# Mathematical Methods in Economics and Business

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## Preparatory Course – Part 2

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# Part 2. Functions of One Variable

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# Motivation

# Motivation

Functions of real variables are among the most important means of investigation and presentation for the description and illustration of economic issues and connections.

One of the central tasks in economics is the analysis of relationships between economic variables. For example it is examined in which way:

- consumption depends on the (national) income: *consumption function*;
- demand depends on the price of a good: *demand function*;
- quantity of a produced good depends on the used factor: *production function*;
- "utility" of a household depends on the amount of the consumed goods: *utlity function*.

# Basic Definitions

# 2.1 Basic Definitions

**Definition: Mapping or Function**

Let $X, Y$ be sets. A rule $f$, which assigns every $x \in X$ exactly one $y \in Y$ is called mapping or function of the set $X$ in the set $Y$. We write:

$$f : X \to Y \qquad \text{or elementwise} \qquad x \in X \mapsto f(x) = y \in Y$$

**Definition: Domain and Range**

A (real-valued) **function** of a real variable $x$ with **domain** $D$ is a rule that assigns a unique real number to each real number $x$ in $D$. As $x$ varies over the whole domain, the set of all possible resulting values $f(x)$ is called the **range/image** of f.

# 2.1 Basic Definitions

**Definition: Monotonicity**

Let $I \in \mathbb{R}$ be an interval and $f : I \to \mathbb{R}$ a function. Then $f(x)$ is called:

- strictly monotonically increasing if for all $x_1, x_2 \in I$ it holds:
  $x_1 < x_2 \Rightarrow f(x_1) < f(x_2)$
- strictly monotonically decreasing if for all $x_1, x_2 \in I$ it holds:
  $x_1 < x_2 \Rightarrow f(x_1) > f(x_2)$
- monotonically increasing if for all $x_1, x_2 \in I$ it holds:
  $x_1 < x_2 \Rightarrow f(x_1) \leq f(x_2)$
- monotonically decreasing if for all $x_1, x_2 \in I$ it holds:
  $x_1 < x_2 \Rightarrow f(x_1) \geq f(x_2)$.

# Graphs of Functions

# 2.2 Graphs of Functions

Functions (of one variable) can be presented in three different ways:

**Function Table or Table of Values:**

The respective function values of selected points in the domain are provided in a table.

**Functional Equation:**

Equation of the form $y = f(x)$; where $y$ is called dependent variable, $x$ is called independent variable or argument of $f$.

**Graphical Illustration:**

The graph of the function is illustrated in a Cartesian coordinate system.

# 2.2 Graphs of Functions

## §32a (1) EStG: Income Tax Rate

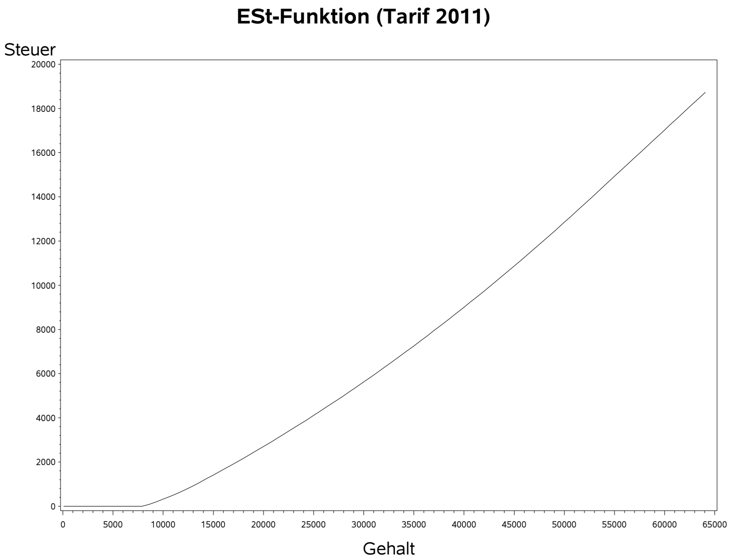
[1]Collective income tax is calculated based on taxable income. [2]Subject to §§32b, 32d, 34, 34a, 34b and 34c it shall be in euros for the taxable income

1. until 7 834 Euro (personal exemption):
   0;
2. from 7 835 Euro to 13 139 Euro:
   $(939.68 \cdot y + 1\,400) \cdot y$;
3. from 13 140 Euro to 52 551 Euro:
   $(228.74 \cdot z + 2\,397) \cdot z + 1\,007$;
4. from 52 552 Euro to 250 400 Euro:
   $0{,}42 \cdot x$ - 8 064;
5. from 250 401 Euro :
   $0{,}45 \cdot x$ - 15 576.

[3]„y“ denotes one ten thousandth of the 7 834 Euro part of the taxable income exceeding the personal exemption, rounded down to a full euro amount. [4]„z“ is one ten thousandth of the exceeding 13 139 Euro part of the rounded down to a full euro amount of the taxable income. [5]„x“ is the taxable income rounded down to a full euro amount. [6]The resulting tax amount shall be rounded down to the nearest full euro amount.

Quelle: `www.gesetze-im-internet.de/estg/__32a.html`

# 2.2 Graphs of Functions



# 2.2 Graphs of Functions

- Tabular summaries of a functional relation are regularly found in empirically collected data (example: demand function).
  By means of econometric methods, functional relations are estimated based on data points.
- In econometric applications the notation $y = y(x)$ is often found, meaning the symbol of the function relation is identical to the symbol of the dependent variable.
- The graph of a function is represented in set notation as follows:

$$G_f = \{x, f(x) : x \in D(f) \wedge f(x) \in range\}$$

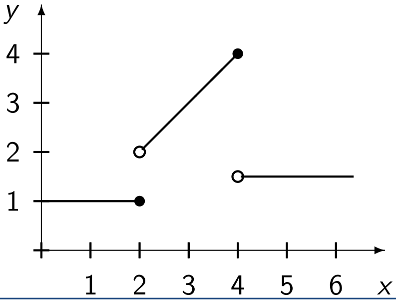
Functions can be classified based on various aspects. We will only consider some function types, which are important in economic applications.

# Digression: Piecewise Functions

**Definition: Piecewise Functions**

If a function is defined on a sequence of disjunct sections, such that a seperate formula for each of these sections of the domain is given, it is called *piecewise defined*.

$$f(x) = \begin{cases} 1 & \text{for} \quad 0 \leq x \leq 2 \\ x & \text{for} \quad 2 < x \leq 4 \\ 1.5 & \text{for} \quad x > 4 \end{cases}$$



# Linear Functions

# 2.3 Linear Functions

Linear functions are the most common type of functions used in economics.

The general form of a linear function is $y = ax + b$, where $a$ is the parameter that defines the slope of the function, and $b$ defines the intercept.

The graph of a linear function is a straight line.

To compute the slope of a straight line in the plane two distinct points on the line are chosen and the difference of the respective ordinate values is related to the difference of the corresponding abscissa values. The slope $a$ of the straight line passing through the points $X = (x_1, x_2)$ and $Y = (y_1, y_2)$ is, therefore, given by $a = \frac{y_2 - y_1}{x_2 - x_1}$

# 2.3 Linear Functions

We consider a so-called linear system of equations which consists of two equations with two unknowns:

$$\begin{aligned} ax + by &= c \\ dx + ey &= f \end{aligned}$$

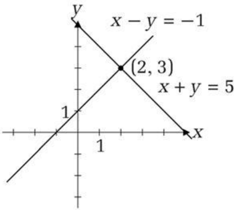
where $a$, $b$, $c$, $d$, $e$, and $f$ are given. The solution of this system of equations can be illustrated graphically by depicting the solution set of each of the two equations as a straight line. The solution set depends on the relation of the two lines:

- if the two lines intersect, the point of intersection is the solution of the system of equations;
- if the lines are parallel, the system has no solution;
- if the two lines are congruent, there exist infinitely many solutions.

# 2.3 Linear Functions

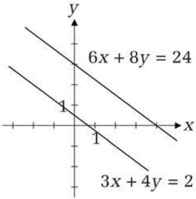
**1. Case:** Unique solution

$$x + y = 5$$
$$x - y = -1$$



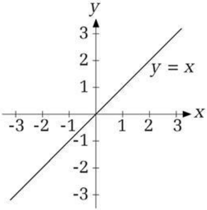

**2. Case:** No solution

$$3x + 4y = 2$$
$$6x + 8y = 24$$



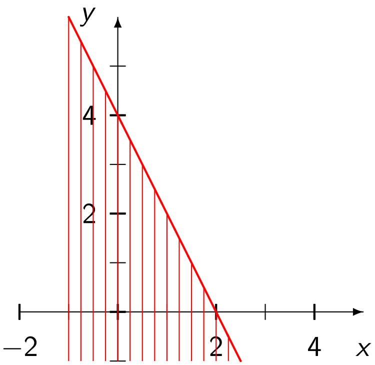

**3. Case:** $\infty$-many solutions

$$x - y = 0$$
$$2x - 2y = 0$$



# 2.3 Linear Functions

Besides the above mentioned linear equations it is also possible to illustrate linear *inequalities* (cf. section 1.5) graphically. In this manner, the set of all pairs of numbers $(x, y)$ that satisfy the inequality $y \leq -2x + 4$ can be represented as:



# 2.3 Linear Functions

Economic examples of linear relations:

- **Linear aggregate consumption function:** $C = a + bY$, where $C =$ aggregate consumption; $Y =$ national income. The parameter $b \in [0; 1]$ is referred to as marginal propensity to consume. It indicates by how many units consumption rises in the respective economy when the income increases by one unit.
- **Market for a good:** Model assumptions: Linear demand function $D = a - bP$ and linear supply function $S = \alpha + \beta P$. The intersection of the two straight lines yields the equilibrium price $P^*$ and the equilibrium quantity $Q^*$.

# Quadratic Functions

# 2.4 Quadratic Functions

If it is reasonable that a variable in some economic models decreases down to some minimum value and then increases, or else increases up to some maximum value and then decreases it is sensible to apply quadratic functions.

General form of a quadratic function: $y = f(x) = ax^2 + bx + c$
where $a, b$, and $c$ are the parameters of the functions and it is assumed that $a \neq 0$.

The graph of a quadratic function is either a parabola that opens upwards ($a > 0$) or downwards ($a < 0$).

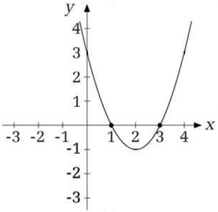
Interesting points of a parabola are:

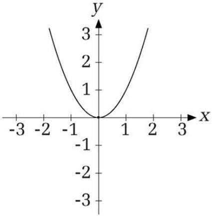
a) the intersections with the abscissa, that can be determined by solving the equation $f(x) = 0$ and

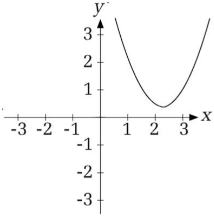
b) the location of the vertex, that can often be determined using the first derivative of the function and by solving the equation $f'(x) = 0$.

# 2.4 Quadratic Functions

A quadratic function can have two, one, or no intersection with the abscissa:








Some economic models lead to quadratic functions for which the location of the extreme point needs to be determined.

For instance, this is the case for the profit function in a monopoly (profit is defined in this case as revenue minus costs). We will consider such examples later in the lecture with the methods of differential calculus.

# Polynomials

# 2.5 Polynomials

Linear and quadratic functions are special cases of a general polynomial function of the type

$$P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x^1 + a_0 \ ,$$

where the coefficients $a_i, i = 1, \ldots, n$ are either constants or parameters of the polynomial.

The **degree of the polynomial** is defined by the highest occurring exponent $n \in \mathbb{N}$.

If $n = 3$, it is called a cubic function :

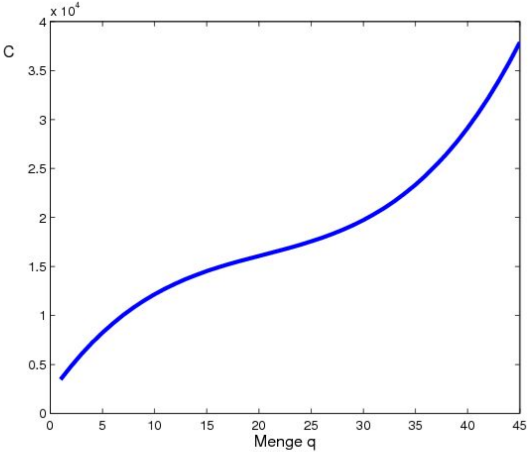
$$y = ax^3 + bx^2 + cx + d$$

The graph of a cubic function can, depending on the choice of $a$, $b$, $c$, and $d$, vary drastically.

# 2.5 Polynomials

Example of a cubic cost function with:

$$C = q^3 - 61\,q^2 + 1500\,q + 2000$$



# 2.5 Polynomials

The cubic cost function represents a typical application in economics and business studies. For polynomials with degree $n$ it holds that they have at most $n$ roots in the range of real numbers or exactly $n$ roots in the range of complex numbers. Denoting the solutions with $x_i, i = 1, \ldots, n$, it holds that:

$$P(x) = \prod_{i=1}^{n} (x - x_i)$$

This representation of a polynomial is called **factorization**.

**Note:** the same solution can appear multiple times; if only real coefficients should be depicted then a modified version applies (*for this see the example in Sydsæter/Hammond (2.A.), p. 145ff; (3.A.), p.140ff*).

# 2.5 Polynomials

The determination of the zeros/solutions/roots is generally not an easy task. It only works analytically for small $n$ and, even then, not always in closed form. Regularly numerical procedures need to be employed.

The representation of the polynomial by factorization can also be used to reduce the degree of the polynomial by one. Is it possible for example to guess a solution $x_1$ then it applies:

$$P(x)/(x - x_1) = (x - x_2) \cdots (x - x_n)$$

Is the polynomial not in the factorized form it is possible to reduce the degree of the polynomial analogously by **polynomial-division**. Here the polynomial is divided in much the same way as a number.

# 2.5 Polynomials

## Example of Polynomial-Division:

$$
\begin{array}{lll}
(-x^3 \quad +4x^2 \quad -x \quad -6) & \div (x-2) = -x^2+2x+3 & \\
\underline{-x^3+2x^2} & \longleftarrow & \boxed{-x^2(x-2)} \\
\qquad 2x^2 - x & & \\
\qquad \underline{2x^2-4x} & \longleftarrow & \boxed{2x(x-2)} \\
\qquad\qquad 3x-6 & & \\
\qquad\qquad \underline{3x-6} & \longleftarrow & \boxed{3(x-2)} \\
\qquad\qquad\quad 0 & &
\end{array}
$$

Thus, it is $(-x^3+4x^2-x-6) \div (x-2) = -x^2+2x+3$.
However, because also $-x^2+2x+3 = -(x+1)(x-3)$, it results:

$$-x^3+4x^2-x-6 = -(x+1)(x-3)(x-2)$$

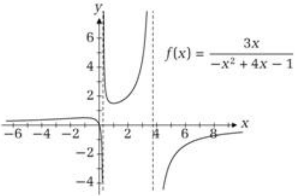
## 2.5 Polynomials

The ratio of two polynomials $P(x)$ and $Q(x)$ is called **rational function** of the form

$$f(x) = \frac{P(x)}{Q(x)} \; .$$

Characteristic for the graph of a rational function are poles and/or asymptotes.

As an example the function $f(x) = \dfrac{3x}{-x^2 + 4x - 1}$ is shown here:

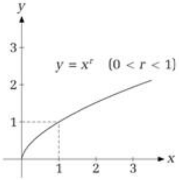


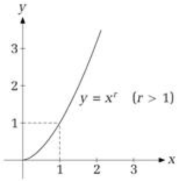
# 2.5 Polynomials

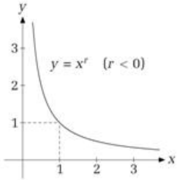
The general form of a power function is: $f(x) = Ax^r$ where $x \geq 0$ is set without loss of generality (w.l.o.g.) and $A$ and $r$ are arbitrary constants.

The shape of the graph depends crucially on the value of $r$:

| *Rootfct.* | *Parabola* | *Hyperbola* |
|---|---|---|
| $0 < r < 1$ | $r > 1$ | $r < 0$ |



# Exponential Functions

# 2.6 Exponential Functions

Growth and shrinkage processes play a major role in economics. To model or describe such phenomena, exponential functions are frequently used.

The general form of an exponential function is $f(x) = A\,a^x$, where $A$ and $a > 0$ are the parameters of the function.

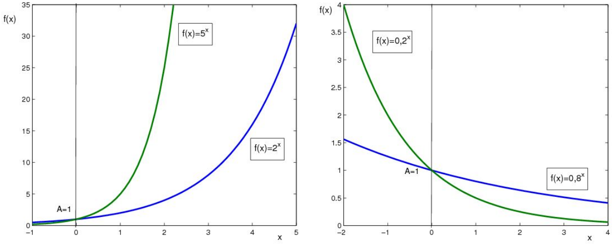
Remarks:

- The interpretation of growth processes gets particularly clear when using the time with label $t$ as independent variable.
- Note the difference between the power function $f(x) = x^a$ on the one hand and the exponential function $f(x) = a^x$ on the other hand!
- In case of the exponential function the independent variable is in the exponent.

# 2.6 Exponential Functions

Graphs of the exponential function for a varying base *a*:

$a > 1$ $\qquad\qquad$ $0 < a < 1$




*Make yourself clear how the above graphs change for varying a.*

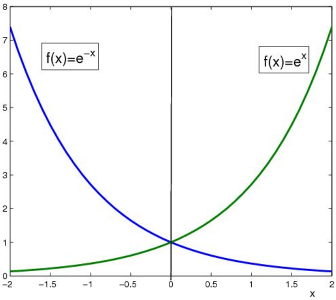
Example: Growth of a bacterial culture.

# 2.6 Exponential Functions

An important special case constitutes the **natural exponential function** with base $e = 2.718...$, Euler's number.

Notation wise either $f(x) = e^x$ or $f(x) = \exp(x)$ is used to denote this function.

The natural exponential function is either monotonically increasing or monotonically decreasing:



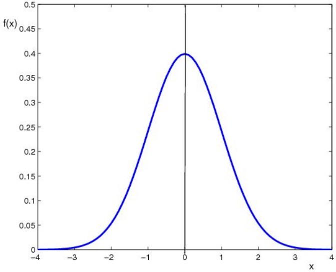
# 2.6 Exponential Functions

In statistics the natural exponential function plays a crucial role as density function of the normal distribution.

The density function of the standard normal distribution follows as

$$f(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) .$$

The graph of the density function of the standard normal distribution is the well known Gaussian bell curve:



# Logarithmic Functions

# 2.7 Logarithmic Functions

Taking the logarithm is besides taking the root a second inversion of exponentiation. By taking the root the base of the power is determined, whereas by taking the logarithm the exponent of the power is determined.

**Logarithm**

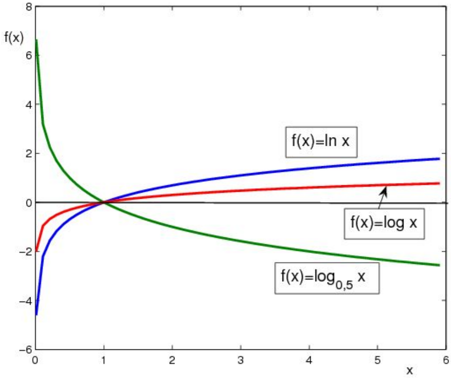
The number $x$ with $b^x = a$ is called the logarithm of $a$ to the base $b$ and is denoted $log_b(a)$.

The general form of a logarithmic function is $f(x) = \log_a x$ where $a > 0$ and $a \neq 1$ is the base of the function.
Of particular importance as bases are the values

- $a = 10$ (common logarithm; notation $\log x$) and
- $a = e$ (natural logarithm; notation $\ln x$)

# 2.7 Logarithmic Functions

The logarithmic function is either monotonically increasing (for bases $a > 1$) or monotonically decreasing (for bases $0 < a < 1$):




Furthermore, it holds that the larger the base the flatter the logarithmic function.

# 2.7 Logarithmic Functions

What does logarithm actually mean?

$\rightarrow$ The word logarithm means the same as exponent or superscript.

Starting point for the understanding of the logarithm is the equivalence of the exponential equation $x = a^y$ and $y = \log_a x$, the logarithm of $x$ to the base $a$.

That is, $y$ is the exponent with which the base $a$ must be exponentiated to get $x$. For every arbitrary base $a \in \mathbb{R}^+ \setminus \{1\}$ and the strictly positive exponents $x$ and $y$ the logarithmic laws hold.

**Logarithmic Laws**

$\log_a(x \cdot y) = \log_a x + \log_a y$ $\qquad$ $\log_a(x/y) = \log_a x - \log_a y$

$\log_a x^r = r \cdot \log_a x \qquad (r \in \mathbb{R})$

# 2.7 Logarithmic Functions

From these rules of calculation two important special cases follow:

- $\log_a(1/x) = \log_a(x^{-1}) = -\log_a x$
- $\log_a(\sqrt[n]{x}) = \log_a(x^{1/n}) = \frac{1}{n}\log_a x$

**Note:** There are no transformations for $\log(x+y)$ or $\log(x-y)$.

In general: $\log_a a^x = x$ and $a^{\log_a x} = x$.

Especially for the common or the natural logarithm it holds:

- $\log 10^x = x$ and $10^{\log x} = x$
- $\ln e^x = x$ and $e^{\ln x} = x$.

It holds: $\ln 1 = 0$ and $\log 1 = 0$.

# 2.7 Logarithmic Functions

Pocket calculators and also mathematical software packages only contain the common and natural logarithm.

Starting point for the so-called change of base is again the exponential equation $a^y = x$.
Taking $\log_b$ on both sides and transforming equivalently yields:

$$\log_a x = \frac{1}{\log_b a} \log_b x \ ,$$

i.e. the logarithm of $x$ to the base $a$ is proportional to the logarithm of $x$ to the base $b$! Sensibly we choose for the base $b$ either 10 or $e$ and get

**Formulas for the Change of Base**

$$\log_a x = \frac{\log x}{\log a} = \frac{\ln x}{\ln a} \ .$$

# 2.7 Logarithmic Functions

The Duplication Problem:

The logarithm turns out to be useful for answering questions of the kind „*How long does it take for a stock/ population, that grows at a fixed rate, to double, triple,...?*'.

This will be illustrated here using an example from financial mathematics.

Consider a (savings) account with an investment amount $K_0$; the interest rate is fixed at $p\%$ and the amount of interest is reinvested. How long does it take until the money in the account has doubled solely through interest payments?

Starting point: $K_t = K_0 \cdot q^t$; from this follows through appropriate transformation, exploiting the information that $K_t = 2 \cdot K_0$:

$$t = \frac{\ln 2}{\ln q} \; .$$

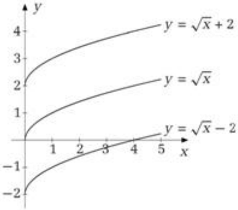
# Composite, Inverse, and Implicit Functions

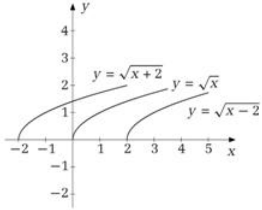
# 2.8 Composite, Inverse, and Implicit Functions

The shifting of graphs of a function and the analysis of the resulting effects are the basic instruments of many economic studies. It is, thus, important that you know how a "right shift" can occur.

We consider the following two important cases (for $c > 0$):

- $y = f(x) + c \quad \rightarrow$ Shift along the ordinate;
- $y = f(x + c) \quad \rightarrow$ Shift along the abscissa;



# 2.8 Composite, Inverse, and Implicit Functions

For real functions, the arithmetic operations of addition, subtraction, multiplication and division can be applied. Examples for this can be found e.g. in Sydæter / Hammond (2.A., 3.A.) chapter 5.2.

Of particular importance in this context is the operation of **composing/chaining** functions, which we want to take a closer look at here.

A so-called **exterior function** $z = f(y)$ and a so-called **kernel/interior function** $y = g(x)$ yield a **composite/chained function** as follows:

$$z = f(g(x)) = f \circ g(x) \qquad \text{where} \quad x \in D(g) \; .$$

Requirement for the composition: The range of the interior function $g$ must be a subset of the domain of the exterior function $f$, i.e. it must hold: $range(g) \subset D(f)$.

# 2.8 Composite, Inverse, and Implicit Functions

An important application of the concept of composite functions are monotone transformations of functions.

**Definition: Monotone Transformation**

Let $I \subset \mathbb{R}$, $g$ be a real-valued function and $f : I \to \mathbb{R}$ be a strictly monotonously increasing function. Then the composite function

$$f \circ g(x) = f(g(x))$$

constitutes a (positive) monotone transformation of $g(x)$.

Examples for monotone transformations:

- Addition of an arbitrary constant;
- Multiplication of a <u>positive</u> number;
- Exponentiation of an <u>odd</u> number;
- Taking the logarithm;
- Forming the exponential function.

# 2.8 Composite, Inverse, and Implicit Functions

**Definition: Symmetry**

In general three types of symmetry can be distinguished.

If for all $x$ in the domain of $f$ it holds:

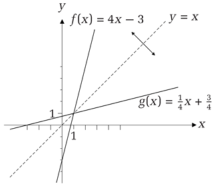
1. $f(-x) = f(x)$, then $f$ is called an *even* function, $f$ is *symmetric about the y-axis*.
2. $f(-x) = -f(x)$, then $f$ is called an *odd* function, $f$ is *symmetric about the origin*.
3. $f(a+x) = f(a-x)$, then $f$ is called *symmetric about the line* $x = a$

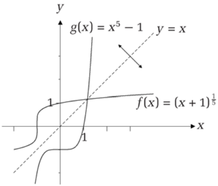
# 2.8 Composite, Inverse, and Implicit Functions

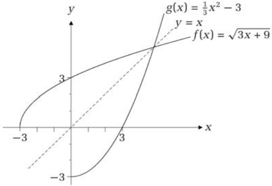
**Definition: Inverse Function**

A function $y = f(x)$ with $x \in D(f); y \in range(f)$ is called unique, if there exists for every value $y$ exactly one value $x$. For the unique function $y = f(x)$ there exists an **inverse function** $x = g(y) = f^{-1}(y)$. It holds $D(f^{-1}) = range(f)$ and $range(f^{-1}) = D(f)$.

The graphs of a function and its inverse are symmetric about the line $x = y$.




Theorem: Let $f$ be a strictly monotone function in $D(f)$. Then it exists an inverse function for $f$, $f^{-1}$, with $D(f^{-1}) = range(f)$. The inverse does not hold.

# 2.8 Composite, Inverse, and Implicit Functions

Up to this point we only considered functions for which the dependent variable $y$ was explicitly described as a function of the independent variable $x$ in the form of $y = f(x)$. In economic applications, however, situations regularly occur in which a function is defined by an equation of the form:

$$F(x, y) = c$$

Such a function is called **implicitly defined function**.

**Note:** Explicitly defined functions can always be transformed into an implicit form, the other way around this is not always possible. Examples of implicitly defined functions (graphs of equations) are circle and ellipse equations. For a center point $(x_0, y_0)$ it holds for these:

$$r = \sqrt{(x - x_0)^2 + (y - y_0)^2} \qquad \text{or} \qquad \frac{(x - x_0)^2}{a} + \frac{(y - y_0)^2}{b} = 1$$