

EBERHARD KARLS
UNIVERSITÄT
TÜBINGEN



WIRTSCHAFTS- UND
SOZIALWISSENSCHAFTLICHE
FAKULTÄT

Dr. Julie Schnaitmann

S414
Advanced Mathematical Methods
Exercises

VECTOR CALCULUS

EXERCISE 1 Derivation rules

Derive the following expressions:

- | | |
|---|--|
| a) $\frac{d}{dx} (-12x^3 + 5x^4 + 1)$ | b) $\frac{d}{dx} (2x^7 - 6^x)$ |
| c) $\frac{d}{dx} (2x^7 \cdot 6^x)$ | d) $\frac{d}{dx} (\sqrt[3]{x} \cdot \ln(x))$ |
| e) $\frac{d}{dx} \left[\frac{x^4 + 3x^2}{x - 1} \right]$ | f) $\frac{d}{dv} \left[\frac{-e^v \cdot \sin(v)}{2v^2 + 1} \right]$ |
| g) $\frac{d}{dw} (w \cdot \exp(2w^8 + 4))$ | h) $\frac{d}{dx} (\sqrt{2x^4 + \sin(x)})$ |
| i) $\frac{d}{du} (\ln(1 + u^4))$ | j) $\frac{d}{dx} (x^2 \cdot \sin(2x))$ |
| k) $\frac{d}{dx} \left[\left(x^{1/2} - \frac{1}{x^2} \right)^3 \right]$ | l) $\frac{d}{dx} (x \cdot \ln(x) \cdot \ln(x^2))$ |

EXERCISE 2 General partial derivatives

Find the following derivatives:

- | | | |
|--|---|---|
| a) $\frac{\partial}{\partial x} \ln(x^2 + 2y)$ | b) $\frac{\partial^2}{\partial x^2} \ln(x^2 + 2y)$ | c) $\frac{\partial^3}{\partial y \partial x^2} \ln(x^2 + 2y)$ |
| d) $\frac{\partial^3}{\partial x^2 \partial y} \exp(2x + y^2)$ | e) $\frac{\partial^n}{\partial x^n} \exp(2x + y^2)$ | |

EXERCISE 3 Total partial derivative

Determine for the function

$$z = e^{x^2} + y^2 e^{xy}$$

with $x = 2t + 3s$ and $y = t^2 s^3$ the total partial derivative $\left(\frac{\partial z}{\partial s} \right)$.

EXERCISE 4 Open Sets

Determine which of the following sets are open:

- a) $U = \{(x, y) | 2 < x^2 + y^2 < 3\} \subseteq \mathbb{R}^2$ b) $U = \{(x, y) | x + y = 2\} \subseteq \mathbb{R}^2$
c) $U = \{(x, y, z) | xyz > 0\} \subseteq \mathbb{R}^3$

EXERCISE 5 Real-valued and vector-valued function

Find the domain and the range of the function:

$$f(x, y) = \frac{x - 3y^2}{x^2 - y}$$

EXERCISE 6 Real-valued and vector-valued function

Find the domain of the function $\mathbf{F} : U \subseteq \mathbb{R}^2 \rightarrow \mathbb{R}^3$ defined by

$$\mathbf{F}(x, y) = \left(\frac{y}{x^2 + y^2}, \frac{x}{x^2 + y^2}, \sqrt{xy} \right)$$

EXERCISE 7 Gradient

Determine the gradient of $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ which is given by $(x, y) \mapsto e^{xy} + \sin xy$:

EXERCISE 8 Derivative of a function

Compute the derivative of the function $\mathbf{F}$ at the point $\mathbf{a}$:

- a) $\mathbf{F}(x, y) = (y, x, 11), \mathbf{a} = (0, 0)$
b) $\mathbf{F}(x, y, z) = (\ln(x^2 + y^2 + z^2), 2xy + z), \mathbf{a} = (1, 1, 0)$
c) $f(x, y, z) = \|x\mathbf{i} + y\mathbf{j} + z\mathbf{k}\|^2, \mathbf{a} = (a_1, a_2, a_3)$

EXERCISE 9 Taylor rule

Derive the fourth order Taylor expansion of the function $f(x) = \cos(x)$ around a suitable expansion point x_0 .

Solution Exercise 1:

- a) $-36x^2 + 20x^3$
- b) $14x^6 - \ln(6) \cdot 6^x$
- c) $14x^6 \cdot 6^x + 2x^7 \ln(6)6^x$
- d) $\frac{1}{3} \frac{\ln(x)+3}{\sqrt[3]{x^2}}$
- e) $\frac{3x^4-4x^3+3x^2-6x}{(x-1)^2}$
- f) $e^{v \frac{-(\sin(v)+\cos(v))(2v^2+1)+\sin(v) \cdot 4v}{(2v^2+1)^2}}$
- g) $\exp(2w^8 + 4)(16w^8 + 1)$
- h) $\frac{1}{2} \cdot \frac{1}{\sqrt{2x^4+\sin(x)}} \cdot (8x^3 + \cos(x))$
- i) $\frac{4u^3}{1+u^4}$
- j) $2x(\sin(2x) + x \cos(2x))$
- k) $3 \cdot (x^{\frac{1}{2}} - \frac{1}{x^2})^2 \cdot (\frac{1}{2}x^{-\frac{1}{2}} + 2 \cdot \frac{1}{x^3})$
- l) $(\ln(x) + 1) \ln(x^2) + 2 \ln(x)$

Solution Exercise 2:

- a) $\frac{2x}{x^2+2y}$
- b) $\frac{-2x^2+4y}{(x^2+2y)^2}$
- c) $\frac{12x^2-8y}{(x^2+2y)^3}$
- d) $8y \exp(2x + y^2)$
- e) $2^n \exp(2x + y^2)$

Solution Exercise 3:

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x} \frac{\partial x}{\partial s} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial s} = 3(\exp(x^2)2x + y^3 \exp(xy)) + 3t^2 s^2 (2y \exp(xy) + xy^2 \exp(xy))$$

Solution Exercise 4:

- a) U is open in $\mathbb{R}^2$
- b) U is not open in $\mathbb{R}^2$
- c) U is open in $\mathbb{R}^3$

Solution Exercise 5:

$$f : \mathbb{R}^2 \setminus \{(x, y) | y = x^2\} \rightarrow \mathbb{R}$$

Solution Exercise 6:

$$f : \{(x, y) \in \mathbb{R}^2 : \{x \geq 0 \cap y \geq 0\} \cup \{x \leq 0 \cap y \leq 0\} \setminus \{x = 0 \cap y = 0\}\} \rightarrow \mathbb{R}^3$$

Solution Exercise 7:

$$\nabla f(x, y) = (e^{xy} + \cos xy)(y\mathbf{i} + x\mathbf{j})$$

Solution Exercise 8:

a) $\begin{pmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 0 \end{pmatrix}$

b) $\begin{pmatrix} 1 & 1 & 0 \\ 2 & 2 & 1 \end{pmatrix}$

c) $(2a_1 \quad 2a_2 \quad 2a_3)$

Solution Exercise 9:

$$x_0 = 0$$
$$T(x) = 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4$$