i} : i < r\}$ be a purely existential $\mathcal{L}^\lambda$-formula, $0 < \sigma \leqq s$, $k, r > 0$, let $G_\sigma = \exists xB$ be generated by F_σ, and let $\Xi_0, \Xi_1, ..., \Xi_{\vartheta-1}$ be a sequence of substitution-chains $[x_1/y_1]...[x_k/y_k]$, for any $y_1, ..., y_k$, in $\mathcal{L}^\lambda$; thus $\vartheta \leqq \lambda^k$. For any $i < r$, $\iota < \vartheta$, let $L_{\sigma,i,\iota} = L_{\sigma,i}\Xi_\iota$, and let $\overrightarrow{L}_{\sigma,\vartheta} = L_{\sigma,0,0}, ..., L_{\sigma,r-1,\vartheta-1}$ be the canonical enumeration of the set $\{L_{\sigma,i,\iota} : i < r \;\&\; \iota < \vartheta\}$. We call '*$\sigma$-canonical*' any multinomial W_σ of either the shape $\bigvee\{\bigwedge\{Ⓢ_{\sigma,i,\iota} : i < r\} : \iota < \vartheta\}$ for $\vartheta > 0$ or else $Ⓢ_\sigma \vee \bigvee\{\bigwedge\{Ⓢ_{\sigma,i,\iota} : i < r\} : \iota < \vartheta\}$, in the extended complex domain $\text{UM}(r \cdot \vartheta + s, l)$ (mod index-renaming). By the same token, we call '*σ-canonical*' the adjacent realization Z_σ that arises from W_σ by rewriting $Ⓢ_\sigma$ to G_σ and every $Ⓢ_{\sigma,i,\iota}$ to $L_{\sigma,i,\iota}$. In the sequel we call such W_σ, Z_σ a 'σ-canonical pair'. (Recall that the complex special symbols $Ⓢ_{\sigma,i,\iota}$ are treated differently than $Ⓢ_\sigma$, in that $Ⓢ_{\sigma,i,\iota}$ are not supposed to be sentences anymore.)

(30) Definition. For any $U, V \in \text{UM}(s, \lambda)$ and normal $\mathcal{L}^\lambda$-realizations $X = U[\overrightarrow{O} :=$

$\overrightarrow{G}]$, $Y = V[\overrightarrow{O'} := \overrightarrow{G'}]$, we let $\langle U, X\rangle \trianglerighteq \langle V, Y\rangle$ (in words: *U, X reduces to V, Y*) if either $\langle U, X\rangle = \langle V, Y\rangle$ or else the following two conditions hold. First, there exists a reduction chain $X = X_0 \rightsquigarrow X_1 \rightsquigarrow ... \rightsquigarrow X_p = Y$ from X to Y, in NSC_λ. Second, the adjacent multinomial reduction chain $U = U_0 \rightsquigarrow U_1 \rightsquigarrow ... \rightsquigarrow U_p = V$ from U to V, where $X_i = U_i[\overrightarrow{O^i} := \overrightarrow{G^i}]$, is admissible in $\mathsf{NSC}_\lambda(\overrightarrow{Ⓢ}_s)$, which is specified as follows. Arguing by induction on p, we define the latter chain and specify our requirements. Let $X_i \rightsquigarrow X_{i+1} = X_i[Q \hookrightarrow Q']$, where $X_i = U_i[\overrightarrow{O^i} := \overrightarrow{G^i}]$ is normal, be any reduction in the chain in question, $Q \hookrightarrow Q'$ being the underlying rewrite rule q from the list NSC_λ, $0 < q \leqq 11$. Suppose that some G^i_r from $\overrightarrow{G^i}$ occurs (as subformula) in or coincides with Q. Then $q \neq 5, 6, 7$ must be the case. Moreover, let $O^i_r = Ⓢ_\sigma$ be the correlated occurrence in U_i. It is readily seen that G^i_r determines one or two of its natural 'successors' in Q' which coincide(s) with G^i_r (mod bound renaming). If $q \neq 8, 9$ or else G^i_r differs from the 'main' subformula $\forall x D$, resp. $\forall x A$ (as exposed in the above description of NSC_λ), then we correlate $Ⓢ_\sigma$ with every G^i_r-successor in question. Otherwise, we note that D, resp. A, is an iterated disjunction (possibly improper) of semiexistential formulas generated by G^i_r and, then, we correlate $Ⓢ_\sigma$ with each semiexistential disjunct in question, according to the rule 4^+, resp. 9^+. By recursion/induction on p, this procedure allows us to construct the desired multinomial reduction chain from U to V, in $\mathsf{NSC}_\lambda(\overrightarrow{Ⓢ}_s)$. Dealing with multinomials in the extended complex domains $\text{UM}(r \cdot \vartheta + s, \lambda)$ we specify the above definition of $\langle U, X\rangle \trianglerighteq \langle V, Y\rangle$, accordingly. That is, the correctness of the adjacent multinomial reduction chain $U \rightsquigarrow^* V$ is fully determined by the one of $X \rightsquigarrow^* Y$ with regard to the assignments $Ⓢ_{\sigma,i,\iota} := L_{\sigma,i,\iota}$, whose last "substitution index" ι is supposed to change under the rule 10, accordingly.

(31) Lemma. *For any $\mathcal{L}^\lambda$-formulas (possibly empty) and variables (resp.) X, Y and x, y, for any purely existential and semiexistential $\mathcal{L}^\lambda$-formulas (resp.) F and G, B, and for any $U, V, W \in \text{UM}(s, \lambda)$ and normal $\mathcal{L}^\lambda$-realizations $X = U[\overrightarrow{O} := \overrightarrow{G}]$, $Y = V[\overrightarrow{O'} := \overrightarrow{G'}]$, $Z = W[\overrightarrow{O''} := \overrightarrow{G''}]$, the following hold.*

1. *$X \vee Y$ is generated by F iff so are both X and Y.*
2. *If $\exists x X$ is generated by F, then so is $X[x/y]$.*
3. *If $G \rightsquigarrow^* X \vee B \vee Y$ then $B[\exp] \rightarrow G[\exp]$ (holds and) is provable $\mathbf{HBL}_\lambda$.*
4. *The relation $\trianglerighteq$ is transitive, i.e. if $\langle U, X\rangle \trianglerighteq \langle V, Y\rangle$ and $\langle V, Y\rangle \trianglerighteq \langle W, Z\rangle$ then $\langle U, X\rangle \trianglerighteq \langle W, Z\rangle$. Furthermore, the nested analogue of $\trianglerighteq$-transitivity still holds with respect to composite multinomials and realizations (see above Definition 27).*

Proof. This is readily seen. ∎

(32) Lemma. *Let F_σ be as in Definition 29, and let G be any $\mathcal{L}^\lambda$-formula generated by F_σ. Then there exists a σ-canonical pair W_σ, Z_σ and a $U \in \mathrm{UM}(r \cdot \lambda^k + s, \lambda)$ together with a normal $\mathcal{L}^\lambda$-realization $X = U[\overrightarrow{O} := \overrightarrow{G}]$ such that $G = X$ and $\langle W_\sigma, Z_\sigma \rangle \trianglerighteq \langle U, X \rangle$.*

Proof. The proof runs by an easy induction on the length of the reduction chain $F_\sigma \rightsquigarrow^* G$ in NSC_λ. As long as the reductions preserve at least one existential quantifier (of F_σ), we let $U = \circledS_\sigma$. Each time F_σ generates another, i.e. non-existential formula F', we assign $\circledS_\sigma$ to all maximal existential subformulas of F', while labelling every literal $L_{\sigma,i}\Xi_\iota$ occurring beyond any scope of $\exists$ by the corresponding new symbol $\circledS_{\sigma,i,\iota}$. By Lemma 31(1,2) it readily follows that doing so we eventually arrive at the required normal extension $\langle U, G \rangle$ of the resulting σ-canonical pair W_σ, Z_σ. ∎

(33) Lemma. *Let W_σ, Z_σ be a σ-canonical pair, $X = U[\overrightarrow{O} := \overrightarrow{G}]$ a normal $\mathcal{L}^\lambda$-realization, $\overrightarrow{G}$ being correlated with $\overrightarrow{O}, \overrightarrow{F}$ where $\overrightarrow{O} = O_1, ..., O_t$, and let $0 < r \leqq t$ be fixed such that $O_r = \circledS_\sigma$. Let $U^\# = U[O_r := W_\sigma] \in \mathrm{UM}(r \cdot \vartheta + s, \lambda)$ and $X^\# = X[G_r := Z_\sigma]$ be the corresponding composite multinomial and realization, respectively, and let $\overrightarrow{G^\#}$ be the list of semiexistential formulas correlated with the corresponding composite lists $\overrightarrow{O^\#}, \overrightarrow{F^\#}$ of $\circledS$-occurrences and purely existential formulas in $U^\#$ and $X^\#$, respectively. Suppose there exists a normal $\mathcal{L}^\lambda$-realization $Y = V[\overrightarrow{O'} := \overrightarrow{G'}]$ such that $\langle U^\#, X^\# \rangle \trianglerighteq \langle V, Y \rangle$ and $V[\overrightarrow{O'} := \overrightarrow{G'}[\exp]]$ is 1-order valid. Then there exists a normal $\mathcal{L}^\lambda$-realization $Z = W[\overrightarrow{O''} := \overrightarrow{G''}]$ such that $\langle U, X \rangle \trianglerighteq \langle W, Z \rangle$ and $W[\overrightarrow{O''} := \overrightarrow{G''}[\exp]]$ is 1-order valid.*

Proof. This proof is more involved. Let $\langle U^\#, X^\# \rangle \trianglerighteq \langle V, Y \rangle$ be as in the lemma. In order to construct a desired $\langle W, Z \rangle$, we wish, loosely speaking, to rewrite every complex special symbol $\circledS_{\sigma,i,\iota}$ occurring in V "back" to $\circledS_\sigma$, under the intended realization $\circledS_\sigma := \bigwedge\{L_{\sigma,i,\iota} : \iota < r\}$. But we also wish to "inherit" 1-order validity of $V[\overrightarrow{O'} := \overrightarrow{G'}[\exp]]$ to the resulting semiboolean realization $W[\overrightarrow{O''} := \overrightarrow{G''}[\exp]]$. That is to say, we wish to "postpone" every distributive split of $\bigwedge\{\circledS_{\sigma,i,\iota} : \iota < r\}$ occurring in the "original" multinomial reduction chain $U^\# \rightsquigarrow^* V$. To this end, we have to *collapse* these "original" distributive branches. Moreover, every branch in the modified reduction chain has to absorb (say, as new disjuncts) information about "parallel" branches in the "original" reduction chain, for all $\circledS_{\sigma,i,\iota}$ which were separated by the corresponding multinomial applications of the rewrite-rule 3. Moreover, we delete those occurrences $\circledS_{\sigma,i,\nu}$, $\nu \neq \iota$, which occur in the new $\forall$-scopes created by the "parallel" applications of the rewrite-rule 10, in order to preserve "original" substitution-chains Ξ_ι. It is readily seen that such deletings don't destroy the resulting validity. For obvious reasons, the algorithm producing such W runs by recursion on distributive depth in question. The precise description must include the (nested version of) standard proof theoretical notions of *successors* and *predecessors* with respect to reductions and reduction chains involved (see e.g. [6] for nested specifications). For the sake of brevity, we only sketch the algorithm, and in fact only one crucial case, as follows. Without loss of generality we assume $r = 2$.

Consider $U^\# = A \vee B(y) \vee (Ⓢ_{\sigma,0,\iota} \wedge Ⓢ_{\sigma,1,\iota})$ and the correlated literals $L_{\sigma,0,\iota}(y)$ and $L_{\sigma,1,\iota}(x,y)$. Consider multinomial reduction chain of the shape:

$$
\begin{aligned}
&U^\# = A \vee B(y) \vee (Ⓢ_{\sigma,0,\iota} \wedge Ⓢ_{\sigma,1,\iota}) \rightsquigarrow \\
&(A \vee B(y) \vee Ⓢ_{\sigma,0,\iota}) \wedge (A \vee B(y) \vee Ⓢ_{\sigma,1,\iota}) \rightsquigarrow \\
&(A \vee B(y) \vee Ⓢ_{\sigma,0,\iota}) \wedge (A' \vee B(y) \vee Ⓢ_{\sigma,1,\iota}) \rightsquigarrow \\
&(A \vee \forall x(B(x) \vee Ⓢ_{\sigma,0,\varepsilon})) \wedge (A' \vee B(y) \vee Ⓢ_{\sigma,1,\iota}) \rightsquigarrow \\
&(A \vee \forall x(B(x)' \vee Ⓢ_{\sigma,0,\varepsilon})) \wedge (A' \vee B(y) \vee Ⓢ_{\sigma,1,\iota}) = V,
\end{aligned}
$$

where $A \rightsquigarrow A'$ and $B(x) \rightsquigarrow B(x)'$ and $\Xi_\varepsilon = [y/x]$, i.e. $L_{\sigma,0,\varepsilon} = L_{\sigma,0,\iota}(x)$. This chain is collapsed as follows (extensively using the contraction rewrite-rule 4) :

$$
\begin{aligned}
&\widetilde{U}^\# = A \vee B(y) \vee Ⓢ_\sigma = \\
&A \vee B(y) \vee Ⓢ_\sigma \rightsquigarrow \\
&A \vee A \vee B(y) \vee Ⓢ_\sigma \rightsquigarrow \\
&A \vee A' \vee B(y) \vee Ⓢ_\sigma \rightsquigarrow \\
&A \vee A' \vee B(y) \vee B(y) \vee Ⓢ_\sigma \rightsquigarrow \\
&A \vee A' \vee B(y) \vee \forall x B(x) \vee Ⓢ_\sigma \rightsquigarrow \\
&A \vee A' \vee B(y) \vee \forall x B(x) \vee \forall x B(x) \vee Ⓢ_\sigma \rightsquigarrow \\
&A \vee A' \vee B(y) \vee \forall x B(x) \vee \forall x B(x)' \vee Ⓢ_\sigma = \widetilde{V} =: W.
\end{aligned}
$$

The adjacent realization reduction chain that is obtained from the above by rewriting $Ⓢ_\sigma$ to $L_{\sigma,0,\iota}(y) \wedge L_{\sigma,1,\iota}(x,y)$ remains correct. Moreover, if the correlated semiboolean expansion $V[\overrightarrow{O'} := \overrightarrow{G'}[\exp]]$ is 1-order valid, then clearly so is the collapsed semiboolean expansion $W[\overrightarrow{O''} := \overrightarrow{G''}[\exp]]$.

Having defined the whole recursive collapsing procedure it is not hard to establish the required soundness by induction on the length of given reduction chains $\langle U^\#, X^\# \rangle \trianglerighteq \langle V, Y \rangle$, also using Lemma 31(4) (we refer to [6] for basic nested techniques). ■

(34) Lemma. *For any $U \in \text{UM}(s,l)$ let $X = U[\overrightarrow{O} := \overrightarrow{G}]$ be a normal $\mathcal{L}^\lambda$-realization, where as before $\overrightarrow{G} = G_1, ..., G_t$ is correlated with $\overrightarrow{O}, \overrightarrow{F}$ ($\overrightarrow{F} = F_{\sigma(1)}, ..., F_{\sigma(t)}$, $\overrightarrow{G} = G_1, ..., G_t$ being any lists of respectively purely existential and semiexistential $\mathcal{L}^\lambda$-formulas). Suppose X is provable in $\mathbf{HBL}_\lambda$. Then there exist a normal $\mathcal{L}^\lambda$-realization $Y = V[\overrightarrow{O'} := \overrightarrow{G'}]$ such that $\langle U, X \rangle \trianglerighteq \langle V, Y \rangle$ and $V[\overrightarrow{O'} := \overrightarrow{G'}[\exp]]$ is 1-order valid.*

Proof. By Theorem 23, we can safely replace the assumption 'X is provable in $\mathbf{HBL}_\lambda$' by 'X is derivable in NSC_λ'. Now the proof runs, for all dimensions s simultaneously, by induction on the length of derivation of X, i.e. reduction chain $X \rightsquigarrow^* \top$, in NSC_λ. Moreover, for the sake of brevity, we drop the rewrite-rules $1, 2$ of NSC_λ, while assuming that all formulas under consideration are equal mod $\vee$-commutative and $\vee$-associative laws. The basis of induction is trivial, for $\top = \top[\exp]$ is valid. Let $X \rightsquigarrow X' = X[Q \hookrightarrow Q']$ be the first reduction in $X \rightsquigarrow^* \top$, $Q \hookrightarrow Q'$ being the rewrite rule q $(3 \leqq q \leqq 11)$ of NSC_λ (abbr.: $q : Q \hookrightarrow Q'$). First consider

Case 1. Suppose Q is a subformula of some G_r $(0 < r \leqq t)$.

We observe that $G_r \rightsquigarrow G'_r = G_r[Q \hookrightarrow Q']$ holds, G'_r still being semiexistential, and hence $X' = U[\overrightarrow{O} := \overrightarrow{G'}]$ is a normal $\mathcal{L}^\lambda$-realization, where $\overrightarrow{G'}$ is obtained from $\overrightarrow{G}$ by rewriting G_r to G'_r. Moreover, $\langle U, X\rangle \trianglerighteq \langle U, X'\rangle$. By the induction hypothesis and Lemma 31(4), this yields the result for a proper subformula Q. If $Q = G_r$ $(0 < r \leqq t)$, we argue analogously, except that in cases $q = 8, 11$ we note that Q' can split into disjunctions of several semiexistential formulas. If this takes place, we apply the adjacent rule 4^+ of $\mathsf{NSC}_\lambda(\overrightarrow{Ⓢ}_s)$, while associating $Ⓢ_{\sigma(r)}$ with every disjunct in question. To complete the proof, we arrive at

Case 2. Suppose that all G_r $(0 < r \leqq t)$ which have nonempty disjunctive intersections with Q occur in it as proper subformulas. Generally, we argue as in the previous case, but there are more exceptions where $q : Q \hookrightarrow Q'$ changes the shape of some G_r in question. This can happen when: 1) $q = 5$, or 2) $q = 6$, or 3) $q = 7$, or 4) G_r is the $\forall x A$ of 9, or else 5) G_r is the $A \wedge B$ of 3. Consider these exceptions.

1. *Subcase 1).* We have $Q = \top \vee A$, $Q' = \top$. Let U_Q be the sub-multinomial of U whose realization is Q, and let U' be a multinomial that arises from U by rewriting U_Q to a chosen "dummy" occurrence $Ⓢ_\delta$. Then clearly X' is a normal realization of U'. Moreover, by the induction hypothesis, there exists a normal realization $Z = W[\overrightarrow{O''} := \overrightarrow{G''}]$ such that $\langle U', X'\rangle \trianglerighteq \langle W, Z\rangle$ and $W[\overrightarrow{O''} := \overrightarrow{G''}[\exp]]$ is 1-order valid. Let $\overrightarrow{\forall} Q$ be the universal closure of Q, and let $\overrightarrow{\forall} U_Q$ be the corresponding universal closure of U_Q ($\overrightarrow{\forall} Q$ and $\overrightarrow{\forall} U_Q$ both having the same prefix $\overrightarrow{\forall}$). Now everywhere in the given reduction chains $X' \rightsquigarrow^* Z$ and $U' \rightsquigarrow^* W$ we rewrite Q' and $Ⓢ_\delta$, and all its successors, to $\overrightarrow{\forall} Q$ and $\overrightarrow{\forall} U_Q$, respectively. Denote the resulting modification of W, Z by V, Y, and let $\overrightarrow{O}$ be the correlated list of all Ⓢ-occurrences in V. It is readily seen that Y is a normal realization of V such that $\langle U, X\rangle \trianglerighteq \langle V, Y\rangle$ and $V[\overrightarrow{O'} := \overrightarrow{G'}[\exp]]$ is 1-order valid, as required.

2. *Subcase 2).* By the assumption we have $Q = \top \wedge \top$, $Q' = \top$ such that $Q = G_u \wedge G_v$ and $G_u = G_v = \top$. We argue as in previous subcase, except rewriting Q' and $Ⓢ_\delta$ directly to Q and $U_Q = Ⓢ_{r(u)} \wedge Ⓢ_{r(v)}$, respectively.

3. *Subcase 3).* We have $Q = L \vee \neg L$, $Q' = \top$. This subcase is entirely analogous to Subcase 1).

4. *Subcase 4).* We have $Q = \forall x A \vee D$, $Q' = \forall x(A \vee D)$, x is not free in D. By the same token, the only question is how to pass from Q to Q' under the restrictions on the adjacent multinomial reduction. In the only nontrivial case, the semiexistential formula $\forall x A$ splits into a (possibly iterated) disjunction A of semiexistential disjuncts (cf. analogous subcases $q = 8, 11$ of the previous case). If this takes place, we associate $Ⓢ_{\sigma(r)}$ with every disjunct in question and use the adjacent rule 9^+.

5. *Subcase 5).* We have $Q = C \vee (A \wedge B)$, $Q' = (C \vee A) \wedge (C \vee B)$. Denote by U_Q and U_C the sub-multinomials of Q whose realizations are Q and C, respectively. Suppose $G_r = A \wedge B$ is generated by $F_{\sigma(r)}$. Hence, by Lemma 35, there exists a $\sigma(r)$-canonical pair $W_{\sigma(r)}, Z_{\sigma(r)}$ and a multinomial $U_{A\wedge B} = U_A \wedge U_B$ (of dimension $s' > s$) whose correlated normal realization coincides with $A \wedge B$ and such that $\langle W_{\sigma(r)}, Z_{\sigma(r)}\rangle \trianglerighteq \langle U_{A\wedge B}, A \wedge B\rangle$. Consider the corresponding composite multinomial $U^{\#} = U[O_r := U_{A\wedge B}]$ whose correlated composite normal realization coincides with X. Obviously, A and B are the correlated normal realizations of U_A and U_B, respectively. Moreover, the adjacent rewrite-rule $U_C \vee U_{A\wedge B} \hookrightarrow (U_C \vee U_A) \wedge (U_C \vee U_B)$ is legitimate in the extended formalism of $U_{A\wedge B}$. Furthermore, the correlated normal composite realization of the resulting reduced composite multinomial $U' = U^{\#}[U_C \vee U_{A\wedge B} \hookrightarrow (U_C \vee U_A) \wedge (U_C \vee U_B)]$ coincides with X' (whose derivation is shorter that the one of X). Hence, by the induction hypothesis with respect to the extended formalism of $\overrightarrow{Ⓢ_{s'}}$, there exists a normal realization $Z = W[\overrightarrow{O'''} := \overrightarrow{G'''}]$ such that $\langle U', X'\rangle \trianglerighteq \langle W, Z\rangle$ and $W[\overrightarrow{O'''} := \overrightarrow{G'''}[\exp]]$ is 1-order valid. Hence, by definition, we also have $\langle U, X\rangle \trianglerighteq \langle W, Z\rangle$, in the extended formalism in question. By Lemma 33, the same conclusion holds true in the original formalism of $\overrightarrow{Ⓢ}_s$, i.e. there exists a normal realization $Y = V[\overrightarrow{O''} := \overrightarrow{G''}]$ such that $\langle U, X\rangle \trianglerighteq \langle V, Y\rangle$ and $V[\overrightarrow{O''} := \overrightarrow{G''}[\exp]]$ is 1-order valid, Q.E.D. ∎

(35) Claim. *Lemma 19 holds.*

Proof. Note that $\textsc{up}(s,\lambda) \subset \textsc{um}(s,\lambda)$. Furthermore, any $U \in \textsc{up}(s,\lambda)$ is $\wedge$-free. Now suppose that $X = U[Ⓢ_1 := S_1, ..., Ⓢ_s := S_s]$ is provable in $\mathbf{HBL}_\lambda$, where $U \in \textsc{up}(s,\lambda)$ and purely existential $\mathcal{L}^\lambda$-sentences $S_1, ..., S_s$ are as in Lemma 19. Hence, by Lemma 37, there exist a $V \in \textsc{um}(s,\lambda)$ and a normal $\mathcal{L}^\lambda$-realization $Y = V[\overrightarrow{O''} := \overrightarrow{G''}]$ such that $\langle U, X\rangle \trianglerighteq \langle V, Y\rangle$ and $V[\overrightarrow{O''} := \overrightarrow{G''}[\exp]]$ is 1-order valid. Note that V is $\wedge$-free, since so is U. Consider the adjacent multinomial reduction chain $U \rightsquigarrow^* V$ in $\mathsf{NSC}_\lambda(\overrightarrow{Ⓢ}_s)$. Observe that the only reason why we can't infer $U \sqsubseteq V$ is due to possible applications of the formula-rewrite rule 10 which introduce extra occurrences of $\forall$. So we drop/reverse all reductions according to such 10. It is readily seen that the resulting modified reduction chain $U \rightsquigarrow^* V'$ is still correct, in $\mathsf{NSC}_\lambda(\overrightarrow{Ⓢ}_s)$. This is because on the one hand no variable occurs free in the initial realization X, and on the other hand the Ⓢ-free part of the initial polynomial U is the disjunction of

$\forall v_l H_i(v_l)$ for all variables v_l allowed, hence being closed under bound renaming, in $\mathcal{L}^\lambda$. By the same token, since dropping 10 can only improve validity, the adjacent modified realization $V'[\overrightarrow{O'} := \overrightarrow{G'}[\exp]]$ is still 1-order valid. But then, by Lemma 31(3), the corresponding "original" realization $V'[\overrightarrow{Ⓢ} := \overrightarrow{S}[\exp]]$ is 1-order valid, since the adjacent reductions from $X \rightsquigarrow^* Y$ in NSC_λ are valid as inverse implications. From this we easily arrive at a required $W \in \textsc{up}(s, \lambda)$ such that $U \sqsubseteq W$ and $W[Ⓢ_1 := S_1^0, ..., Ⓢ_s := S_s^0]$ is 1-order valid. In fact, such W is isomorphic to V' mod disjunctive equivalence, while by definition $S_\sigma[\exp] = S_\sigma^0$ holds for all $0 < \sigma \leqq s$.

The other direction of Lemma 19 is straightforward, although not trivial. To begin with, we notice that, in any finite variable domain, every formula Z has the uniquely determined (mod disjunctive equivalence) $\wedge\forall$-normal form $Z^\star$ (abbr.: CUNF) to which Z is convertible by successively applying, as long as possible, the ordinary boolean formula-rewrite distributive rules, i.e. the rule 3 of NSC_λ along with its $\vee$-commutative variant, plus the new formula-rewrite $\forall\wedge$-homomorphic rule $\forall x(A \wedge B) \hookrightarrow \forall xA \wedge \forall xB$ which is readily admissible by the rules $8, 10$ of NSC_λ. These conversions are *confluent* and *strategy-invariant*, i.e. applicable in any chosen order (see [6] for an exhaustive exposition). The resulting CUNF $Z^\star$ has the shape $\bigwedge\{\partial_k : k < \theta\}$ where every ∂_k is entirely $\wedge$-free. Now suppose $W \in \textsc{up}(s, \lambda)$ such that $U \sqsubseteq W$ and $Z = W[Ⓢ_1 := S_1^0, ..., Ⓢ_s := S_s^0]$ is 1-order valid. Let $Z^\star = \bigwedge\{\partial_k : k < \theta\}$ be the CUNF of Z. Thus for every $k < \theta$, ∂_k is valid. Now ∂_k is built up from atoms by using only disjunction and universal quantification. From this it readily follows that ∂_k is valid iff applying one or more rules 9 it contains a subformula of the shape $\bigvee\{L_j : j < \eta\}$ such that there are $p, q < \eta$ with $L_p = \neg L_q$. Hence 1-order validity of ∂_k implies its derivability in NSC_λ via $7, 5, 9$ and, if necessary, 8 for $A = \top$. Having this, we easily conclude, by 6, that 1-order validity of $Z^\star$ implies its derivability in NSC_λ. Since CUNF expansions are admissible in NSC_λ (see above), Z is derivable in NSC_λ, and hence, by 11, so is the correlated existential closure $Y = W[Ⓢ_1 := S_1, ..., Ⓢ_s := S_s]$. Hence by Theorem 23, Y is provable in $\mathbf{HBL}_\lambda$. On the other hand, from $U \sqsubseteq W$, $X = U[Ⓢ_1 := S_1, ..., Ⓢ_s := S_s]$ and $Y = W[Ⓢ_1 := S_1, ..., Ⓢ_s := S_s]$ we infer that X reduces to Y by a chain of 9 simulating the relation $\sqsubseteq$ by rewriting every occurrence $Ⓢ_\sigma$ to S_σ $(0 < \sigma \leqq s)$; this operation is legitimate in NSC_λ, since no free variable occurs in S_σ. Moreover, we know that $\sqsubseteq$ is provable in $\mathbf{HBL}_\lambda$ as inverse implication. Hence X is provable in $\mathbf{HBL}_\lambda$, Q.E.D. ∎

(36) Remark. This claim completes the negative proof of Theorem 2 and thereby confirms the results exposed in Subsection 1.1. Now consider Tarski λ-variable formalism with equality (abbr.: $\mathbf{TL}_\lambda$) underlying [21]. Recall that $\mathbf{TL}_\lambda$ is stronger than $\mathbf{HBL}_\lambda$. Hence by the positive part of Theorem 2 (resp. 4), the sentence $\mathfrak{F}_{m,n}$ (resp. $\mathfrak{T}_{m,n} = 1$, mod canonical 3-variable translation) is surely provable in $\mathbf{TL}_{n+3}$. On the other hand, we notice that for any $\lambda > 2$, the general Leibniz law for λ-variable formulas is admissible in $\mathbf{HBL}_{\lambda+1}$ and, generally, every λ-variable formula provable in $\mathbf{TL}_\lambda$ is provable in $\mathbf{HBL}_{\lambda+1}$ (see [4, 5]). Hence from the negative part of Theorem 2 (resp. 4) we infer that $\mathfrak{F}_{m,n}$ (resp. $\mathfrak{T}_{m,n} = 1$, mod canonical 3-variable translation) is not provable in $\mathbf{TL}_{n+1}$. This yields a combinatorial 1-order "almost"-solution to the original Problem 2.12 of [11]. Notably, $\mathfrak{F}_{m,n}$ was obtained by proof theoretical analysis of the algebraic arguments used in [17], which thereby essentially improves indirect loose bounds of provability claimed in [17] with regard to the same algebraic construction.[4] Furthermore, $\mathfrak{F}_{m,n}$ is equality-free, which enables us to remove "almost" from the above description, provided that the following conjecture holds.

(37) Conjecture. *For every $\lambda > 3$, every equality-free sentence provable in some of Henkin-Tarski λ-variable formalisms with equality, as exposed in [8, 12] a/o [21], is also provable in the corresponding formalism $\mathbf{HBL}_\lambda$. This conjecture is known to hold for $\lambda = 4$. But in the general case, the proof would require deeper proof theoretical insights into finite-variable logics with the general Leibniz law.*

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