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The Definitional View of Atomic Systems in Proof-Theoretic Semantics

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Abstract: Atomic systems, that is, sets of rules containing only atomic formulas, play an important role in proof-theoretic notions of logical validity. We consider a view of atomic systems as definitions that allows us to discuss a proposal made by Prawitz (2016). The implementation of this view in the base case of an inductive definition of validity leads to the problem that derivability of atomic formulas in an atomic system does not coincide with the validity of these formulas. This is due to the fact that, on the definitional view of atomic systems, there are not just production rules, but both introduction and elimination rules for atoms, which may even generate non-normalizable atomic derivations. This shows that the way atomic systems are handled is a fundamental issue of proof-theoretic semantics.

Keywords: Proof-theoretic semantics, Validity of atomic formulas, Definitions, Definitional reflection, Dummett's fundamental assumption

1 Introduction

The proof-theoretic semantics of logical constants can be given by an inductive definition of a notion of validity for logical formulas. Such an inductive definition contains a semantic clause for each logical constant under consideration. In its base case the validity of atomic formulas (in short: atoms) is defined in terms of derivability of these formulas in atomic systems. Atomic systems can be sets of atoms or sets of atomic rules, that is, sets of rules that contain only atoms. Atomic rules can be production rules, but one may also consider atomic rules that can discharge atomic assumptions, or even generalize further and consider higher-level atomic rules that can discharge assumed atomic rules (see Section 2).

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Atomic systems can be given different interpretations. One possibility is to view atomic systems as *knowledge bases* that are invariant with respect to extensions with further knowledge. Propositions that are valid for a given knowledge base should remain valid if this knowledge base is extended. Under this view, consequence with respect to atomic systems S can be explained by making reference to such extensions of atomic systems: A proposition B is a consequence of a proposition A with respect to an atomic system S (in short: B is an S-consequence of A) if, and only if, for all extensions S' of S, it holds that whenever A is valid w.r.t. S', then B is valid w.r.t. S'. Thus S-consequence is monotone with respect to extensions of atomic systems, if they are understood as knowledge bases.

Another interpretation of atomic systems is given by the *definitional view*, where atomic systems are understood as definitions of certain atoms. Under this view we do not expect monotonicity of S-consequence with respect to extensions, since extending a definition will in general change the meaning of the defined atoms. An explanation of S-consequence should then no longer refer to extensions of atomic systems.

The definitional view of atomic systems is preferred by Prawitz (2016), who sees an atomic system (which he calls a base) "as determining the meanings of the atomic sentences" (ibid., p. 15). According to him, extensions should not be considered in the definitional view: "To consider extensions of the given base [...] is natural when a base is seen as representing a state of knowledge, but is in conflict with the view adopted here that a base is to be understood as giving the meanings of the atomic sentences." (ibid., fn. 12, p. 18).

We implement the definitional view by using a theory of definitions based on the principle of definitional reflection[2] (see Section 3). What we consider to be essential for the definitional interpretation of atomic systems is that the atomic rules for a specific atom completely determine its meaning. This does not exclude partial or non-wellfounded definitions. Complete determination of meaning consists rather in the fact that once we write down an atomic system as a definition of certain atoms it is assumed that the atomic system is complete in the sense that nothing else defines these atoms. Under this assumption (which corresponds to the extremality condition in standard inductive definitions) the principle of definitional reflection is justified, and a certain kind of derivability relation for atoms is induced.

Based on this kind of derivability we then inductively define a proof-

[2] See Hallnäs (1991) and Schroeder-Heister (1993).

theoretic notion of validity (see Section 4). The base case in this inductive definition defines the validity of an atom with respect to an atomic system by its derivability in that atomic system; inductive clauses for logical constants are given and logical validity is defined. We then show that for atoms validity does not coincide with derivability. This indicates that the definitional view of atomic systems might not be the proper foundation for proof-theoretic validity.

That the non-definitional view of atomic systems as knowledge bases, where arbitrary extensions of atomic systems are considered, is problematic, was already shown in Piecha, de Campos Sanz, and Schroeder-Heister (2015). Together these results show that the role played by atomic systems in a definition of validity for complex formulas is far from trivial. This problem has been widely neglected in proof-theoretic semantics.

2 Atomic systems

We use letters $a, b, c, \ldots, a_1, a_2, \ldots$ for atoms. The most simple kind of atomic systems (besides sets of atoms) are sets of production rules for atoms. Such a *(first-level) atomic system* S is a (possibly empty) set of atomic rules of the form

$$\frac{a_1 \quad \ldots \quad a_n}{b}$$

Such rules are also called *first-level rules*. The set of premisses $\{a_1, \ldots, a_n\}$ in a rule can be empty, in which case the rule is called an *atomic axiom* and is of *level 0*.

In the context of proof-theoretic semantics, atomic systems of this kind were considered, for example, by Prawitz (1971) and Dummett (1991). But one does not have to stop at first-level atomic systems. We also consider second-level and arbitrary higher-level atomic systems.

A *second-level atomic system* S is a (possibly empty) set of atomic rules of the form

$$\frac{[\Gamma_1] \qquad\qquad [\Gamma_n]}{a_1 \quad \ldots \quad a_n}{b}$$

where the a_i and b are atoms, and the Γ_i are finite sets of atoms. Such a *second-level rule* expresses that from the premisses $a_1, \ldots, a_n$ one may conclude b, where b need no longer depend on assumptions belonging to Γ_i on which the premisses a_i might still depend, for each i; that is, assumptions in Γ_i may be discharged. If the sets $\Gamma_1, \ldots, \Gamma_n$ are empty, then the rule is

a first-level rule. If the set of premisses $\{a_1, \ldots, a_n\}$ is empty in addition, then the rule is an axiom.

We now further generalize second-level atomic systems to the higher-level case by allowing for atomic rules that can discharge not only atoms but also atomic rules as assumptions.[3] We use the following linear notation to introduce *higher-level rules*:

1. Every atom a is a rule of level 0.
2. If $R_1, \ldots, R_n$ are rules ($n \geq 1$), whose maximal level is ℓ, and a is an atom, then $(R_1, \ldots, R_n \rhd a)$ is a rule of level $\ell + 1$.

In tree notation, higher-level rules have the form

$$\frac{\begin{array}{ccc}[\Gamma_1] & & [\Gamma_n] \\ a_1 & \ldots & a_n\end{array}}{b}$$

where the a_i and b are atoms, and the Γ_i are finite sets $\{R_1^i, \ldots, R_k^i\}$ of rules, which may be empty. The set of premisses $\{a_1, \ldots, a_n\}$ of such a rule can again be empty, in which case the rule is an axiom. A *higher-level atomic system* S is a (possibly empty) set of higher-level rules.

We now define the notion of *derivation* for higher-level atomic systems:

1. For a level-0 rule a,
$$\frac{}{a}\, a$$
is a *derivation* of a from $\{a\}$.
2. Now consider a level-$(\ell{+}1)$ rule $(\Gamma_1 \rhd a_1), \ldots, (\Gamma_n \rhd a_n) \rhd b$. Suppose that for each i $(1 \leq i \leq n)$ a derivation
$$\begin{array}{c}\Sigma_i \cup \Gamma_i \\ \mathscr{D}_i \\ a_i\end{array}$$
of a_i from $\Sigma_i \cup \Gamma_i$ is given. Then
$$\frac{\begin{array}{ccc}\Sigma_1 & & \Sigma_n \\ \mathscr{D}_1 & & \mathscr{D}_n \\ a_1 & \ldots & a_n\end{array}}{b}\,(\Gamma_1 \rhd a_1), \ldots, (\Gamma_n \rhd a_n) \rhd b$$
is a *derivation* of b from $\Sigma_1 \cup \ldots \cup \Sigma_n \cup \{(\Gamma_1 \rhd a_1), \ldots, (\Gamma_n \rhd a_n) \rhd b\}$.

[3] Atomic rules of this kind are thus a special case of the higher-level rules in Schroeder-Heister (1984), which are not restricted to atomic formulas.

An atom b is *derivable* from Σ in a higher-level atomic system S, if there is a derivation of b from $\Sigma \cup S$. This is written as follows: $\Sigma \vdash_S b$.

We give an example derivation for the higher-level atomic system

$$S^\star \left\{ \begin{array}{r} a \\ ((a \rhd b) \rhd c) \rhd d \\ ((b \rhd c) \rhd f) \rhd g \end{array} \right.$$

and the set of assumptions $\Sigma = \{e, ((d, e) \rhd f)\}$:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{a}\,\langle a\rangle}{b}\,[a \rhd b]^1}{c}\,[b \rhd c]^2}{d}\,{}_1\langle((a \rhd b) \rhd c) \rhd d\rangle \qquad \dfrac{}{e}\,e}{f}\,(d, e) \rhd f}{g}\,{}_2\langle((b \rhd c) \rhd f) \rhd g\rangle$$

We use angle brackets $\langle\,\rangle$ to indicate that a rule from the atomic system $S^\star$ is applied, and we use square brackets $[\,]$ together with numerals to indicate the discharge of assumed rules. In the left branch we start by introducing the premiss a with the 0-level rule $a \in S^\star$. Assuming the rule $a \rhd b$ we conclude b, and assuming the rule $b \rhd c$ we obtain c from b. We have thus derived c under the assumption of rules $a \rhd b$ and $b \rhd c$. An application of $((a \rhd b) \rhd c) \rhd d \in S^\star$ yields d and discharges the first assumption $a \rhd b$ (as indicated by the numeral 1). In the right branch we use the assumption $e \in \Sigma$ to get e. Now the rule $(d, e) \rhd f \in \Sigma$ can be applied to obtain f, which still depends on the assumed rule $b \rhd c \notin \Sigma$. This assumption is discharged in the last step (as indicated by the numeral 2) by applying $((b \rhd c) \rhd f) \rhd g \in S^\star$ to conclude g. The derivation thus shows $\Sigma \vdash_{S^\star} g$.

3 The definitional view of atomic systems

Atomic systems can be understood as inductive definitions[4] of atomic formulas. Consider an atomic system

$$S \left\{ \begin{array}{c} \Gamma_1 \rhd a \\ \vdots \\ \Gamma_k \rhd a \end{array} \right.$$

[4] See Aczel (1977).

of k higher-level atomic rules. The atomic rules $\Gamma_i \rhd a$ can be read as *definitional clauses* for the atom a with *defining conditions* Γ_i, for $1 \leq i \leq k$, and the whole set S of atomic rules can thus be read as a *definition* of the atom a.

The defining conditions in definitional clauses can be empty. Clauses of this form, which we write as $\emptyset \rhd a$, or simply as a, are read as the base clauses in an inductive definition of a. Clauses of the form $\Gamma_i \rhd a$, for non-empty Γ_i, are the inductive clauses of such a definition.

A direct application of a definitional clause $\Gamma_i \rhd a$ consists in passing from the defining condition Γ_i of a to the defined atom a:

$$\frac{\Gamma_i}{a}\,\langle \Gamma_i \rhd a \rangle$$

We refer to rules of this kind as (steps of) *definitional closure*. They figure as *introduction rules for atoms*.

Definitional closure alone is not characteristic for the definitional view of atomic systems. What distinguishes the definitional view from other views of atomic systems is the fact that in the case of a definition of an atom a it is assumed in addition that nothing else defines a. This extremality condition is usually not stated explicitly in a definition; however, by referring to something as a definition one always tacitly assumes it.

The extremality condition justifies an additional reasoning principle for definitions: For an atom a defined by

$$S \begin{cases} \Gamma_1 \rhd a \\ \quad \vdots \\ \Gamma_k \rhd a \end{cases}$$

one can pass from a to an arbitrary atom c whenever c can be obtained from each of the defining conditions Γ_i of a (for $1 \leq i \leq k$). That is, in addition to definitional closure one can argue by *definitional reflection*:

$$\frac{a \qquad \begin{matrix}[\Gamma_1] \\ c\end{matrix} \quad \ldots \quad \begin{matrix}[\Gamma_k] \\ c\end{matrix}}{c}$$

(In linear notation: $(a, (\Gamma_1 \rhd c), \ldots, (\Gamma_k \rhd c)) \rhd c$.) Note that this rule is not given by any definitional clause in S (as it is the case for definitional closure); it only becomes available by reflecting on S as a whole. Each instance of

definitional reflection is an *elimination rule for the atom* a, where a is the *major premiss* of the rule.

In general, the formula c need not even be atomic. If atomic systems are used in the context of logical rules, or if additional rules are available that allow to manipulate higher-level rules, then definitional reflection can take the form

$$\frac{a \qquad \begin{matrix}[\Gamma_1] & & [\Gamma_k] \\ C & \dots & C\end{matrix}}{C}$$

for arbitrary formulas C.[5] Definitional reflection can also take this general form if a is an undefined atom, that is, if S does not contain any clauses of the form $\Gamma \rhd a$. As the set of defining conditions of a is empty in this case, one can infer any formula C from a by definitional reflection. In other words, under the definitional view of atomic systems S a principle of *ex falso quodlibet*

$$a \vdash_S C$$

is available if at least one atom a is not defined by S.

To simplify the introduction of the rules of definitional closure and definitional reflection we considered atomic systems S with definitional clauses for only one atom a. In general, however, a *definition* can be any finite atomic system of the following form:

$$S \left\{ \begin{array}{ccc} \Gamma_1^1 \rhd a_1 & & \Gamma_1^n \rhd a_n \\ \vdots & \cdots & \vdots \\ \Gamma_{k_1}^1 \rhd a_1 & & \Gamma_{k_n}^n \rhd a_n \end{array} \right.$$

Note that such a definition can in general not be divided into separate definitions for each a_i, since the definitional clauses might be entangled in the sense that a_i occurs in the defining conditions of another atom a_j. We also note that such a definition need not have base clauses $\emptyset \rhd a_i$, and is therefore not necessarily well-founded. Moreover, the restriction of definitions to finite atomic systems guarantees that some atom, for example $\bot$, is always undefined, and that therefore *ex falso quodlibet* $\bot \vdash_S C$ is available for any definition S.

As an example for definitional reasoning using definitional closure and definitional reflection consider the following definition (the specific form of

[5] For rules of definitional reflection see Hallnäs (1991, 2006) and Schroeder-Heister (1993).

the distinct sets of higher-level rules Γ, Δ and Σ does not matter in what follows):

$$S^+ \left\{ \begin{array}{ll} \Gamma \rhd a & \Gamma \rhd b \\ \Delta \rhd a & \Delta \rhd b \\ & \Sigma \rhd b \end{array} \right.$$

The two applications of definitional closure

$$\frac{\Gamma}{b}\,\langle \Gamma \rhd b \rangle \qquad \text{and} \qquad \frac{\Delta}{b}\,\langle \Delta \rhd b \rangle$$

show that b can be inferred from assumptions Γ as well as from assumptions Δ. Since Γ and Δ are exactly the defining conditions of a this means that b can be derived from each of the defining conditions of a. Hence definitional reflection can be applied to a, discharging the assumptions Γ and Δ:

$$1\,\frac{a \qquad \dfrac{[\Gamma]^1}{b}\,\langle \Gamma \rhd b \rangle \qquad \dfrac{[\Delta]^1}{b}\,\langle \Delta \rhd b \rangle}{b}\,(1)$$

(def. reflection on S^+)

This derivation shows that $a \vdash_{S^+} b$ holds. Note that this cannot be shown by definitional closure alone; one has to use definitional reflection in addition.

Having explained the definitional view of atomic systems, we now discuss some consequences of this view in proof-theoretic semantics.[6]

4 The definitional view in proof-theoretic semantics

Proof-theoretic semantics of logical constants can be given in different ways. One approach, which is due to Prawitz (1971, 1973, 1974, 2014), is to define a notion of validity for derivations that are constructed from arbitrary inference rules.[7] Alternatively one can define notions of proof-theoretic validity for formulas.[8] This approach, which we follow here, allows for a perspicuous formulation of the definition of validity that still captures the main ideas of the derivations-based approach.

[6] It should be mentioned that definitional reflection develops its full power, in particular from the computational point of view (see Hallnäs & Schroeder-Heister, 1990, 1991), when clauses for atoms with (free) variables are considered. We have here confined ourselves to atoms as sentence letters without any internal structure, as this suffices to make our point.

[7] See Schroeder-Heister (2006, 2012a).

[8] See Kreisel (1961), Gabbay (1976, 1981), Piecha et al. (2015) and Piecha (2016).

We consider a notion of validity for formulas $A, B, C, \ldots$ that are constructed from atoms with the logical constants $\rightarrow$, $\vee$ and $\wedge$. It is intended to be a notion of validity for minimal propositional logic. In the following we first inductively define the relation of *S-validity* ($\models_S$), which is a relation relative to atomic systems S, by clauses (S1)-(S5). Logical validity, in short: *validity* ($\models$), is then defined in clause (S6) as S-validity for all atomic systems S; for Γ being a set of formulas we write $\models_S \Gamma$ for $\{\models_S A_i \mid A_i \in \Gamma\}$:

(S1) $\models_S a \;:\Longleftrightarrow\; \vdash_S a$,

(S2) $\models_S A \rightarrow B \;:\Longleftrightarrow\; A \models_S B$,

(S3) $\Gamma \models_S A \;:\Longleftrightarrow\; (\models_S \Gamma \implies \models_S A)$,

(S4) $\models_S A \vee B \;:\Longleftrightarrow\; \models_S A$ or $\models_S B$,

(S5) $\models_S A \wedge B \;:\Longleftrightarrow\; \models_S A$ and $\models_S B$,

(S6) $\Gamma \models A \;:\Longleftrightarrow\; \forall S : \Gamma \models_S A$.

The S-validity of atoms a is defined by clause (S1) as derivability of a in an atomic system S. For the definitional view considered here this means that an atom a is S-valid if, and only if, a is derivable in S by definitional closure and definitional reflection.

Clause (S2) defines S-validity of implications $A \rightarrow B$, that is, $\models_S A \rightarrow B$, by *$S$-consequence* $A \models_S B$. The latter is defined by clause (S3). By combining clauses (S2) and (S3) we see that implication is explained as follows:

$$\models_S A \rightarrow B \;:\Longleftrightarrow\; (\models_S A \implies \models_S B)$$

Clauses (S4) and (S5) are straightforward, and clause (S6) then gives us a proof-theoretic notion of logical validity based on atomic systems understood as definitions.

In the given notion of validity we have not considered extensions $S' \supseteq S$ of atomic systems, where an atomic system S' is an extension of an atomic system S if $S' = S$ or if S' results from S by adding atomic rules. In doing so we follow Prawitz's suggestion[9] that considering extensions is in conflict with the definitional view of atomic systems. Alternatively, one could define S-consequence using extensions by the following clause:

$$\Gamma \models_S A \;:\Longleftrightarrow\; \forall S' \supseteq S : (\models_{S'} \Gamma \implies \models_{S'} A) \qquad (\text{S3}_{\text{ext}})$$

[9] Cf. Prawitz (2016), fn. 12, p. 18.

This would prevent that an S-consequence $\Gamma \models_S A$ holds just because some atom on which Γ depends is not valid in S. Without extensions, that is, for clause (S3), this is not the case: Consider the empty definition $S = \emptyset$. It is $\not\models_S a$, since $\not\vdash_S a$; hence $\models_S a$ implies $\models_S b$ trivially, and therefore $a \models_S b$ by clause (S3). In general, clause (S3_{ext}) guarantees monotonicity of validity in the sense that, if a is S-valid, it is S'-valid for any extension S' of S. This makes a conceptual difference depending on whether atomic systems are viewed as knowledge bases or as definitions. A knowledge base, unlike a definition, is supposed to be monotone. In fact, under the definitional view of atomic systems already atomic derivability fails to be monotone with respect to extensions of atomic systems. For example, for definition S^+ we had $a \vdash_{S^+} b$. Extending S^+ by $\Theta \rhd a$ to

$$S^{++} \begin{cases} \Gamma \rhd a & \Gamma \rhd b \\ \Delta \rhd a & \Delta \rhd b \\ \Theta \rhd a & \Sigma \rhd b \end{cases}$$

would block definitional reflection for a, since b cannot be derived from Θ, which, however, is an additional defining condition of a in S^{++}. Hence $a \not\vdash_{S^{++}} b$, although $a \vdash_{S^+} b$.

We can now pose the question how derivability of atoms in a definition S from assumed rules relates to S-consequence between a set of assumed formulas and atoms. First we observe that atomic rules R can be represented by formulas R^* over $\{\rightarrow, \wedge\}$ by using a translation * defined as follows:

1. $a^* := a$, for atoms a.

2. $(R_1, \ldots, R_n \rhd a)^* := R_1^* \wedge \ldots \wedge R_n^* \rightarrow a$, for a rule $R_1, \ldots, R_n \rhd a$.

We write S^* for the set of formulas representing the rules in a given atomic system S. This device is only needed because S-consequence is defined as a relation between sets of formulas and formulas, whereas derivability is a relation between sets of rules and atoms.

Let now Δ^* be some set of formulas representing a set of rules Δ, and let S be an arbitrary atomic system. We can then ask whether under the definitional view S-validity is *stable* in the sense that the following biconditional holds for any atomic system S:

$$\Delta^* \models_S a \iff \Delta \vdash_S a$$

That *atomic completeness*

$$\Delta^* \vDash_S a \implies \Delta \vdash_S a$$

does *not* hold was shown in Piecha and Schroeder-Heister (2016). The atomic system $S = \{a \rhd a\}$ gives a simple counterexample: Obviously $\nvdash_S a$, and hence $\nvDash_S a$ by clause (S1). Therefore $a \vDash_S b$ by clause (S3). However, we have $a \nvdash_S b$. This is because any application of definitional reflection on this atomic system S with major premiss a has the form

$$\cfrac{\begin{array}{cc} & [a]^1 \\ a & C \end{array}}{C}\,{}_1$$

and is therefore useless, since the minor premiss C already contains what is to be established.

For *atomic soundness*

$$\Delta \vdash_S a \implies \Delta^* \vDash_S a$$

the situation is a bit more complicated. If one considered S-validity with extensions by having S-consequence defined by clause (S3_{ext}) instead of clause (S3), then already the empty atomic system $S = \emptyset$ gives a counterexample:[10] Since, for example, a is not defined, we have $a \vdash_S b$ for any atom b by definitional reflection. Now we can extend S to $S' = S \cup \{a\}$ for which we have $\vdash_{S'} a$ and, by clause (S1), also $\vDash_{S'} a$. But clearly $\nvdash_{S'} b$, and thus $\nvDash_{S'} b$ by clause (S1). Therefore $\forall S' \supseteq S : (\vDash_{S'} a \implies \vDash_{S'} b)$ does not hold; hence $a \nvDash_S b$ by clause (S3_{ext}).

However, for the notion of S-validity that is under scrutiny here, we do not consider extensions of atomic systems. In this case, a possible counterexample for atomic soundness is given by the atomic system

$$S = \{(a \rhd b) \rhd a\}$$

The derivation

$$\mathscr{D}\left\{\cfrac{\begin{array}{cc} & \overline{}\; a \\ a & \cfrac{a}{b}\,[a \rhd b]^1 \end{array}}{b}\,{}_1 \text{ (def. reflection on } S)\right.$$

[10] Cf. Piecha and Schroeder-Heister (2016).

shows $a \vdash_S b$, and the derivation

$$\mathscr{D}'\left\{\begin{array}{c} \dfrac{\dfrac{}{a}[a]^2 \quad \dfrac{}{b}[a \rhd b]^1}{b}\,1 \text{ (def. reflection on } S) \\ \dfrac{b}{a}\,2 \text{ (def. closure) } \langle (a \rhd b) \rhd a \rangle \end{array}\right.$$

which introduces the atom a in the last step by definitional closure shows $\vdash_S a$. Thus by the latter also $\vDash_S a$, by clause (S1). Now, if one allowed for derivations that are *not* normalizable, then the non-normal derivation

$$\mathscr{D}^\dagger\left\{\begin{array}{c} \dfrac{\mathscr{D}'\left\{\begin{array}{c} \dfrac{\dfrac{}{a}[a]^2 \quad \dfrac{}{b}[a \rhd b]^1}{b}\,1 \text{ (def. refl. on } S) \\ \dfrac{b}{a}\,2 \text{ (def. closure)} \end{array}\right. \qquad \mathscr{D}'\left\{\begin{array}{c} \dfrac{\dfrac{}{a}[a]^2 \quad \dfrac{}{b}[a \rhd b]^1}{b}\,1 \text{ (def. refl. on } S) \\ \dfrac{b}{a}\,2 \text{ (def. closure)} \end{array}\right. \quad \dfrac{}{b}[a \rhd b]^3}{b}\,3 \text{ (def. refl. on } S) \end{array}\right.$$

would show $\vdash_S b$. This derivation results from substituting the closed derivation $\mathscr{D}'$ for the two open assumptions a in derivation $\mathscr{D}$. By clause (S1) we would then have $\vDash_S b$, and therefore $a \vDash_S b$ by clause (S3). Hence S does not give us a counterexample to atomic soundness.

That derivation $\mathscr{D}^\dagger$ is not in normal form is due to the fact that the major premiss a in the final application of definitional reflection is a maximal formula: it is introduced by an application of definitional closure in the last step of $\mathscr{D}'$, and is immediately eliminated in the last step of $\mathscr{D}^\dagger$. Moreover, $\mathscr{D}^\dagger$ is not normalizable, since b cannot be introduced by definitional closure (since there is no definitional clause for b in S).

If, on the other hand, we require that all derivations be normal, then $\mathscr{D}$ and $\mathscr{D}'$ cannot be combined into $\mathscr{D}^\dagger$. In this case there cannot be a closed derivation of b in S, that is, $\nvdash_S b$. Hence $\nvDash_S b$ by clause (S1), while $\vDash_S a$. Therefore $a \nvDash_S b$, which means that S is in this case a counterexample to atomic soundness.

This points to a deeper problem of S-validity under the definitional view of atomic systems. In the non-atomic realm of logically complex formulas or statements one usually imposes Dummett's *fundamental assumption* "that, if we have a valid argument for a complex statement, we can construct a valid argument for it which finishes with an application of one of the introduction rules governing its principal operator" (Dummett, 1991, p. 254; cf. also ch. 12). For the atomic realm, where applications of definitional closure are

the introduction rules for defined atoms, this means that the base clause (S1) in the definition of S-validity has to be replaced by the following clause:

$$\vDash_S a \ :\Longleftrightarrow \vdash_S a, \text{ where } a \text{ is derived by definitional closure in the last step.} \qquad (\text{S1}')$$

This enforces that the fundamental assumption holds for S-validity. Alternatively, the same effect can be achieved by demanding for definitional reflection

$$\frac{a \qquad \begin{matrix}[\Gamma_1] \\ c\end{matrix} \quad \ldots \quad \begin{matrix}[\Gamma_k] \\ c\end{matrix}}{c}$$

that is, for the elimination rule for atoms a, that the major premiss a must always be the conclusion of an assumed rule; this includes the case that the major premiss is an assumed formula.[11] This forces all derivations to be in normal form, and consequently ensures that S-validity complies with the fundamental assumption.

What this tells us about the definitional view of atomic systems in proof-theoretic semantics is the following: If we impose the same assumption that we make for the validity of complex formulas also on the validity of atomic formulas, which is a natural thing to do, then S-validity is not stable on the atomic level, that is, neither atomic completeness nor atomic soundness holds. A restriction from higher-level to first-level atomic systems might rule out counterexamples to atomic soundness, but atomic completeness would still fail, as the given first-level counterexample shows.

An objection one might raise against the clause $(a \rhd b) \rhd a$ is that it is circular in that it defines a in terms of a (as is the case also with the clause $a \rhd a$) and moreover paradoxical, by defining a by $\neg a$. The latter is easily seen if we write the undefined atom b as absurdity $\bot$ and $a \rhd \bot$ as $\neg a$, so that the clause becomes $\neg a \rhd a$. However, we do not consider this a problem here, as we do not want to put restrictions on the form clauses are allowed to take. We prefer to claim *definitional freedom* in that respect. Apart from the fact that in logic programming clauses such as $\neg a \rhd a$ have always been considered and are thus not unusual at all, it gives us a most welcome tool with considerable expressive power. By means of clauses of

[11] This corresponds to the feature that major premisses of elimination rules for logical constants only occur as assumptions. It is sometimes considered in discussions of general elimination rules; for example, according to Tennant (2015, p. 746) "all major premises for eliminations stand proud, with no proof-work above them (that is to say, they occupy leaf-nodes of the proof-tree)".

this kind and the principle of definitional reflection we can develop a natural theory of semantical and set-theoretical paradoxes, which are characterized by non-normalizing atomic derivations.[12] The idea of a 'partial' assignment of meaning as present in self-referential definitions is at the core of the theory of definitional reasoning and its principle of definitional reflection. It is analogous to the consideration of partial recursive functions in recursive function theory (see Hallnäs, 1991).

It is important to note that the failure of stability on the atomic level says nothing about the soundness and completeness of a calculus for minimal or intuitionistic logic with respect to logical validity. Our results apply only to S-validity, that is, to validity with respect to a chosen atomic system. The fact that stability on the atomic level is lacking for approaches based on definitional reasoning shows that a satisfactory theory of proof-theoretic validity based on definitional reasoning with atoms is still a desideratum. It might be the case that a hybrid theory incorporating both a knowledge-base view and a definitional view of atomic systems might be an option, but at the present stage of research this is not much more than a speculation. In any case the theory of atomic reasoning deserves much more attention in proof-theoretic semantics than devoted to it so far.

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[12] See Schroeder-Heister (2012b). This approach goes back to Prawitz (1965, appendix B) and Tennant (1982).

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