

# Advanced Mathematical Methods

WS 2019/20

## 1 Linear Algebra

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WIRTSCHAFTS- UND  
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# Outline: Linear Algebra

## 1.8 Eigenvalues and eigenvectors

# Readings

- ▶ Knut Sydsaeter, Peter Hammond, Atle Seierstad, and Arne Strøm. *Further Mathematics for Economic Analysis*. Prentice Hall, 2008 Chapter 1

# Online Resources

MIT course on Linear Algebra (by Gilbert Strang)

- ▶ Lecture 21: Eigenvalues and Eigenvectors  
<https://www.youtube.com/watch?v=IXNXrLcoerU>
- ▶ Lecture 22: Powers of a square matrix and Diagonalization  
<https://www.youtube.com/watch?v=13r9QY6cmjc>

## 1.8 Eigenvalues and eigenvectors

assume a scalar  $\lambda$  exists such that

$$\mathbf{Ax} = \lambda\mathbf{x}$$

$\lambda$ : eigenvalue

$\mathbf{x}$ : eigenvector

find  $\lambda$  via the homogenous linear equation system

$$(\mathbf{A} - \lambda\mathbf{I})\mathbf{x} = \mathbf{0}$$

## 1.8 Eigenvalues and eigenvectors

The properties of a quadratic homogenous linear equation system imply that:

- ▶ in any case a solution does exist;
- ▶ if  $\det(\mathbf{A} - \lambda \mathbf{I}) \neq 0$ , then  $\bar{\mathbf{x}} = \mathbf{0}$  is the trivial solution;
- ▶ only if  $\det(\mathbf{A} - \lambda \mathbf{I}) = 0$  there is a non-trivial solution.

## 1.8 Eigenvalues and eigenvectors

Determination of the eigenvalues via *characteristic equation*:

$$|\mathbf{A} - \lambda \mathbf{I}| = 0 \iff (-1)^n \lambda^n + \alpha_{n-1} \lambda^{n-1} + \dots + \alpha_1 \lambda + \alpha_0 = 0$$

for every (real or complex) eigenvalue  $\lambda_i$  of the  $(n \times n)$ -Matrix  $\mathbf{A}$  we can calculate the respective eigenvector  $\mathbf{x}_i \neq \mathbf{0}$  solving the homogenous linear equation system

$$(\mathbf{A} - \lambda_i \mathbf{I}) \mathbf{x}_i = \mathbf{0} . \quad (1)$$

The properties of homogenous linear equation systems imply that the solution of eq. (1) is not unambiguous, i.e. for the eigenvalue  $\lambda_i$  we can find infinitely many eigenvectors  $\mathbf{x}_i$

## 1.8 Eigenvalues and eigenvectors

**A** und **B** (quadratic matrices of order  $n$ ) are similar if a regular  $(n \times n)$  - matrix **C** exists, such that

$$B = C^{-1}AC .$$

Special case: **symmetric matrices**

For a symmetric  $(n \times n)$ -matrix **A** it holds that the normalized eigenvectors  $\tilde{x}_j$  with  $j = 1, \dots, n$  have the property

- (1)  $\tilde{x}'_j \tilde{x}_j = 1$  for all  $j$  and
- (2)  $\tilde{x}'_i \tilde{x}_j = 0$  for all  $i \neq j$ .



## 1.8 Eigenvalues and eigenvectors

### Principle axis theorem

collecting the normalized eigenvectors  $\tilde{\mathbf{x}}_j$  ( $j = 1, \dots, n$ ) in a new matrix  $\mathbf{T} = [\tilde{\mathbf{x}}_1 \cdots \tilde{\mathbf{x}}_n]$  with the property  $\mathbf{T}^{-1} = \mathbf{T}'$  yields the diagonalization of  $\mathbf{A}$  as follows:

$$\mathbf{D} = \mathbf{T}'\mathbf{A}\mathbf{T} = \mathbf{T}^{-1}\mathbf{A}\mathbf{T} = \begin{bmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \dots & 0 & \lambda_n \end{bmatrix}$$

## 1.8 Eigenvalues and eigenvectors

### Properties of eigenvalues

- 1) The product of the eigenvalues of a  $(n \times n)$ -matrix yields its determinant:  $|\mathbf{A}| = \prod_{i=1}^n \lambda_i$ .
- 2) From 1.) it follows that a singular matrix must have at least one eigenvalue  $\lambda_j = 0$ .
- 3) The matrices  $\mathbf{A}$  and  $\mathbf{A}'$  have the same eigenvalues.
- 4) For a non-singular matrix  $\mathbf{A}$  with eigenvalues  $\lambda$  we have:  
 $|\mathbf{A}^{-1} - \frac{1}{\lambda} \mathbf{I}| = 0$ .
- 5) Symmetric matrices have only real eigenvalues.
- 6) The rank of a symmetric matrix  $\mathbf{A}$  is equal to the number of eigenvalues different from zero.