Topoi
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Logics of Proof-Theoretic Validity

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Abstract
In proof-theoretic semantics, the validity of atomic formulas is defined as their derivability in systems of atomic rules. We distinguish two types of such systems and two variants of semantics of formulas, one based on introduction rules for logical constants and one based on elimination rules. We thus define four semantics with their respective consequence relations. As these are not necessarily closed under substitution of arbitrary formulas for atoms, we consider the substitution-closed subsets of these four consequence relations in addition. We show which logics, in the sense of formal systems, are complete for seven of these notions. This systematizes in one place four previous results with three new results. For the semantics based on elimination rules intuitionistic or classical logic is complete, depending on the type of atomic rules, whereas the semantics based on introduction rules characterize intermediate logics. This answers several open questions about the main notions of proof-theoretic validity, although for one semantics of the introduction rule type we can only make a conjecture about possible logics. Further notions of proof-theoretic validity are suggested by considering alternative systems of atomic rules from the perspective of logic programming.

Keywords Proof-theoretic semantics · Intuitionistic logic · Intermediate logics · Classical logic · Prawitz's completeness conjecture · Inquisitive logic · Gödel–Dummett logic · Kreisel–Putnam logic · Medvedev's logic · Logic programming

1 Introduction

One of the central tasks of proof-theoretic semantics is the definition of a notion of validity in terms of proofs. This may be (1) the validity of proofs themselves—sometimes called the validity of arguments, if proofs are understood as valid arguments; (2) the validity of formulas, sentences, propositions or judgements; (3) the validity of inferences, inference steps or inference rules; (4) the validity of consequences—sometimes called the validity of sequents or of consequence statements, if consequences are understood as valid sequents or consequence statements; etc.

All entities just mentioned except proofs can in principle be subjected to classical model-theoretic semantics. What is crucial in proof-theoretic semantics is that the semantic explanation proceeds by invoking derivations or proofs as a basic ingredient.

Proof-theoretic validity is defined with respect to atomic structures, also called atomic systems or bases, which correspond to classical structures in the model-theoretic case. Logical or universal validity is defined, as in the model-theoretic case, as validity with respect to all bases. This means that we are led to the problem of semantical completeness we are used to in logic: Is there a formal system such that what is universally valid is derivable in the formal system and vice versa? Thus the model-theoretic and proof-theoretic approaches are structurally similar: we start from a notion of validity with respect to bases, then define universal validity as validity for all bases, and then ask for a formal system that is complete for this notion of universal validity.

As Prawitz was the one who in the late 1960s and the 1970s opened up this field of research, we denote the validity-based proof-theoretic semantics as *Prawitz semantics* and the completeness of a formal system with respect to such a semantics as *Prawitz completeness*. We use this as a generic term beyond the specific semantics that Prawitz himself consid-

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ered, thus distinguishing Prawitz semantics from Prawitz's semantics. Prawitz (1973, 2014) conjectured that the formal system of intuitionistic logic is complete with respect to a certain semantics of proofs developed by him.

However, we do not use "Prawitz semantics" for all sorts of validity-based proof-theoretic semantics, but only for those in which the bases consist of inference rules for atomic formulas ("atoms") and are ordered by the natural subset ordering. These bases are thus a generalization of Herbrand bases. While an Herbrand base is a set of atomic formulas and thus may be considered a set of axioms, bases in our sense may contain atomic inference rules for the generation of atoms. Unlike classical structures, atomic bases are set-theoretically ordered, and this ordering may enter the semantics (in contradistinction to Herbrand bases, whose set-theoretical order plays no semantical role). We here consider only a single ordering—the natural set-theoretical order—and not several frames of ordered bases. The latter we have in Kripke semantics, which we distinguish here from Prawitz semantics. Moreover, we only consider the set of all possible bases and not any subsets thereof. In our terminology, Prawitz semantics is a single-frame semantics, where this single frame is the powerset of the set $\mathcal{R}$ of all atomic rules with the subset ordering. It turns out that considering multiple-frame semantics has certain welcome features, when carried out in proof-theoretic terms, in particular when it comes to completeness (see Stafford and Nascimento (2023), Schroeder-Heister (2024)). However, this would be a proof-theoretic variant of Kripke completeness that we do not here call Prawitz completeness.

Another semantic variant not considered here are single-frame semantics based on a proper subset of $\mathcal{R}$, even though this represents an interesting semantical topic in itself (see Stafford (2021, 2024)). In the following, we will distinguish between inference rules of different levels. Depending on which level of inference rules we admit in bases, the set $\mathcal{R}$ of all inference rules means something different. More precisely, we will distinguish between the set $\mathcal{R}_1$ of rules of maximal level 1 and the set $\mathcal{R}_2$ of rules of maximal level 2, so that $\mathcal{R}_1 \subsetneq \mathcal{R}_2$. Thus technically, if $\mathcal{R}_2$ is the set of all atomic rules considered anywhere, we are considering one specific subset of it, namely $\mathcal{R}_1$.

Whereas Prawitz in his own papers considers primarily validity of proofs and only secondarily the validity of sentences and consequences, we here consider the validity of consequences, as this framework is currently at the center of interest and well-suited for the framing of completeness claims. Thus we define "$\Gamma \vDash_S A$", which means that A is a valid consequence of the set of sentences Γ with respect to the atomic system S, where the atomic system S is a set of inference rules for the generation of atomic formulas. We confine ourselves to propositional logic (except for Sect. 4, where we consider atomic systems as logic programs), as this suffices to make the points we want to make. This means that atomic systems serve for the generation of propositional letters. When we speak of atoms, we mean such letters, here denoted by $a, b, c, \ldots, p, q, r, \ldots$, with and without indices. Occasionally, we use $\bot$ as a propositional letter in the notation of atomic rules, which corresponds to the logical constant of absurdity, which is denoted by $\bot$ as well.

Historical remark on Prawitz's approach. Prawitz himself favors a different form of semantics. In his work in the late 1960s and 1970s, he defined validity for arguments, which have the structure of proofs. A would follow from Γ with respect to an atomic system S ($\Gamma \vDash_S A$ in our notation), if there is an S-valid argument of A from Γ with respect to S. In the definition of an S-valid argument Prawitz referred to constructive transformations of arguments corresponding to what in the proof theory of natural deduction he called 'reductions'. For example, a closed (i.e., assumption-free) argument for a conjunction is S-valid, if it *reduces* to S-valid proofs of the two conjuncts, followed by the introduction of the conjunction. This approach introduces an additional proof-theoretic concept, that of reduction, into the notion of validity, which goes beyond the provability in atomic systems. This concept allows for additional distinctions within the notion of validity, such as the consideration of reductions uniform with respect to certain parameters (such as the base S). Prawitz also tends to consider atomic bases to be definitions, which calls our below clause ($\vDash_S$) and the corresponding monotonicity of validity into question. All these items cannot be discussed here. The reader is referred to Piecha and Schroeder-Heister (2016b), Schroeder-Heister (2024), Piccolomini d'Aragona (2024), and Stafford (2021). In our own context, as mentioned above, we nevertheless speak of Prawitz completeness, as the general approach to validity-based proof-theoretic semantics and the problem of semantic completeness in these semantics must be credited to Prawitz. It was him who put the question of semantic completeness in the area of proof-theoretic semantics on the agenda, and conjectured it for IPC with respect to the proof-theoretic semantics favored by him.

1.1 Atomic Systems

We distinguish two types of atomic systems according to the sort of rules they contain: level-1 and level-2 systems. Level-1 systems consist of level-1 rules (or 'first-level' or 'production' rules), that is, rules leading from atoms to atoms, and of level-0-rules (or 'axioms'), which just generate a single atom. Level-1 rules have the form

$$\frac{a_1 \quad \dots \quad a_n}{b}$$

for $n \geq 1$, linearly written as $(a_1, \dots, a_n) \rhd b$, and level-0 rules (axioms) have the form

$$\frac{}{b}$$

linearly written as $\rhd b$ or simply b. Together they form the set $\mathcal{R}_1$ of rules. Level-2 systems consist of level-0, level-1 or level-2 rules, which together make up the set $\mathcal{R}_2$ of rules. Level-2 rules (or 'second-level' rules) are rules which allow one to discharge assumptions when they are applied. This is quite analogous to the implication introduction or disjunction elimination rules in natural deduction, but used for atoms only. They have the form

$$\frac{\begin{matrix}[\Gamma_1] \\ a_1\end{matrix} \quad \dots \quad \begin{matrix}[\Gamma_n] \\ a_n\end{matrix}}{b} \tag{1}$$

for $n \geq 1$ and finite sets $\Gamma_1, \dots, \Gamma_n$ of atoms that may be discharged when the rule is applied. Here some, but not all Γs can be empty (otherwise it would be a level-1 rule). In linear notation we write such a rule as

$$((\Gamma_1 \rhd a_1), \dots, (\Gamma_n \rhd a_n)) \rhd b.$$

This can be generalized to higher-level rules, that is, level-k rules for any k (see Schroeder-Heister (1984, 2014), Piecha et al. (2015)). However, as in our context level-k rules for $k > 2$ can be coded by level-2 rules, we do not consider higher-level rules in detail, although in some cases we use them for better readability. One might distinguish axioms-only systems as level-0 systems from level-1 and level-2 systems, but this is not in the focus of this paper (see, however, the remarks on axioms-only systems in Schroeder-Heister (2024), § 7). In the following, we consider notions of proof-theoretic validity with respect to level-1 and to level-2 bases. This distinction is crucial, as we will see that the semantics with respect to these two sorts of bases behave differently.

Remark on the meaning of second-level rules As second-level rules in the general sense of the schema (1), in particular when only dealing with atoms, are not very common, we give some explanation how they are to be understood. Consider as an example the level-2 system

$$S \begin{cases} \rhd a & \text{(Rule 1)} \\ b \rhd c & \text{(Rule 2)} \\ (b \rhd c) \rhd d & \text{(Rule 3)} \\ (a, d) \rhd e & \text{(Rule 4)} \end{cases}$$

The derivation

$$\dfrac{\dfrac{}{a}\,\text{(Rule 1)} \qquad \dfrac{\dfrac{[b]^1}{c}\,\text{(Rule 2)}}{d}\,\text{(Rule 3)}^1}{e}\,\text{(Rule 4)}$$

shows $\vdash_S e$. The assumption b was discharged by the application of Rule 3.

Coding of higher-level rules by second-level rules. In the context of atomic bases, it it easy to see that rules of any level can be reduced to rules of level 2. Suppose we have indefinitely many fresh atoms $e, e_1, e_2, \dots$ for coding purposes at our disposal, something that can easily be arranged in our bases. Consider an arbitrary rule

$$R: \quad ((\Gamma_1 \rhd a_1), \dots, (\Gamma_n \rhd a_n)) \rhd b$$

of level $k > 2$, where the Γ_i may contain rules of any level $< k - 1$ and not necessarily atoms only. We introduce the following additional rules:

$$\Gamma_i \rhd e_i \qquad e_i \rhd \Gamma_i$$

where $e_i \rhd \Gamma_i$ is a set of rules obtained by replacing every rule $\Delta \rhd c$ in Γ_i with $\Delta \cup \{e_i\} \rhd c$ and with $e_i \rhd c$ if Δ is empty. These new rules are of maximal level $k - 1$. When we add them to our base we can replace R with the level-2 rule

$$((e_1 \rhd a_1), \dots, (e_n \rhd a_n)) \rhd b.$$

By iterating this procedure we obtain a level-2 base conservative over the original one. As an example, consider the level-3 rule

$$((a_1 \rhd a_2) \rhd a_3) \rhd b$$

which is replaced with the following three level-1 and level-2 rules:

$$(a_1 \rhd a_2) \rhd e \qquad (e, a_1) \rhd a_2 \qquad (e \rhd a_3) \rhd b.$$

1.2 Proof-Theoretic Validity

Our formulas are built up from an infinite set of atoms (which are propositional letters) by the logical constants $\bot, \wedge, \vee, \rightarrow$ in the usual way. We consider two variants of semantics of formulas by defining for each variant a finite consequence relation $\Gamma \vDash A$ between a finite set of formulas Γ and a formula A. This is carried out by first defining consequence relations $\Gamma \vDash_S A$ with respect to bases S, and then $\Gamma \vDash A$

by generalization over S as a universal (i.e., logical) consequence relation.

The two semantic variants are distinguished according to whether we consider the canonical conditions of introduction (I) rules (in the natural-deduction understanding) or the canonical consequences of elimination (E) rules as meaning-giving for the logical connectives. On the I-rule approach, the clauses run as follows:

$$
\begin{array}{rl}
\text{(At)} & \vDash_S a \;:\Longleftrightarrow\; \vdash_S a. \\
(\bot) & \vDash_S \bot \;:\Longleftrightarrow\; \text{For all } a : \vDash_S a. \\
(\wedge) & \vDash_S A \wedge B \;:\Longleftrightarrow\; \vDash_S A \text{ and } \vDash_S B. \\
(\vee\text{-I}) & \vDash_S A \vee B \;:\Longleftrightarrow\; \vDash_S A \text{ or } \vDash_S B. \\
(\rightarrow) & \vDash_S A \rightarrow B \;:\Longleftrightarrow\; A \vDash_S B. \\
(\vDash_S) & \Gamma \vDash_S A \;:\Longleftrightarrow\; \text{For all } S' \supseteq S\text{: } (\vDash_{S'} \Gamma \Longrightarrow \vDash_{S'} A). \\
(\vDash) & \Gamma \vDash A \;:\Longleftrightarrow\; \text{For all } S : \Gamma \vDash_S A.
\end{array}
$$

Note that in clause ($\vDash_S$) we are referring to extensions S' of a base S. This is the place where the set-theoretic ordering of bases comes into play. Technically, this clause guarantees monotonicity of validity with respect to extensions of bases. Conceptually, its idea is that bases represent states of knowledge, and that what follows from this knowledge continues to follow when additional knowledge is acquired. This clause (then with respect to an accessibility relation rather than the superset relation) is the well-known implication clause in Kripke semantics. It should be emphasized, however, that our semantics are different from Kripke semantics as it is a single-frame semantics, which on the Kripke approach would only represent a limiting case. Incidentally, it is obvious that universal validity is the same as validity in the empty base, that is, $\Gamma \vDash A$ iff $\Gamma \vDash_{\emptyset} A$.

We note a simple result about atomic assumptions, which immediately follows from (At) and ($\vDash_S$).

Lemma 1.1 *For atomic $a_1, \ldots, a_n, b$, we have $a_1, \ldots, a_n \vDash_S b \iff a_1, \ldots, a_n \vdash_S b$.*

Remark on definitions. An alternative view of bases would be to look at them not as knowledge bases but as incorporating definitions. This would change the situation, as with definitions we do not expect monotonicity. This cannot be dealt with here. But see the discussions in Piecha and Schroeder-Heister (2016b), Schroeder-Heister (2024) and Sect. 4 below.

Remark on finiteness of consequence. Our consequence relations $\Gamma \vDash_S A$ and $\Gamma \vDash A$ are only defined for finite Γ, that is, they are by definition compact. We can argue for this restriction by referring to the fact that they should code some consequence notion for arguments in a general sense (see Prawitz), and arguments, like proofs, proceed from a finite number of assumptions. However, there might be related semantic conceptions, for which compactness is not a trivial matter. This topic cannot be discussed here. Stafford raises this question in Stafford (2021, § 4) and mentions systems not based on the full powerset of bases, which are not compact. Clarifying these issues in all detail is still a desideratum.

Remark on the treatment of absurdity and negation. Note that we understand absurdity $\bot$ as a logical constant which expresses the inconsistency of a base S in the sense of the derivability of all atoms in this base. Alternatively, we could understand it as an atomic constant from which in every base every atom follows. We do not understand it, as in Kripke semantics, as something that is never true.
Defining $\bot$ as a non-atomic logical constant by the (infinitary) introduction rule

$$\frac{\{a \mid a \text{ atomic}\}}{\bot}$$

(see Dummett (1991), Chapter 13) leads to the loss of finiteness of logical consequence in the sense that the following implication does *not* hold: If $\{a \mid a \text{ atomic}\} \vDash \bot$, then there is a finite subset Γ_{fin} of $\{a \mid a \text{ atomic}\}$, such that $\Gamma_{\text{fin}} \vDash \bot$. Since finiteness holds for derivability in IPC, Prawitz completeness would not hold for IPC.
We cannot discuss this issue further here, though it is highly significant in proof-theoretic semantics.

When in ($\vDash$) and ($\vDash_S$) we universally quantify over all bases, we must presuppose that the domain of quantification is clear. We said above that we are always considering the powerset of the set of all rules, where each element of this powerset is a base. However, in view of the distinction between the rule sets $\mathcal{R}_1$ and $\mathcal{R}_2$ and correspondingly between level-1 and level-2 bases, we now distinguish between the set of all level-1 bases (i.e., the powerset of $\mathcal{R}_1$) and the set of all level-2 bases (i.e., the powerset of $\mathcal{R}_2$). This gives us two kinds of consequence relations, depending on whether in ($\vDash$) and ($\vDash_S$) we quantify over all level-1 bases or over all level-2 bases. Universal consequence with respect to all level-1 bases will henceforth be denoted by $\vDash^{\mathsf{I1}}$ and consequence with respect to all level-2 bases by $\vDash^{\mathsf{I2}}$. Here I1 stands for 'introduction rule semantics with respect to level-1 bases' and I2 for 'introduction rule semantics with respect to level-2 bases'. The distinction between level-1 bases and level-2 bases is crucial, as semantical completeness heavily relies on it.

Our second variant of semantics puts the elimination rules in front as the primary meaning-determining inference rules. Except in the case of disjunction, the canonical consequences of elimination rules are the same as the canonical conditions of introduction rules. Therefore the E-rule approach has the same clauses as shown above, with the exception of the disjunction clause, which is replaced with the clause

(∨-E) $\vDash_S A \vee B \;:\Longleftrightarrow\;$ For every atom c and every $S' \supseteq S$: if $A \vDash_{S'} c$ and $B \vDash_{S'} c$, then $\vDash_{S'} c$.

Note that in contradistinction to the disjunction elimination rule in natural deduction, we quantify over atoms c rather than arbitrary formulas C. Otherwise this would not work as a clause in an inductive definition. However, this is no real restriction. It is easy to see that the following holds:

Lemma 1.2 $\vDash_S A \vee B \iff$ *For every formula C and every $S' \supseteq S$: if $A \vDash_{S'} C$ and $B \vDash_{S'} C$, then $\vDash_{S'} C$.*

In the formal system of intuitionistic propositional logic IPC we cannot atomize the minor premisses and conclusion of disjunction elimination. We then would, for example, no longer be able to prove $p \vee q$ from $q \vee p$. However, in the case of the semantical clause, this atomization is possible without any loss, as the Lemma shows.

Historical remark on the E-rule approach. In the context of proof-theoretic semantics, the system just described is due to Sandqvist (2015). However, Sandqvist did not present it as a system based on elimination rules. He propagated his approach as a semantics with a deviant clause for disjunction rather than as an E-rule-based approach with its natural disjunction clause. Actually, there have been a couple of approaches in proof-theoretic semantics based on elimination rules (see Dummett (1991), Prawitz (2007), Schroeder-Heister (2015), Litland (2012), Oliveira (2021)). Even the fact that, when systems with disjunction are considered, it is not only definitionally necessary, but also conceptually possible and meaningful to atomize the minor premiss of disjunction elimination, had been considered, at the earliest by Dummett (1991), and later by Ferreira (2006), Prawitz (2007), Litland (2012), and Schroeder-Heister (2015) (see Schroeder-Heister (2024), § 9). Since Sandqvist's paper (2015) the E-rule approach with its specific clause for disjunction dominates a great deal of the discussion in proof-theoretic semantics (see Pym et al. (2024), Gheorghiu and Pym (2023, 2025), Eckhardt and Pym (2024)), which is due to the fact that in computer science, reductive logic, and categorial semantics, the E-rule approach has certain advantages over the I-rule approach. This followup-discussion focused on Sandqvist's specific E-rule approach to 'base-extension' semantics, because this provides a semantics for IPC. Despite this focus, most proof-theoretic semantics have bases and extensions available.

As for the introduction rule variant of our semantics, we can, for the elimination rule approach, distinguish between consequence with respect to level-2 bases and with respect to level-1 bases as its subset. Thus we arrive at universal consequence with respect to all level-1 bases denoted by $\vDash^{\mathsf{E1}}$ and consequence with respect to all level-2 bases by $\vDash^{\mathsf{E2}}$. Here E1 stands for 'elimination rule semantics with respect to level-1 bases' and E2 for 'elimination rule semantics with respect to level-2 bases'. As in the introduction case, the distinction between level-1 bases and level-2 bases is crucial.

In the following we will discuss four semantic approaches based on the two distinctions made so far: that between level-1 and level-2 bases, and that between a semantics related to introduction rules and a semantics related to elimination rules. This gives us four relations of universal (logical) consequence, namely $\vDash^{\mathsf{I1}}, \vDash^{\mathsf{I2}}, \vDash^{\mathsf{E1}}, \vDash^{\mathsf{E2}}$. For each of those we want to characterize its logic syntactically and thus prove completeness.

By the *logic* of a consequence relation $\vDash$ we understand its tautologies, i.e., the set of formulas following from the empty set of assumptions: $\{A \mid \vDash A\}$. As our consequence relations are finite, or in another terminology, satisfy compactness, or are a 'deductive system', the logic of $\vDash$ characterizes the relation $\vDash$, as every $\Gamma \vDash A$ can be expressed as $\vDash \bigwedge \Gamma \to A$, where $\bigwedge \Gamma$ is the conjunction of all elements of Γ.

To simplify notation, the logics of $\vDash^{\mathsf{I1}}, \vDash^{\mathsf{I2}}, \vDash^{\mathsf{E1}}, \vDash^{\mathsf{E2}}$ are denoted as I1, I2, E1, E2, respectively. We are looking for formal systems generating these logics. More precisely, for each of these logics L, we would like to show L = F, where F is known to be the set of formulas derivable in a particular formal system and can thus be identified with this system. Such a result would also be considered a completeness result for this formal system with respect to proof-theoretic validity in our sense. For example, if we can show L = IPC, this means that the intuitionistic propositional calculus, here identified with the set of its derivable formulas, is complete for $\vDash$, where L is the logic of $\vDash$. If F lies between intuitionistic and classical logic it is called an *intermediate logic*.

1.3 Closure Under Substitution

There is a further distinction we must make. The logics (sets of formulas) I1, I2, E1, E2 are not necessarily closed under substitution. They are, of course, closed under renaming of atoms. For example, if $\neg(p \vee q) \to (\neg p \wedge \neg q)$ is in I1, then $\neg(r \vee s) \to (\neg r \wedge \neg s)$ is in there as well. However, the closure under substitution with complex formulas may fail. For example, it can be shown that $\neg\neg p \to p$ is in I1, but $\neg\neg(q \vee r) \to (q \vee r)$ is not in I1. Thus we do not consider closure under substitution to be mandatory for a logic. We call logics that are not closed under substitution *weak logics*. However, we consider the substitution-closed subset of a logic to be an interesting logic in itself. If L $= \{A \mid \vDash A\}$ is the logic of $\vDash$, then its substitution-closed subset Ls is defined as

$\mathsf{Ls} := \{A \mid \vDash A$, and $\vDash A'$ for every substitution instance A' of $A\}$.

Obviously, Ls is the logic of the consequence relation $\vDash'$, which is defined as

$$\Gamma \vDash' A \;:\Longleftrightarrow\; \text{for every set of substitution instances } \Gamma', A' \text{ of } \Gamma, A \text{ it holds that } \Gamma' \vDash A'.$$

By the suffix s we denote the substitution-closed sublogic (= subset) of the logic in question. This means that in addition to I1, I2, E1, E2, we consider their substitution-closed sublogics I1s, I2s, E1s, E2s. The letter s should remind one of 'substitution' or 'strict' or 'structural' (the latter is a common designation for consequence relations closed under substitution).

Note that if, given a logic L, we have a completeness result L = F for a syntactically specified system F whose inference rules are schematic, i.e., closed under substitution, then Ls = L, that is, L is closed under substitution. This will be the case in the elimination based approaches. If IPC is the intuitionistic propositional calculus and CPC the classical one, then it will turn out that E1 = E1s = CPC, and E2 = E2s = IPC.

Conversely, if for a logic L we do not have closure under substitution Ls = L, there can be no completeness L = F for a syntactically specified system F with schematic inference rules. This fact can be used to refute certain completeness claims such as I1 = IPC (which may be considered a variant of Prawitz's completeness conjecture for validity-based proof-theoretic semantics).

Philosophical remark on closure under substitution. This raises the more general question whether logicality should always imply closure under substitution. The view that logics should be closed under substitution can be found in early works on modern logic such as Bolzano (1973):

> But suppose that there is just a single idea in it which can be arbitrarily varied without disturbing its truth or falsity, i.e., if all the propositions produced by substituting for this idea any other idea we pleased are either true altogether or false altogether, presupposing that they have a denotation. […] I permit myself, then, to call propositions of this kind, borrowing an expression from Kant, *analytic*. (Bolzano (1973), §148, p. 192)

Tarski (1956) argued:

> Consider any class K of sentences and a sentence X which follows from the sentences of this class. From an intuitive standpoint it can never happen that both the class K consists only of true sentences and the sentence X is false. Moreover, since we are concerned here with the concept of logical, i.e. *formal*, consequence, and thus with a relation which is to be uniquely determined by the form of the sentences between which it holds, this relation cannot be influenced in any way by empirical knowledge, and in particular by knowledge of the objects to which the sentence X or the sentences of the class K refer. The consequence relation cannot be affected by replacing the designations of the objects referred to in these sentences by the designations of any other objects. (Tarski (1956), pp. 414–415)

And more recently Stalnaker (1977) wrote:

> A substitution instance of an argument is an argument of the same form, and arguments are logically valid only if all arguments of the same form are logically valid. (Stalnaker (1977), p. 332)

However, there are other notions of uniform substitution. For example, Humberstone offers the following:

> […] closure under uniform substitution of propositional variables (rather than arbitrary formulas) for propositional variables. (Humberstone (2011), p. 188)

In view of Prawitz completeness for intuitionistic logic and the lack of closure under substitution for I1 we note that the BHK interpretation as, e.g., formulated by Heyting (1971) (see de Campos Sanz and Piecha (2014)) explicitly demands closure under substitution as follows:

> A logical formula with proposition variables, say $\mathfrak{A}(p, q, \ldots)$, can be asserted, if and only if $\mathfrak{A}(\mathfrak{p}, \mathfrak{q}, \ldots)$ can be asserted for arbitrary propositions $\mathfrak{p}, \mathfrak{q}, \ldots$; that is, if we possess a method of construction which by specialization yields the construction demanded by $\mathfrak{A}(\mathfrak{p}, \mathfrak{q}, \ldots)$. (Heyting (1971), p. 103)

From this point of view, Prawitz semantics in the form of I1 would have to be discarded as a semantics for intuitionistic logic from the beginning.
Logics without closure under substitution (which are also called "weak" or "non-structural") are discussed in Stafford (2021).

1.4 Completeness Results

In what follows we present completeness results for the eight systems considered here, partly by reference to the literature and partly by proofs of our own. Before going into technical details, we list the results.

Theorem 1.3 (Main Theorem) *For the logics of our eight semantics the following holds:*

1. E2 = E2s = IPC. *Here* IPC *is intuitionistic propositional logic.*
2. E1 = E1s = CPC. *Here* CPC *is classical propositional logic.*
3. I2 = GKPL. *Here* GKPL *is generalized Kreisel–Putnam logic, which is an intermediate logic satisfying the generalized Kreisel–Putnam axiom.*
4. I2s = ML. *Here* ML *is Medvedev's logic, which is known to be the largest intermediate logic having the disjunction property (see Maksimova (1986)).*
5. I1 = SL. *Here* SL *is an intermediate logic defined below (see Sect. 3.3).*
6. I1s = ?. *Here ? is an intermediate logic, whose axioms we are not yet able to describe. It does not contain Medvedev's logic* ML. *We guess that it contains Rose's system* $\mathfrak{P}$ *(see Rose (1953)), also known as Scott's logic.*

We remark two striking features:

First, the semantics based on elimination rules is extremely well-behaved: For bases with second-level rules, we have completeness of intuitionistic logic with respect to this semantics, and for bases with first-level rules only, we have completeness of classical logic.

Second, the semantics based on introduction rules, which is normally considered the most natural approach in proof-theoretic semantics, are much more complicated. With respect to these semantics intuitionistic logic is incomplete and classical logic is unsound. Instead these semantics determine certain intermediate systems.

The order of items in the main theorem is chosen accordingly: With the simple and straightforward cases first and the not so simple cases second. In what follows in this paper, we discuss these items in turn, either by giving a proof ourselves, or by referring to the literature where the reader can find a proof, or by explicating why we do not have a definite answer (in the last case). Section 2 deals, in its two subsections, with the elimination rule approaches (Main Theorem, items 1 and 2), and Sect. 3 deals, in its four subsections, with the introduction rule approaches (Main Theorem, items 3 to 6).

In order to repeat what should be clear by now: in this paper we are interested in characterizing the logics of certain proof-theoretic consequence relations by certain formal systems. This immediately establishes the incompleteness of different (non-equivalent) formal systems with respect to these notions of consequence. It thus contributes to the big discussion of incompleteness results in proof-theoretic semantics, in particular the incompleteness of intuitionistic logic IPC with respect to certain semantics based on introduction rules (see Piecha et al. (2015), Piecha (2016), Piecha and Schroeder-Heister (2019), Stafford (2021, 2024)). However, here we are interested in positive results: If IPC is incomplete for a certain semantics, then which logic is complete, i.e., is *the* logic of this semantics in the realm of intermediate logics, and likewise for other systems? (IPC has always been the standard example of constructive logics. That is why researchers including Prawitz and ourselves have been interested so much in its completeness. IPC is not our target system here.)

2 The Logics of the Elimination Rule Approaches

As mentioned above, the logics of the elimination rule approaches are extremely well-behaved. If we use second-level bases, then we obtain intuitionistic logic. If we admit only first-level bases, then we obtain classical logic. The first result was proved by Sandqvist (2015), for the second one we present a proof ourselves. As intuitionistic logic IPC and classical logic CPC are closed under substitution, closure under substitution is not an issue for these two semantics.

2.1 The Logic of E2 and E2s: Intuitionistic Logic IPC

Sandqvist (2015) proved the completeness of intuitionistic logic with respect to the proof-theoretic semantics based on elimination rules. He shows that, in our terminology, E2 = E2s = IPC. Sandqvist himself described his semantics quite unspecifically as a "base-extension semantics" with a deviant clause for disjunction, which then needs a specific justification. Considered as a straightforward elimination rule semantics it fits perfectly into our taxonomy of semantical systems. For a short historical discussion of systems based on elimination rules see our historical remark after Lemma 1.2.

We would like to add an observation concerning the notion of disjunction in E2. Even though disjunction is defined via its elimination rule, we have the logical disjunction property, as we have it in IPC, simply due to the completeness result. Thus globally, when quantifying over all bases S, we have that $\vDash A \vee B$ implies $\vDash A$ or $\vDash B$. However, locally, when considering single bases S, we do not always have this property, that is, we do not necessarily have that $\vDash_S A \vee B$ implies $\vDash_S A$ or $\vDash_S B$. This difference between the global (logical) and local meaning of disjunction is something that deserves further discussion.

2.2 The Logic of E1 and E1s: Classical Logic CPC

In this subsection we show that E1 = E1s = CPC, that is, we exactly obtain classical logic when confining ourselves to first-level bases, rather than including second-level bases leading to intuitionistic logic. Compared to the second-level case where much coding is involved, and essentially some sort of intuitionistic logic is incorporated at the atomic level in the form of second-level rules (see Sandqvist (2015)), our proof is surprisingly simple.

We first observe the following, which is an easy exercise.

Lemma 2.1 E1 *(and thus* E1s*) contains all laws of intuitionistic logic* IPC. *In other words,* IPC *is sound for* $\vDash^{\mathsf{E1}}$.

Moreover, we have the following.

Lemma 2.2 E1 *(and thus* E1s*) contains no laws extending* CPC. *In other words,* CPC *is complete for* $\vDash^{\mathsf{E1}}$.

Proof Let A be a non-theorem of CPC. Then there is a valuation v such that $v(A) = 0$. Let $S = \{\rhd a \mid v(a) = 1\} \cup \{a \rhd b \mid v(a) = 0 \text{ and } b \text{ atomic}\}$. Then S has no non-trivial extensions and thus $\vDash_S a$ if $v(a) = 1$ and $\vDash_S \neg a$ if $v(a) = 0$. Hence $\nvDash A$. □

Now we can prove

Lemma 2.3 $\vDash^{\mathsf{E1}}$ $p \vee \neg p$ *for any atom* p.

Proof According to the definition of $\vee$, we must show that, for any S, from $a \vDash_S^{\mathsf{E1}} c$ and $a \to \bot \vDash_S^{\mathsf{E1}} c$ we can obtain $\vDash_S^{\mathsf{E1}} c$. Now $a \vDash_S^{\mathsf{E1}} c$ says that we can derive c from a in S (Lemma 1.1), and $a \to \bot \vDash_S^{\mathsf{E1}} c$ says that we can derive c in every extension S' of S such that $a \vDash_{S'} b$ for every b. Now take S' to be $S \cup S''$ where $S'' = \{a \rhd b \mid b \text{ atomic}\}$. Then $a \to \bot \vDash_S^{\mathsf{E1}} c$ implies that c is derivable in $S \cup S''$. If in this derivation any atomic rule of S'' is used, its premiss a is derivable in S (just consider a top application of a rule from S'', if there is more than one application). Using the derivability of c from a in S due to $a \vDash_S^{\mathsf{E1}} c$, we obtain a derivation of c in S, which means $\vDash_S^{\mathsf{E1}} c$. If none of the rules of S'' is used in the derivation of c in $S \cup S''$, then we have a derivation of c in S anyway. □

Note that for this proof to go through it is critical that we are working with level-1 bases and thus with atomic axioms and production rules only. If S contained a level-2 rule, it could well be that an application of a rule $a \rhd b$ in S'' would be used within an application of a level-2 rule such as $(a \rhd b) \rhd c$, without a derivation of a being available.

Lemma 2.3 says that $p \vee \neg p \in \mathsf{E1}$. We now show that $p \vee \neg p \in \mathsf{E1s}$, that is, that the *tertium non datur* holds for arbitrarily complex formulas. In view of Lemmas 2.1 and 2.2 this establishes our claim that E1 = E1s = CPC, as we can validate all classical tautologies.

Theorem 2.4 $\vDash^{\mathsf{E1}}$ $A \vee \neg A$ *for any* A.

Proof According to Lemma 2.1 all laws of IPC are contained in E1. Recall the following theorems of IPC:

1. $A \to B \vdash A \to (C \to B)$,
2. $(A \vee B) \to C \dashv\vdash (A \to C) \wedge (B \to C)$,
3. $\neg(A \vee B) \to C \vdash \neg A \to (\neg B \to C)$,
4. $(A \wedge B) \to C \dashv\vdash A \to (B \to C)$,
5. $\neg(A \wedge B) \to C \vdash (\neg A \to C) \wedge (\neg B \to C)$,
6. $(A \to B) \to C \vdash (\neg A \to C) \wedge (B \to C)$,
7. $\neg(A \to B) \to C \vdash A \to (\neg B \to C)$.

We proceed by induction on A showing that if $\vDash_S^{\mathsf{E1}} A \to C$ and $\vDash_S^{\mathsf{E1}} \neg A \to C$ then $\vDash_S^{\mathsf{E1}} C$. For the sake of having a sufficiently strong induction hypothesis, this claim is stronger than the assertion of the theorem in that C is an arbitrary formula rather than just an atom c (as would be sufficient for the theorem).

As the induction base, we use Lemma 2.3. Together with Lemma 1.2, this proves that if $\vDash_S^{\mathsf{E1}} p \to C$ and $\vDash_S^{\mathsf{E1}} \neg p \to C$, then $\vDash_S^{\mathsf{E1}} C$ for arbitrary C.

If $\vDash_S^{\mathsf{E1}} (A \wedge B) \to C$ and $\vDash_S^{\mathsf{E1}} \neg(A \wedge B) \to C$, then by (4) and (5) we have $\vDash_S^{\mathsf{E1}} A \to (B \to C)$ and $\vDash_S^{\mathsf{E1}} \neg A \to C$ and $\vDash_S^{\mathsf{E1}} \neg B \to C$. So by (1), $\vDash_S^{\mathsf{E1}} \neg A \to (B \to C)$, which by application of the induction hypothesis gives us $\vDash_S^{\mathsf{E1}} B \to C$. Together with $\vDash_S^{\mathsf{E1}} \neg B \to C$, and by another application of the induction hypothesis we obtain $\vDash_S^{\mathsf{E1}} C$.

The proof continues in the same manner for disjunction and implication.

If $\vDash_S^{\mathsf{E1}} (A \vee B) \to C$ and $\vDash_S^{\mathsf{E1}} \neg(A \vee B) \to C$, then by (2) and (3) we have $\vDash_S^{\mathsf{E1}} \neg A \to (\neg B \to C)$ and $\vDash_S^{\mathsf{E1}} A \to C$ and $\vDash_S^{\mathsf{E1}} B \to C$. So by (1), $\vDash_S^{\mathsf{E1}} A \to (\neg B \to C)$, which by application of the induction hypothesis gives us $\vDash_S^{\mathsf{E1}} \neg B \to C$, and by another application of the induction hypothesis we obtain $\vDash_S^{\mathsf{E1}} C$.

If $\vDash_S^{\mathsf{E1}} (A \to B) \to C$ and $\vDash_S^{\mathsf{E1}} \neg(A \to B) \to C$, then by (6) and (7) we have $\vDash_S^{\mathsf{E1}} A \to (\neg B \to C)$ and $\vDash_S^{\mathsf{E1}} \neg A \to C$ and $\vDash_S^{\mathsf{E1}} B \to C$. So by (1), $\vDash_S^{\mathsf{E1}} \neg A \to (\neg B \to C)$, which by application of the induction hypothesis gives us $\vDash_S^{\mathsf{E1}} \neg B \to C$, and by another application of the induction hypothesis we obtain $\vDash_S^{\mathsf{E1}} C$.

It is trivial that if $\vDash_S^{\mathsf{E1}} \bot \to C$ and $\vDash_S^{\mathsf{E1}} \neg\bot \to C$, then $\vDash_S^{\mathsf{E1}} C$, because $\neg\bot$, i.e., $\bot \to \bot$, is a tautology. □

3 The Logics of the Introduction Rule Approaches

Whereas E-rule semantics directly lead to intuitionistic and classical logic, that is, to the two central logics in foundational discussions, the semantical conceptions based on I-rules result in various systems of intermediate logic. This is philosophically unexpected, as systems of proof-theoretic semantics since Gentzen, which are predominantly built on I-rules, often consider intuitionistic logic to be the target of semantical justification. For I-rule semantics this target needs to be revised. For three of the four semantics considered here, we can give a precise answer in terms of a corresponding intermediate system, and for the fourth system we can at least give some approximative hints.

3.1 The Logic of I2: General Kripke–Putnam Logic GKPL

The logic of I2 is generalized Kreisel–Putnam logic GKPL (also called *general inquisitive logic*), as shown in Stafford (2021) based on work in Piecha et al. (2015), Ciardelli and Roelofsen (2011), and Punčochář (2016). We will sketch the proof here. We do this because in Sect. 3.3 we will use the same proof strategy and it is hoped seeing it sketched here first will aid the reader. In broad strokes we will (i) identify an important theorem GKP of I2 and important properties of I2, (ii) show that the logic GKPL has the important properties, (iii) show that only one logic satisfies GKP and the properties.

Let us recall that a logic is *weak* if it is not closed under all substitutions but is closed under a subset of substitutions that restrict the syntactic form of the formulas substituted, and recall that an *intermediate* logic (which might be weak) is any logic extending intuitionistic logic and is contained in classical logic. We will be interested in weak intermediate logics containing restrictions of the following axiom:

Definition 3.1 The *Gödel–Dummett axiom* is:

$$(A \to B \lor C) \to [(A \to B) \lor (A \to C)] \qquad \text{(GD)}$$

If we add this axiom to IPC, we get *Gödel–Dummett logic* (GDL).

> *Remark on other axiomatizations.* Gödel–Dummett logic is also known as the extension of IPC by the linearity axiom
>
> $$(A \to B) \lor (B \to A)$$
>
> or by the generalized fourth De Morgan axiom
>
> $$((A \land B) \to C) \to ((A \to C) \lor (B \to C)).$$
>
> Both these axioms are equivalent to GD.

This logic is quite a stong intermediate logic and was used in the demonstration that intuitionistic logic cannot be viewed as a truth-valued logic for any n truth values (see Gödel (1932)).

We can restrict the axiom GD to negated formulas A as follows:

Definition 3.2 The *Kreisel–Putnam axiom* is:

$$(\neg A \to B \lor C) \to [(\neg A \to B) \lor (\neg A \to C)] \qquad \text{(KP)}$$

If we add this axiom to IPC we get *Kreisel–Putnam logic* (KPL).

Kreisel–Putnam logic is still closed under substitution. It was introduced to demonstrate the falsity of a conjecture about the form of intuitionistically invalid formulas made by Kreisel and Putnam (1957).

3.1.1 Important Properties of I2

There are sets of axioms that we can add to IPC that are not closed under substitution, such that the resulting logics lie between GDL and KPL. The set of axioms we are interested in restricts substitution to disjunction free formulas.

Definition 3.3 The *generalized Kreisel–Putnam axiom* is:

$$(A \to B \lor C) \to [(A \to B) \lor (A \to C)] \qquad \text{(GKP)}$$

where A is disjunction free.

This holds because we can translate between formulas and rules. We know from previous results that for every disjunction free formula, there is an equivalent set of rules (see Piecha et al. (2015)); for example,

$$\vDash^{\mathsf{I2}}_{S \cup \{(p_1, \ldots, p_n) \rhd q\}} A \iff (p_1 \land \ldots \land p_n) \to q \vDash^{\mathsf{I2}}_{S} A.$$

Because of this every atomic base S is equivalent to a set of formulas. Let $*$ take rules to formulas and # take formulas to rules.

Fact 3.4 *The following equivalences hold:*

1. $\Gamma \vDash^{\mathsf{I2}}_{S \cup \{R\}} A \iff \Gamma, *(R) \vDash^{\mathsf{I2}}_{S} A.$
2. $\Gamma \vDash^{\mathsf{I2}}_{S \cup \{\#(B)\}} A \iff \Gamma, B \vDash^{\mathsf{I2}}_{S} A.$

These translations allow us to prove:

Theorem 3.5 (see Piecha et al. (2015)) I2 *satisfies* GKP.

There are three other properties of I2 which are important to our proof. First we know some of the theorems of I2. It can be easily seen that IPC is sound with respect to I2:

Theorem 3.6 IPC $\subseteq$ I2.

Theorem 3.7 (see Piecha and Schroeder-Heister (2019)) I2 *has the disjunction property.*

Theorem 3.8 (see Piecha et al. (2015), p. 331, Lemma 4) I2 *has the same disjunction free theorems as* IPC.

3.1.2 The Axiomatization

If we add GKP to IPC we will get a weak logic.

Definition 3.9 The *generalized Kreisel–Putnam logic* is:

$$\mathsf{IPC} + \mathsf{GKP} \tag{GKPL}$$

This extension of intuitionistic logic can already be found in the literature. It is a generalization of the logic of inquisitive semantics (Punčochář 2016).

3.1.3 Important Properties of GKPL

Theorem 3.10 (see Punčochář (2016)) GKPL *has the disjunction property.*

Theorem 3.11 (see Punčochář (2016)) GKPL *has the same disjunction free theorems as* IPC.

So both I2 and GKPL contain IPC and GKP, and both have the disjunction property and the same disjunction free fragment as IPC. We will now show that only one logic has these properties.

3.1.4 Normal Form

We can define a normal form for GKPL-theorems such that every formula can be written as the disjunction of disjunction free formulas.

Definition 3.12 Define f_{I2} as a function from formulas to formulas.

$$\begin{aligned}
f_{\mathsf{I2}}(a) &= a,\\
f_{\mathsf{I2}}(\bot) &= \bot,\\
f_{\mathsf{I2}}(A \vee B) &= f_{\mathsf{I2}}(A) \vee f_{\mathsf{I2}}(B),\\
f_{\mathsf{I2}}(A \wedge B) &= \bigvee_{i \in \mathbb{A}} \bigvee_{j \in \mathbb{B}} (A_i \wedge B_j), \text{ where}\\
& \quad f_{\mathsf{I2}}(A) = \bigvee_{i \in \mathbb{A}} A_i \text{ and } f_{\mathsf{I2}}(B) = \bigvee_{j \in \mathbb{B}} B_j,\\
f_{\mathsf{I2}}(A \to B) &= \bigvee_{(j_1,\ldots,j_n) \in \mathbb{B}^n} \bigwedge_{i \in \mathbb{A}} (A_i \to B_{j_i}),\\
& \quad \text{where } f_{\mathsf{I2}}(A) = \bigvee_{i \in \mathbb{A}} A_i,\ |\mathbb{A}| = n \text{ and}\\
& \quad f_{\mathsf{I2}}(B) = \bigvee_{j \in \mathbb{B}} B_j.
\end{aligned}$$

As an example, consider $f_{\mathsf{I2}}((a \wedge (b \vee c)) \to (d \vee e))$. It is $f_{\mathsf{I2}}(a \wedge (b \vee c)) = (a \wedge b) \vee (a \wedge c)$. So we need to consider every way that $a \wedge b$ and $a \wedge c$ can be paired with d and e. We get:

$$\begin{aligned}
&[(a \wedge b) \to d] \wedge [(a \wedge c) \to d] \vee\\
&[(a \wedge b) \to d] \wedge [(a \wedge c) \to e] \vee\\
&[(a \wedge b) \to e] \wedge [(a \wedge c) \to d] \vee\\
&[(a \wedge b) \to e] \wedge [(a \wedge c) \to e].
\end{aligned}$$

We cannot put every theorem of IPC into f_{I2} normal form, but with the generalized Kreisel–Putnam axiom GKP we can do this for GKPL. As I2 is a (possibly equal) superset of GKPL every formula is equivalent to a formula in f_{I2} normal form.

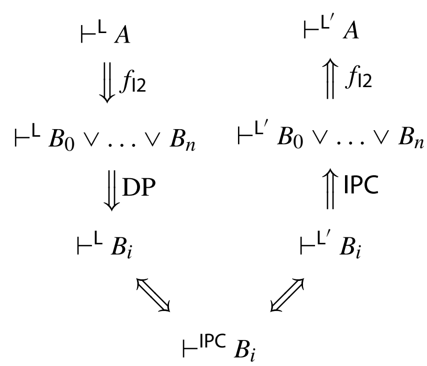
3.1.5 Proving I2 = GKPL

Next we show that there is only one logic meeting the following conditions. This result develops work of Ciardelli and Roelofsen (2011, Theorem 3.2.36).

Theorem 3.13 (see Stafford (2021), Corollary 2.12) *There is only one weak intermediate logic that*

1. *has the disjunction property,*
2. *every theorem is equivalent to a theorem in f_{I2} normal form,*
3. *has the same disjunction free theorems as* IPC.

Proof The proof is relatively simple. Take a theorem A of the first logic L. Apply f_{I2} to transform it into the normal form. Use the disjunction property (DP) to select the true disjunct B_i. That disjunct is a theorem of IPC, and thus also a theorem of the second logic L′. Introduce the earlier disjuncts and apply the normal form transformation f_{I2} again to return to the original formula A.



□

By Theorems 3.10 and 3.11 we have:

Theorem 3.14 (see Stafford (2021)) I2 = GKPL.

This turns out to be a generalization of the logic of inquisitive semantics. The actual logic of inquisitive semantics requires the addition to GKPL of double negation elimination for atomic formulas:

$$\neg\neg p \to p \tag{atomicDNE}$$

This is another schema which does not allow arbitrary substitutions.

3.1.6 An Additional Completeness Result

There is a proof-theoretic semantics for inquisitive logic (IPC + GKP + atomicDNE). Recall that I2 is equivalent to a system extended by rules of higher level: I∞. We obtain inquisitive logic by restricting I∞ as follows:

Definition 3.15 Define Ir based on $\mathsf{I}\infty$ as follows:

$$\mathsf{Ir} = \{A \mid \forall S \subseteq \mathcal{R}_\infty, \text{ if } ((p \rhd \bot) \rhd \bot) \rhd p) \in S, \text{ for all atomic } p, \text{ then } \vDash^{\mathsf{I2}}_S A\}.$$

Here $\bot$ is understood as an atom such that the rules $\bot \rhd a$, for all a, are in S.

It can be shown that $\mathsf{Ir} = \mathsf{IPC} + \mathsf{GKP} + \mathsf{atomicDNE}$. We will not give the details here; the interested reader is directed to Stafford (2021). But the proof is similar to that outlined above.

3.2 The Logic of I2s: Medvedev's Logic ML

Recall that GKPL is a weak logic. If we close it under substitution, we get Gödel–Dummett logic (GDL). If we close IPC + GKP + atomicDNE under substitution, we get classical logic (CPC). The maximal sublogic of IPC + GKP + atomicDNE closed under substitution is Medvedev's logic ML of finite problems; see Ciardelli and Roelofsen (2011). The maximal sublogic of GKPL closed under substitution is also ML (see Punčochář (2016)[1]). Thus

$\mathsf{I2s} = \mathsf{Irs} = \mathsf{ML}$.

ML is an interesting intermediate logic in its own right (Medvedev 1962). It was developed by Medvedev to capture Kolmogorov's problem interpretation of the BHK clauses for intuitionistic logic (Kolmogorov 1932). Medvedev's formalization of Kolmogorov's problem interpretation is not the only example of a formalization of the BHK clauses that led to intermediate logics. Kleene's realizability semantics (Kleene 1945) has also ended up above intuitionistic logic (see Plisko (2011)). We can now add proof-theoretic semantics in the form of I2 to this list and see how close it is to Medvedev's attempt.

3.3 The Logic of I1: Production Rule Logic SL

In this section, we take the technique used to provide soundness and completeness of GKPL for I2 and extend it to show the soundness and completeness of a weak intermediate logic for I1, the introduction rule semantics with respect to level-1 (or production) bases. We will (i) state the important theorems 1KP, atomicDNE, and tEQ of I1 as well as important properties of I1, (ii) show that the logic SL has these properties and theorems, and (iii) show that only one logic satisfies them.[2]

[1] There $s(L)$ stands for the closure of the logic under substitution. By Corollary 5, $g(\mathsf{IPC}) = \mathsf{I2}$ (g is found in Definition 11) and so by Corollary 12, $s(g(\mathsf{IPC})) = s(\mathsf{I2}) = \mathsf{ML}$.

[2] SL stands for 'Stafford's logic', as it is called by T.P. and P.S.-H.

Remark on the decidability of I1. That I1 is axiomatizable (unlike I2s) was shown in Stafford (2022). There it was proved that I1 is part of a class of restrictions of I2 that are decidable. However, it was not known that the axiomatization was short enough to be studied.

3.3.1 Important Properties of I1

Recall that we introduced a series of restrictions on GD which allowed us to push implications through disjunctions for a restricted range of formulas.

Let a *production formula* either be an atom a, absurdity $\bot$ or of the form $(a_1 \wedge \ldots \wedge a_n) \to b$ for atoms $a_1, \ldots, a_n$ and b an atom or $\bot$. Production formulas containing $\bot$ are understood as sets $\{(a_1 \wedge \ldots \wedge a_n) \to b \mid b \text{ atomic}\}$.

We introduce a variant of KP:

Definition 3.16 Let each A_i be a production formula. The *production Kreisel–Putnam axioms* are:

$$(\bigwedge A_i \to B \vee C) \to [(\bigwedge A_i \to B) \vee (\bigwedge A_i \to C)] \quad (\mathsf{1KP})$$

Recall also that $*$ takes rules to formulas and $\#$ takes formulas to rules.

Fact 3.17 *If Γ is a set of production formulas, then the following equivalences hold:*

1. $\Gamma \vDash^{\mathsf{I1}}_{S\cup\{R\}} A \iff \Gamma, *(R) \vDash^{\mathsf{I1}}_S A.$
2. $\Gamma \vDash^{\mathsf{I1}}_{S\cup\{\#(B)\}} A \iff \Gamma, B \vDash^{\mathsf{I1}}_S A.$

Based on this, it has been shown that we get Harrop's rule restricted to conjunctions of production formulas.

Fact 3.18 $\vDash^{\mathsf{I1}}$ 1KP.

We take as given the following results:

Fact 3.19 $\mathsf{IPC} \subseteq \mathsf{I1}$.

Fact 3.20 I1 *has the disjunction property.*

Fact 3.21 I1 *has the same disjunction free theorems as* CPC.

An important difference between I1 and I2 is that proofs in I1 can be split in a well behaved way:

Lemma 3.22 (Stafford (2021), Lemma 4.6) *If $\mathcal{D}$ is a proof in an atomic system S, from distinct rules $(p_1, \ldots, p_n) \rhd q, R_0, \ldots, R_m$ with conclusion r, i.e., if $\mathcal{D}$ witnesses*

$$\vdash_{S\cup\{(p_1,\ldots,p_n)\rhd q, R_0,\ldots,R_m\}} r,$$

and all the rules in S and $R_0, \ldots, R_m$ are level-0 or level-1 rules, then if $\mathcal{D}$ contains $(p_1, \ldots, p_n) \rhd q$ it follows that there are $\mathcal{D}_i$ and $\mathcal{D}_{n+1}$ witnessing

$$\vdash_{S\cup\{R_0,\ldots,R_m\}} p_i \quad \text{and} \quad \vdash_{S\cup\{q,R_0,\ldots,R_m\}} r,$$

respectively.

With this in place we know that

$$\mathsf{IPC} + \mathsf{atomicDNE} + \mathsf{1KP} \subseteq \mathsf{I1}.$$

3.3.2 Describing a New Axiom

We need one more axiom to describe the logic of I1. First note that we can define two recursive functions co and pr that give us the conclusions and premisses of atomic rules, respectively:

$$co(\rhd q) = \{q\},$$
$$co((p_1, \ldots, p_n) \rhd q) = \{q\};$$

$$pr(\rhd q) = \varnothing,$$
$$pr((p_1, \ldots, p_n) \rhd q) = \{p_1, \ldots, p_n\}.$$

We will use the abbreviations $co(S) := \bigcup_{R \in S} co(R)$ and $co(A) := co(\#(A))$, and similarly for pr. Now we define a function r transforming sets of rules and an atom into a new production formula with the conjunction of the conclusions of the set of rules as antecedent and the atom as succedent:

$$r(\varnothing, a) = a,$$
$$r(S, a) = \left[\bigwedge co(S)\right] \to a.$$

Using this we define recursively a transformation t on certain implications between production formulas. In particular, if $A_1, \ldots, A_n$ are production formulas, the transformation will take $(A_1 \wedge \ldots \wedge A_n) \to a$ to a formula that only contains $\to$ in production formulas:

$$t(S, a) = \bigvee_{S' \subseteq S} (r(S', a) \wedge \bigwedge_{R \in S'} \bigwedge_{p \in pr(R)} t(S - \{R\}, p)).$$

In words: consider every collection of rules one could have used to prove a and add a formula saying their conclusions imply a. Then recursively for every rule in that collection and all premisses of that rule consider how one proves these premisses from the other rules. For example, $(p \to q) \to s$ would be equivalent to

$$\begin{aligned} t(\{p \rhd q\}, s) &= r(\varnothing, s) \vee [r(\{p \rhd q\}, s) \wedge t(\varnothing, p)] \\ &= s \vee [(q \to s) \wedge t(\varnothing, p)] \\ &= s \vee [(q \to s) \wedge r(\varnothing, p)] \\ &= s \vee ((q \to s) \wedge p). \end{aligned}$$

With this in place we can prove an interesting fact about I1.

Theorem 3.23 $\vDash^{\mathsf{I1}} [(A_1 \wedge \ldots \wedge A_n) \to a] \leftrightarrow t(\{\#(A_1), \ldots, \#(A_n)\}, a)$*, where* $A_1, \ldots, A_n$ *are production formulas.*

Proof The proof proceeds by induction on the number of A_i.

For the base case where there are 0, we want to show $\vDash^{\mathsf{I1}} a \leftrightarrow t(\varnothing, a)$, which follows from $t(\varnothing, a) = r(\varnothing, a) = a$.

Now assume we have proven the result for n and need to show that

$$\vDash^{\mathsf{I1}} [(A_1 \wedge \ldots \wedge A_n \wedge A_{n+1}) \to a] \leftrightarrow t(\{\#(A_1), \ldots, \#(A_n), \#(A_{n+1})\}, a).$$

First assume

$$\vDash^{\mathsf{I1}}_S (A_1 \wedge \ldots \wedge A_n \wedge A_{n+1}) \to a.$$

It follows by Fact 3.17 that S must be such that

$$\vDash^{\mathsf{I1}}_{S \cup \{\#(A_1), \ldots, \#(A_n), \#(A_{n+1})\}} a.$$

Let $\mathcal{D}$ be the proof constructed from the atomic rules. One of the following holds:

1. $\mathcal{D}$ only contains rules in S.
2. $\mathcal{D}$ ends in $\#(A_i)$ for some i.
3. $\mathcal{D}$ contains some of the $\#(A_i)$ but not as the last rule.

In case 1, $S \vdash a$ so $\vDash^{\mathsf{I1}}_S r(\varnothing, a)$ and by disjunction introduction we are done.

In case 2, there is an A_i that ends with a, so $r(\{A_i\}, a) = a \to a$; and $\vDash^{\mathsf{I1}}_S r(\{A_i\}, a)$ holds trivially. It also follows that there are proofs in $S \cup \{\#(A_1), \ldots, \#(A_n), \#(A_{n+1})\} - \#(A_i)$ of each element of $p \in pr(A_i)$. This follows by Lemma 3.22. Therefore

$$\vDash^{\mathsf{I1}}_S [\bigwedge \{A_1, \ldots, A_n, A_{n+1}\} - \{A_i\}] \to p,$$

and thus, by the induction hypothesis, for all $p \in pr(A_i)$:

$$\vDash^{\mathsf{I1}}_S t(\{\#(A_1), \ldots, \#(A_n), \#(A_{n+1})\} - \{\#(A_i)\}, p).$$

Finally, apply conjunction introductions followed by disjunction introductions.

In case 3, there must be a subset

$$S_R \subseteq \{\#(A_1), \ldots, \#(A_n), \#(A_{n+1})\}$$

of rules that occur in a derivation $\mathcal{D}$ and only rules in

$$S - \{\#(A_1), \ldots, \#(A_n), \#(A_{n+1})\}$$

occur below. These are the bottommost occurrences of the A_is in $\mathcal{D}$. By repeated application of Lemma 3.22 we can split $\mathcal{D}$ into a derivation in S from $co(S_R)$ to a and for each $A_i \in S_R$ into derivations in $S - \#(A_i)$ of each element of $pr(A_i)$. As there is a derivation in S from $co(S_R)$ to a it follows that $\vDash^{\mathsf{I1}}_S r(S_R, a)$. The reasoning now proceeds exactly as in case 2.

Next assume that

$$\vDash^{\mathsf{I1}}_S t(\{\#(A_1), \ldots, \#(A_n), \#(A_{n+1})\}, a)$$

and

$$\vDash^{\mathsf{I1}}_S A_1 \wedge \ldots \wedge A_n \wedge A_{n+1}.$$

Due to the disjunction property there must be some subset

$$S_R \subseteq \{\#(A_1), \ldots, \#(A_n), \#(A_{n+1})\}$$

such that

$$\vDash^{\mathsf{I1}}_S r(S_R, a) \wedge \bigwedge_{R \in S_R} \bigwedge_{p \in pr(R)} t(\{\#(A_1), \ldots, \#(A_n), \#(A_{n+1})\} - \{R\}, p).$$

By definition of $r(S_R, a)$ it follows that we have $(p_1 \wedge \ldots \wedge p_m) \to a$ for $p_i \in co(S_R)$. By the induction hypothesis and

$$t(\{\#(A_1), \ldots, \#(A_n), \#(A_{n+1})\} - \{R\}, p)$$

it follows that we have derivations of $p_1, \ldots, p_m$ from subformulas of $A_1 \wedge \ldots \wedge A_n \wedge A_{n+1}$. By combining these derivations we can prove a. □

Let us say that the axiom tEQ is the claim just proven, namely that for all production formulas $A_1, \ldots, A_n$:

$$[(A_1 \wedge \ldots \wedge A_n) \to a] \leftrightarrow t(\{\#(A_1), \ldots, \#(A_n)\}, a). \quad (\mathsf{tEQ})$$

We can now give the axiomatization:

$$\mathsf{IPC} + \mathsf{atomicDNE} + \mathsf{1KP} + \mathsf{tEQ} \quad (\mathsf{SL})$$

Spelled out in full, this means

$$\begin{aligned} &\mathsf{IPC} + \{\neg\neg p \to p \mid p \text{ atomic}\} + \quad (\mathsf{SL})\\ &\{[\textstyle\bigwedge_i A_i \to a] \leftrightarrow t(\bigcup_i \{\#(A_i)\}, a),\\ &\ [\textstyle\bigwedge_i A_i \to (B \vee C)] \to [\bigwedge_i A_i \to B] \vee [\bigwedge_i A_i \to C]\\ &\qquad \mid A_i \text{ are production formulas}\} \end{aligned}$$

We will prove that SL axiomatizes I1.

It it worth noting the similarities between IPC+atomicDNE +1KP and inquisitive logic (IPC+atomicDNE+KP). Though tEQ appears to have no parallel.

3.3.3 Describing a New Normal Form

We have already shown that SL is sound for I1. To show it is complete we will define a disjunctive normal form.

In order to define the normal form we first define a pre-transformation f which skips over $\to$ and gets formulas as close to a disjunctive normal form as IPC will allow.

Definition 3.24 Define f as a function from formulas to formulas.

$$\begin{aligned} f(a) &= a,\\ f(\bot) &= \bot,\\ f(A \vee B) &= f(A) \vee f(B),\\ f(A \wedge B) &= \textstyle\bigvee_{i \in \mathbb{A}} \bigvee_{j \in \mathbb{B}} (A_i \wedge B_j)), \text{ where}\\ f(A) &= \textstyle\bigvee_{i \in \mathbb{A}} A_i \text{ and } f(B) = \bigvee_{j \in \mathbb{B}} B_j,\\ f(A \to B) &= f(A) \to f(B). \end{aligned}$$

Using this pre-transformation f we define our new normal form f_{I1}:

Definition 3.25 Define f_{I1} as a function from formulas to formulas such that

$$\begin{aligned} f_{\mathsf{I1}}(a) &= a,\\ f_{\mathsf{I1}}(\bot) &= \bot,\\ f_{\mathsf{I1}}(A \vee B) &= f_{\mathsf{I1}}(A) \vee f_{\mathsf{I1}}(B),\\ f_{\mathsf{I1}}(A \wedge B) &= \textstyle\bigvee_{i \in \mathbb{A}} \bigvee_{j \in \mathbb{B}} (A_i \wedge B_j)), \text{ where}\\ f_{\mathsf{I1}}(A) &= \textstyle\bigvee_{i \in \mathbb{A}} A_i \text{ and } f_{\mathsf{I1}}(B) = \bigvee_{j \in \mathbb{B}} B_j,\\ f_{\mathsf{I1}}(A \to B) &=\\ &\quad f(\textstyle\bigwedge_{i \in \mathbb{A}} \bigvee_{j \in \mathbb{B}} \bigwedge_{v_j \in \mathbb{B}_j} t(\{\#A_{u_i} \mid u_i \in \mathbb{A}_i\} \cup pr(B_{u_j}), co(B_{u_j}))),\\ &\text{where } f_{\mathsf{I1}}(A) = \textstyle\bigvee_{i \in \mathbb{A}} \bigwedge_{u_i \in \mathbb{A}_i} A_{u_i} \text{ and}\\ &\quad f_{\mathsf{I1}}(B) = \textstyle\bigvee_{j \in \mathbb{B}} \bigwedge_{v_j \in \mathbb{B}_j} B_{v_j}. \end{aligned}$$

It may be hard to discern at first that f_{I1} does in fact give us a disjunctive normal form. But the result is that every formula is the disjunction of conjunctions of production formulas (with 0 being an acceptable number of disjunctions or conjunctions).

The proofs that follow require the following theorems of IPC, in addition to those listed in the proof of Theorem 2.4:

8. $\vdash A \to (B \to A)$,
9. $A \to B \vdash A \wedge C \to B$,
10. $A \leftrightarrow B \vdash (A \vee C) \leftrightarrow (B \vee C)$,
11. $(A \vee B) \wedge C \dashv\vdash (A \wedge C) \vee (B \wedge C)$,
12. $A \to (B \wedge C) \dashv\vdash (A \to B) \wedge (A \to C)$.

Note that these are sufficient to show that

$$\vdash^{\mathsf{IPC}} f(A) \leftrightarrow A.$$

Lemma 3.26 *$f_{\mathsf{I1}}(A)$ is the disjunction of conjunctions of production formulas.*

Proof By induction on the complexity of A. The atomic and disjunctive cases are trivial and the conjunctive case follows from distributivity (11) above.

For the implication case, note that the only place $\to$ appears is in $r(\ldots)$, and the range of this function is always a production formula. The outside f will then put $f_{\mathsf{I1}}(A \to B)$ in disjunctive normal form as no implications will impede the application of distributivity. ◻

Theorem 3.27 $\vdash^{\mathsf{SL}} A \leftrightarrow f_{\mathsf{I1}}(A)$.

Proof We only need to show the implication case by induction on formula complexity as otherwise the translation is just f.

It will be sufficient to show that if $f_{\mathsf{I1}}(A)$ is $\bigvee_i \bigwedge_{u_i} A_{u_i}$ and $f_{\mathsf{I1}}(B)$ is $\bigvee_j \bigwedge_{v_j} B_{v_j}$, then

$$\vdash^{\mathsf{SL}}(A \to B) \leftrightarrow (\bigwedge_i \bigvee_j \bigwedge_{v_j} t(\bigcup_{u_i} \{\#(A_{u_i})\} \cup pr(B_{v_j}), co(B_{v_j}))).$$

By the induction hypothesis

$$\vdash^{\mathsf{SL}}(A \to B) \leftrightarrow f_{\mathsf{I1}}(A) \to f_{\mathsf{I1}}(B).$$

So write $f_{\mathsf{I1}}(A)$ as $\bigvee_i \bigwedge_{u_i} A_{u_i}$ and similarly for $f_{\mathsf{I1}}(B)$. Then

$$\begin{aligned} \vdash^{\mathsf{SL}}(\textstyle\bigvee_i \bigwedge_{u_i} A_{u_i} \to \bigvee_j \bigwedge_{v_j} B_{v_j}) \leftrightarrow & \quad \text{(by 2)}\\ \textstyle\bigwedge_i(\bigwedge_{u_i} A_{u_i} \to \bigvee_j \bigwedge_{v_j} B_{v_j}) \leftrightarrow & \quad \text{(by 1KP)}\\ \textstyle\bigwedge_i \bigvee_j(\bigwedge_{u_i} A_{u_i} \to \bigwedge_{v_j} B_{v_j}) \leftrightarrow & \quad \text{(by 12)}\\ \textstyle\bigwedge_i \bigvee_j \bigwedge_{v_j}(\bigwedge_{u_i} A_{u_i} \to B_{v_j}). & \end{aligned}$$

By (12):

$$\vdash^{\mathsf{SL}} \textstyle\bigwedge_i \bigvee_j \bigwedge_{v_j}(\bigwedge_{u_i} A_{u_i} \to (\bigwedge pr(B_{v_j}) \to co(B_{v_j}))) \leftrightarrow \bigwedge_i \bigvee_j \bigwedge_{v_j}(\bigwedge_{u_i} A_{u_i} \wedge \bigwedge pr(B_{v_j}) \to co(B_{v_j}))$$

Finally, by the axioms of SL we have:

$$\vdash^{\mathsf{SL}} \textstyle\bigwedge_i \bigvee_j \bigwedge_{v_j}(\bigwedge_{u_i} A_{u_i} \wedge \bigwedge pr(B_{v_j}) \to co(B_{v_j})) \leftrightarrow \bigwedge_i \bigvee_j \bigwedge_{v_j}(t(\{A_{u_i}\} \cup pr(B_{v_j}), co(B_{v_j})).$$ ◻

Corollary 3.28 $\vDash^{\mathsf{I1}} A \leftrightarrow f_{\mathsf{I1}}(A)$.

3.3.4 Important Properties of SL

Lemma 3.29 SL *has the same disjunction free theorems as* CPC.

We know that SL satisfies $\{\neg\neg A \to A \mid A \text{ is } \vee \text{ free}\}$ as IPC+atomicDNE does, which ensures it has the same disjunction free fragment as classical logic.

Lemma 3.30 SL *has the disjunction property.*

Proof Let $\vdash^{\mathsf{SL}} A \vee B$. By transforming $A \vee B$ into the normal form f_{I1} we have $\vdash^{\mathsf{SL}} \bigvee A_i \vee \bigvee B_j$ for disjunction free A_is and B_js. We know that I1 has the disjunction property and validates all the axioms. Thus there must be a formula C among the A_is and B_js which it validates. As this C is disjunction free, classical logic proves it. Therefore intuitionistic logic proves $\neg\neg C$. From which it follows that SL proves C and hence must have the disjunction property. ◻

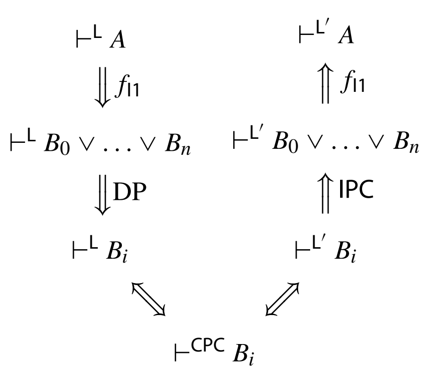
3.3.5 Proving I1 = SL

The following theorem is analogous to Theorem 3.13:

Theorem 3.31 *There is only one weak intermediate logic that*

1. *has the disjunction property,*
2. *every theorem is equivalent to a theorem in f_{I1} normal form,*
3. *has the same disjunction free theorems as* CPC.

Proof In analogy to the proof of Theorem 3.13 we have:



◻

Consequently, we have:

Theorem 3.32 I1 = SL.

3.4 The Logic of I1s: ?

We have characterized the logic of I1 but this axiomatization is not closed under substitution. Thus we can ask what is the maximal sublogic that is. We do not currently know the answer to this question but we can show that I1s $\neq$ I2s and offer a conjecture for where to look next.

As discussed in Sect. 3.2, the logic of I2s is Medvedev's logic of finite problems. While this logic is not axiomatized it is known that KPL is a sublogic of it. So it is sufficient to show that KPL is not a sublogic of I1s to show they are distinct.

We will do this by demonstrating that an instance of KP is not valid in I1.

Lemma 3.33 $\nvDash^{\mathsf{I1}} (\neg(\neg a \wedge \neg b) \to (a \vee b)) \to [(\neg(\neg a \wedge \neg b) \to a) \vee (\neg(\neg a \wedge \neg b) \to b)]$.

Proof This is because $\nvDash^{\mathsf{I1}} [(\neg(\neg a \wedge \neg b) \to a) \vee (\neg(\neg a \wedge \neg b) \to b)]$ but $\vDash^{\mathsf{I1}} (\neg(\neg a \wedge \neg b) \to (a \vee b))$.

First we will show that $\nvDash^{\mathsf{I1}} [(\neg(\neg a \wedge \neg b) \to a) \vee (\neg(\neg a \wedge \neg b) \to b)]$. Note that $\vDash^{\mathsf{I1}}_{\{\rhd a\}} \neg(\neg a \wedge \neg b)$ as for any S such that $\vDash^{\mathsf{I1}}_{S \cup \{\rhd a\}} \neg a \wedge \neg b$ we get $\vDash^{\mathsf{I1}}_{S \cup \{\rhd a\}} \bot$. But $\nvDash^{\mathsf{I1}}_{\{\rhd a\}} b$. So $\nvDash^{\mathsf{I1}} \neg(\neg a \wedge \neg b) \to b$, and the same reasoning holds with a.

Next we show by contradiction that $\vDash^{\mathsf{I1}} \neg(\neg a \wedge \neg b) \to (a \vee b)$. Assume that $\vDash^{\mathsf{I1}}_S \neg(\neg a \wedge \neg b)$ but $\nvDash^{\mathsf{I1}}_S a$ and $\nvDash^{\mathsf{I1}}_S b$ (from which it also follows that $\nvDash^{\mathsf{I1}}_S \bot$). As $\vDash^{\mathsf{I1}}_S \neg(\neg a \wedge \neg b)$ it follows that $\vDash^{\mathsf{I1}}_{S \cup \{a \rhd \bot, b \rhd \bot\}} \bot$. But this means that there must be a derivation using atomic rules in S either of $\bot$, a, or b. This is a contradiction, so $\vDash^{\mathsf{I1}}_S (a \vee b)$. □

This means that the logic of I1s is distinct from the logic of I2s. But it is still possible that it could be one of the many intermediate logics not containing KP. It is also possible that it is intuitionistic logic. Two candidate schemas seem particularly worth checking. These are *Scott's axiom*

$$[(\neg\neg A \to A) \to (\neg A \vee A)] \to (\neg\neg A \vee \neg A)$$

and *Rose's axiom*:

$$\begin{aligned}&[(\neg\neg(\neg A \vee \neg B) \to (\neg A \vee \neg B))\\ &\to (\neg(\neg A \vee \neg B) \vee (\neg A \vee \neg B))]\\ &\to [\neg\neg(\neg A \vee \neg B) \vee \neg(\neg A \vee \neg B)].\end{aligned}$$

Rose's axiom is particularly worth examining because it is not just contained in Medvedev's logic but also in the logics of the realizability semantics discussed in Sect. 3.2. One way to go about counterexample searching for these axioms would be to use the normal form described in Sect. 3.3.

4 Alternative Notions of Atomic Bases: The Logic Programming Perspective

We have shown that the logic one finally obtains not only depends on the clauses for logical constants (such as the introduction vs. elimination interpretation of disjunction), but also on the conception of an atomic system. We considered the difference in atomic inference rules permitted in bases and showed that level-1 bases lead to different logics than level-2 bases. Now the difference between levels of rules is by far not the only distinction that one can draw between bases. Another one would be the distinction between rules considered as "factual" in the sense that they describe our state of knowledge and rules considered as "definitional" in the sense that they are clauses in the definition of a certain concept. In the conception followed here, we have followed the bases-as-states-of-knowledge approach. This is reflected in the fact that we have monotonicity with respect to bases, that is, if $S' \supseteq S$, then $\Gamma \vDash_S A$ implies $\Gamma \vDash_{S'} A$. Correspondingly, our clause ($\vDash_S$) quantified over all extensions of S. This we would have to give up in a bases-as-definitions approach, as the addition of new clauses changes the meaning of the definiendum rather than only adding new knowledge to what we already know. Actually, this is the concept of base favoured by Prawitz in his publications from 1973 on (see Prawitz (1973, 1974, 2016)). We have discussed this at several places (see Piecha and Schroeder-Heister (2016b, 2017)), though a thorough evaluation of the bases-as-definitions approach is still a desideratum.

Here we would like to draw the reader's attention to another possibility, namely the understanding of bases as logic programs (see also Gheorghiu (2024, Chapter 23), Gheorghiu and Pym (2023)). We mention this topic because logic programming is a highly significant approach to atomic reasoning whose significance has not been appropriately appreciated in philosophical logic, and also because it sheds a light on how widely diversified the field of potential atomic bases is. Very many other approaches might be considered as well such as the whole area of non-monotonic reasoning, to which the logic programming approach belongs. We do not want to give any detailed results. In particular we do not present any specific logic which proof-theoretically corresponds to logic programming bases. We just want to point, as an example, to an interesting notion of atomic base.

Level-1 atomic systems S in our sense, i.e., systems of production rules, can immediately be understood as logic programs consisting of definite Horn clauses, i.e., clauses with exactly one positive literal. In logic programming terms these can be *facts*, i.e., level-0 atomic rules, or *rules*, i.e., level-1 rules with premisses. Horn clauses that contain no positive literal are called *goals*. In the context of logic programming, such clauses are usually written with the reverse implication $\leftarrow$ as shown in Table 1.

Here we consider first-order languages with ground terms $k, l, m, \ldots, 0, 1, \ldots$, function terms and identity, i.e., $a, b, a_1, a_2, \ldots, p, q, \ldots$ are now first-order atoms. The atom to the left of $\leftarrow$ ist called the *head* of a clause and the atoms to the right are its *body*.

In standard proof-theoretic semantics the S-validity of

Table 1 Notation

Clause type	Rule type	Clause notation	Rule notation	Linear rule notation
Fact	Level-0	$a \leftarrow$	$\frac{}{a}$	$\rhd a$ or a
Rule	Level-1	$b \leftarrow a_1, \dots, a_n$	$\frac{a_1 \quad \dots \quad a_n}{b}$	$(a_1, \dots, a_n) \rhd b$
Normal rule	Level-2	$b \leftarrow a_1, \dots, \neg a_i, \dots, a_n$	$\frac{a_1 \quad \dots \quad \genfrac{}{}{0pt}{}{[a_i]}{\bot} \quad \dots \quad a_n}{b}$	$(a_1, \dots, a_i \rhd \bot, \dots, a_n) \rhd b$
Goal	Level-1	$\leftarrow a$ or $\bot \leftarrow a$	$\frac{a}{\bot}$	$a \rhd \bot$
Empty clause	Level-0	$\leftarrow$ or $\bot$	$\frac{}{\bot}$	$\rhd \bot$ or $\bot$

ground atoms a is defined by derivability of a in S:

(At) $\vDash_S a \; :\Longleftrightarrow \; \vdash_S a.$

Whereas in the logic programming perspective the notion of derivability is based on SLD-resolution. An atom a is derivable in S if there is an SLD-refutation of the goal $\leftarrow a$ with respect to the logic program S, i.e., an SLD-derivation of the empty clause for given S and $\leftarrow a$ (see, e.g., Lloyd (1993), Chapter 2). The S-validity of ground atoms a is then defined by the SLD-refutability of $\leftarrow a$ with respect to S:

(At-SLD) $\vDash_S a :\Longleftrightarrow \leftarrow a$ is SLD-refutable with respect to S.

Now this definition of atomic validity is not so much deviant from our original (At), as it can be shown that SLD-refutability of $\leftarrow a$ corresponds to derivability in S (completeness of SLD-resolution; see Lloyd (1993), § 2.8). However, this situation changes if we include negation as failure.

Definite Horn clause programs can be extended to *normal programs* (see Lloyd (1993), Chapter 3). These may contain clauses that can also have negated atoms in the body, like

$$b \leftarrow a_1, \dots, \neg a_i, \dots, a_n,$$

which we write as

$$b \leftarrow a_1, \dots, (\bot \leftarrow a_i), \dots, a_n.$$

In our notation they correspond to level-2 rules of the form

$$\frac{a_1 \quad \dots \quad \genfrac{}{}{0pt}{}{[a_i]}{\bot} \quad \dots \quad a_n}{b}$$

where a_i may be discharged. Here $\bot$ is not a logical constant in the sense of this paper, but an atomic constant which expresses *finite failure* in the sense of logic programming. So the ith premiss in this rule is to be read as "a_i fails finitely".

In order to handle negation in normal programs one extends SLD-resolution to SLDNF-resolution by adding the *negation as failure rule* for ground atoms a:

(NF) If the search for an SLD-refutation for the goal $\leftarrow a$ with respect to a normal program S fails after finitely many steps, then conclude $\neg a$.

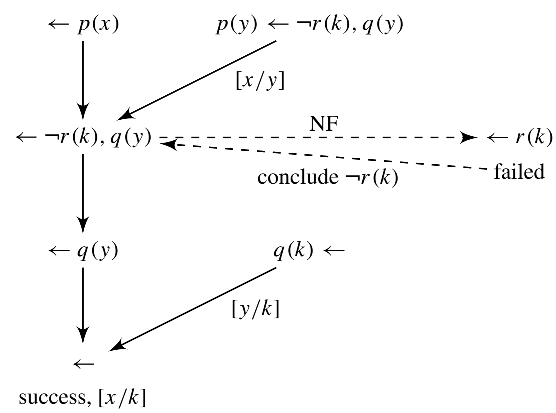
Since NF calls for finite failure it can be seen as a constructive justification for using negated ground atoms in bodies. The S-validity of ground atoms a is then defined as follows:

(At-SLDNF) $\vDash_S a \; :\Longleftrightarrow \; \leftarrow a$ is SLDNF-refutable with respect to S.

As an example for the application of NF consider the normal program

$$S = \{q(k) \leftarrow; \; p(y) \leftarrow \neg r(k), q(y)\}.$$

An SLDNF-refutation of the goal $\leftarrow p(x)$ with respect to S looks as follows:



In the second resolution step the negated ground atom $\neg r(k)$ is selected, which can only be handled by NF. The search for an SLD-refutation for the goal $\leftarrow r(k)$ fails finitely and we can conclude $\neg r(k)$. This yields the resolvent $\leftarrow q(y)$ as a new goal for which we obtain the empty clause in the last step, thus having SLDNF-refuted the initial goal $\leftarrow p(x)$ with the answer substitution $[x/k]$. According to (At-SLDNF) this means that the ground atom $p(k)$ is S-valid.

Note that in this example we have not just used atoms in the sense of propositional variables as in the preceding parts of this paper, but atoms as atomic formulas with individual variables and individual constants as is standard in first-order logic. For the individual variables the resolution-based evaluation procedure in logic programming then tries to compute substitutions for which the original goal can be refuted.

In the presence of NF we do not have monotonicity of S-validity with respect to extensions $S' \supseteq S$. As an example, consider the program

$$S = \{p(k) \leftarrow \ ; \ p(l) \leftarrow \ ; \ q(m) \leftarrow \}.$$

Every search for the goal $\leftarrow p(m)$ fails finitely (i.e., the SLD-tree for this goal fails finitely), and we can thus conclude $\neg p(m)$ with NF. However, for the extension

$$S' = \{p(k) \leftarrow \ ; \ p(l) \leftarrow \ ; \ q(m) \leftarrow \ ; \ p(m) \leftarrow q(m)\}.$$

of S there is now an SLD-refutation of $\leftarrow p(m)$, and $\neg p(m)$ can no longer be inferred. This result is not unplausible at all. On the contrary, that something fails is relative to the limited means available, and this is due to the atomic system S under consideration.

As mentioned above, non-monotonicity of S-validity with respect to extensions $S' \supseteq S$ does not conflict with notions of proof-theoretic validity and actually widens its range of applications. Indeed, if logic programs are understood as (possibly partial) definitions of the heads of clauses, then non-monotonicity is to be expected since extensions of definitions can change the meaning of what has been defined before. Moreover, logic programs allow for a separation between definitional programs (non-monotonic) and knowledge base programs (monotonic) that could be combined in a notion of proof-theoretic validity by distinguishing and keeping track of atoms that have either been defined or are given by a knowledge base (see Piecha and Schroeder-Heister (2016b)).

Remark on computability The extension of logic programs (only definite Horn clauses) to normal programs does not extend the notion of computability given by SLD-resolution. Turing equivalence holds for both kinds of programs. Independent of whether a notion of proof-theoretic validity is based on (At-SLD) or (At-SLDNF) the set of logically valid rules that can be justified is restricted by the notion of (Turing-)computability or partial recursive function.

Remark on the level of rules in the presence of NF The logic programming approach may offer some insights concerning the levels of atomic rules. Program rules with (left-iterated) implicative formulas in the body of clauses can be transformed into sets of rules. For example, the level-2 rule $a \leftarrow (b \leftarrow c)$ is transformed into the set of rules $\{a \leftarrow \neg c\, ;\ a \leftarrow b\}$, where the first rule can equivalently be written as $a \leftarrow (\bot \leftarrow c)$, emphasizing that it is a level-2 rule. By using NF, the level of rules can be reduced further to level 1 since negated gound atoms $\neg a$ in the body of a clause (level-2) are concluded from the failed SLD-tree for the level-1 goal $\leftarrow a$. In this sense there is a reduction from level-2 rules to level-1 rules in the presence of NF.

Further extensions of SLD- or SLDNF-resolution may be considered, e.g., by adding a principle of definitional reflection (see Hallnäs and Schroeder-Heister (1990, 1991), Hallnäs (1991), Schroeder-Heister (1991)), and S-validity of atomic formulas can then be defined accordingly by the resulting derivability (or refutability) relation (see Piecha and Schroeder-Heister (2016b, 2017)).

Note that what we have done here is only to give an illustration of an alternative view of atomic bases. We have nothing yet established about the logic resulting from bases with SLD- or SLDNF-resolution. This is a topic which goes far beyond the range of the present paper.

5 Summary

We have distinguished eight Prawitz semantics E2, E2s, E1, E1s, I2, I2s, I1, I1s and have provided (or reported) completeness proofs for all but I1s. For the latter we have made a conjecture about possible logics. We summarize these results as follows:

Proof of the Main Theorem 1.3 For the logics of our eight Prawitz semantics the following holds:

1. E2 = E2s = IPC, by Sandqvist (2015), where IPC is intuitionistic propositional logic.
2. E1 = E1s = CPC, by Lemma 2.1 and Theorem 2.4, where CPC is classical propositional logic.
3. I2 = GKPL, by Theorem 3.14 (see also Stafford (2021)), where GKPL is generalized Kreisel–Putnam logic, which is an intermediate logic satisfying the generalized Kreisel–Putnam axiom GKP.
4. I2s = ML, by Sect. 3.2, where ML is Medvedev's logic.
5. I1 = SL, by Theorem 3.32, where SL is the intermediate logic axiomatized by IPC + atomicDNE + 1KP + tEQ.

6. l1s = ?, where ? is an intermediate logic that does not contain Medvedev's logic ML, is therefore distinct from l2s, and for which we made a conjecture in Sect. 3.4. □

Here we see the importance of intermediate logics and the way notions of constructivity extend beyond intuitionistic logic.

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Declarations

Conflict of interest The authors declare no conflict of interest.

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