Advanced Mathematical Methods

WS 2019/20

1 Linear Algebra

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Outline: Linear Algebra

1.9 Quadratic forms and sign definitness

Readings

- Knut Sydsaeter, Peter Hammond, Atle Seierstad, and Arne Strøm. *Further Mathematics for Economic Analysis*. Prentice Hall, 2008 Chapter 1

Online Resources

MIT course on Linear Algebra (by Gilbert Strang)

- Lecture 26: Symmetric matrices and positive definiteness
  https://www.youtube.com/watch?v=umt6BB1nJ4w
- Lecture 27: Positive definite matrices and minima – Quadratic forms
  https://www.youtube.com/watch?v=vF7eyJ2g3kU

1.9 Quadratic forms and sign definitness

Definitions

- Degree of a polynomial
- Form of *n*th degree
- special case: quadratic form

$$Q(x_1, x_2) = a_{11}x_1^2 + 2a_{12}x_1x_2 + a_{22}x_2^2$$

1.9 Quadratic forms and sign definitness

A quadratic form $Q(x_1, x_2)$ for two variables x_1 and x_2 is defined as

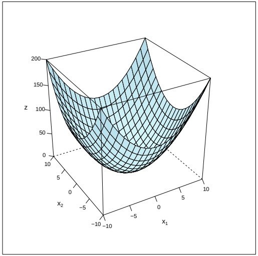
$$Q(x_1, x_2) = \underset{(1\times 2)(2\times 2)(2\times 1)}{\mathbf{x}' \, \mathbf{A} \, \mathbf{x}} = \sum_{i=1}^{2} \sum_{j=1}^{2} a_{ij} \, x_i \, x_j$$

where $a_{ij} = a_{ji}$ and, thus,

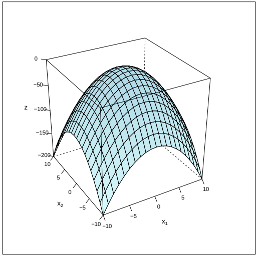
with the symmetric coefficient matrix $\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{bmatrix}$

1.9 Quadratic forms and sign definitness

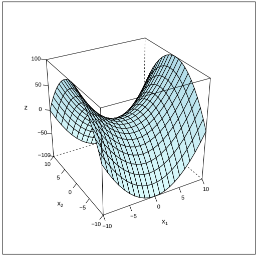
Graph of the positive definite form $Q(x_1, x_2) = x_1^2 + x_2^2$



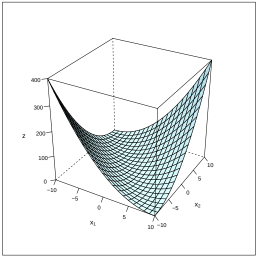
Graph of the negative definite form $Q(x_1, x_2) = -x_1^2 - x_2^2$



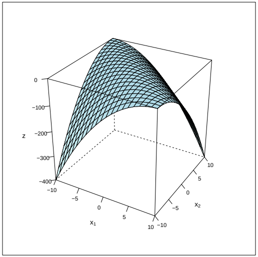
Graph of the indefinite form $Q(x_1, x_2) = x_1^2 - x_2^2$



Graph of the positive semidefinite form $Q(x_1, x_2) = (x_1 + x_2)^2$



Graph of the negative semidefinite form $Q(x_1, x_2) = -(x_1 + x_2)^2$



1.9 Quadratic forms and sign definitness

The quadratic form associated with the matrix $\mathbf{A}$ (and thus the matrix $\mathbf{A}$ itself) is said to be

positive definite,	if $Q = \mathbf{x}'\mathbf{A}\mathbf{x} > \mathbf{0}$	for all $\mathbf{x} \neq \mathbf{0}$
positive semi-definite,	if $Q = \mathbf{x}'\mathbf{A}\mathbf{x} \geq \mathbf{0}$	for all $\mathbf{x}$
negative definite,	if $Q = \mathbf{x}'\mathbf{A}\mathbf{x} < \mathbf{0}$	for all $\mathbf{x} \neq \mathbf{0}$
negative semi-definite,	if $Q = \mathbf{x}'\mathbf{A}\mathbf{x} \leq \mathbf{0}$	for all $\mathbf{x}$

Otherwise the quadratic form is **indefinite**.

Note: For any quadratic matrix $\mathbf{A}$ it holds that $\mathbf{x}'\mathbf{A}\mathbf{x} = \mathbf{x}'\mathbf{B}\mathbf{x}$ with $\mathbf{B} = 0,5 \cdot (\mathbf{A} + \mathbf{A}')$ symmetric.

1.9 Quadratic forms and sign definitness

The quadratic form $Q(\boldsymbol{x})$ is

- positive (negative) definite, if **all** eigenvalues of the matrix $\mathbf{A}$ are positive (negative): $\lambda_j > 0$ $(\lambda_j < 0)$ $\forall j = 1, 2, \ldots, n$;
- positive (negative) semi-definite, if **all** eigenvalues of the matrix $\mathbf{A}$ are non-negative (non-positive): $\lambda_j \geq 0$ $(\lambda_j \leq 0)$ $\forall j = 1, 2, \ldots, n$ and **at least one** eigenvalue is equal to zero;
- indefinite, if two eigenvalues have different signs.

1.9 Quadratic forms and sign definitness

Properties of positive definite and positive semi-definite matrices

1) Diagonal elements of a positive definite matrix are strictly positive. Diagonal elements of a positive semi-definite matrix are nonnegative.
2) If $\mathbf{A}$ is positive definite, then $\mathbf{A}^{-1}$ exists and is positive definite.
3) If $\mathbf{X}$ is $n \times k$, then $\mathbf{X}'\mathbf{X}$ and $\mathbf{X}\mathbf{X}'$ are positive semi-definite.
4) If $\mathbf{X}$ is $n \times k$ and $\text{rk}(\mathbf{X}) = \mathbf{k}$, then $\mathbf{X}'\mathbf{X}$ is positive definite (and therefore non-singular).