
Advanced Mathematical Methods

WS 2025/26

1 Linear Algebra

Dr. Julie Schnaitmann

*Department of Statistics, Econometrics and Empirical
Economics*

EBERHARD KARLS
UNIVERSITÄT
TÜBINGEN



WIRTSCHAFTS- UND
SOZIALWISSENSCHAFTLICHE
FAKULTÄT

Outline: Linear Algebra

1.8 Eigenvalues and eigenvectors

1.9 Quadratic forms and sign definiteness

Readings

- Knut Sydsaeter, Peter Hammond, Atle Seierstad, and Arne Strøm. *Further Mathematics for Economic Analysis*. Prentice Hall, 2008 Chapter 1

Online Resources

MIT course on Linear Algebra (by Gilbert Strang)

- Lecture 21: Eigenvalues and Eigenvectors
<https://www.youtube.com/watch?v=IXNXrLcoerU>
- Lecture 22: Powers of a square matrix and Diagonalization
<https://www.youtube.com/watch?v=13r9QY6cmjc>
- Lecture 26: Symmetric matrices and positive definiteness
<https://www.youtube.com/watch?v=umt6BB1nJ4w>
- Lecture 27: Positive definite matrices and minima – Quadratic forms
<https://www.youtube.com/watch?v=vF7eyJ2g3kU>

1.8 Eigenvalues and eigenvectors

Assume a scalar λ exists such that

$$\mathbf{A}\mathbf{x} = \lambda\mathbf{x}$$

λ : eigenvalue

\mathbf{x} : eigenvector

Find λ via the homogenous linear equation system

$$(\mathbf{A} - \lambda\mathbf{I})\mathbf{x} = \mathbf{0}$$

1.8 Eigenvalues and eigenvectors

The properties of a quadratic homogenous linear equation system imply that:

- in any case a solution does exist;
- if $\det(\mathbf{A} - \lambda \mathbf{I}) \neq 0$, then $\bar{\mathbf{x}} = 0$ is the trivial solution;
- only if $\det(\mathbf{A} - \lambda \mathbf{I}) = 0$ there is a non-trivial solution.

1.8 Eigenvalues and eigenvectors

Determination of the eigenvalues via the *characteristic equation*:

$$|\mathbf{A} - \lambda \mathbf{I}| = 0 \iff (-1)^n \lambda^n + \alpha_{n-1} \lambda^{n-1} + \dots + \alpha_1 \lambda + \alpha_0 = 0$$

for every (real or complex) eigenvalue λ_i of the $(n \times n)$ -Matrix \mathbf{A} we can calculate the respective eigenvector $\mathbf{x}_i \neq 0$ solving the homogenous linear equation system

$$(\mathbf{A} - \lambda_i \mathbf{I}) \mathbf{x}_i = 0. \tag{1}$$

The properties of homogenous linear equation systems imply that the solution of eq. (1) is not unambiguous, i.e. for the eigenvalue λ_i we can find infinitely many eigenvectors \mathbf{x}_i .

1.8 Eigenvalues and eigenvectors

Determination of the eigenvalues via the *characteristic equation*:

$$|\mathbf{A} - \lambda \mathbf{I}| = 0 \iff (-1)^n \lambda^n + \alpha_{n-1} \lambda^{n-1} + \dots + \alpha_1 \lambda + \alpha_0 = 0$$

for every (real or complex) eigenvalue λ_i of the $(n \times n)$ -Matrix \mathbf{A} we can calculate the respective eigenvector $\mathbf{x}_i \neq 0$ solving the homogenous linear equation system

$$(\mathbf{A} - \lambda_i \mathbf{I})\mathbf{x}_i = 0. \tag{1}$$

The properties of homogenous linear equation systems imply that the solution of eq. (1) is not unambiguous, i.e. for the eigenvalue λ_i we can find infinitely many eigenvectors \mathbf{x}_i .

1.8 Eigenvalues and eigenvectors

Determination of the eigenvalues via the *characteristic equation*:

$$|\mathbf{A} - \lambda \mathbf{I}| = 0 \iff (-1)^n \lambda^n + \alpha_{n-1} \lambda^{n-1} + \dots + \alpha_1 \lambda + \alpha_0 = 0$$

for every (real or complex) eigenvalue λ_i of the $(n \times n)$ -Matrix \mathbf{A} we can calculate the respective eigenvector $\mathbf{x}_i \neq 0$ solving the homogenous linear equation system

$$(\mathbf{A} - \lambda_i \mathbf{I})\mathbf{x}_i = 0. \tag{1}$$

The properties of homogenous linear equation systems imply that the solution of eq. (1) is not unambiguous, i.e. for the eigenvalue λ_i we can find infinitely many eigenvectors \mathbf{x}_i .

1.8 Eigenvalues and eigenvectors

A und **B** (quadratic matrices of order n) are similar if a regular $(n \times n)$ - matrix **C** exists, such that

$$\mathbf{B} = \mathbf{C}^{-1} \mathbf{A} \mathbf{C} .$$

Special case: **symmetric matrices**

For a symmetric $(n \times n)$ -matrix **A** it holds that the normalized eigenvectors $\tilde{\mathbf{x}}_j$ with $j = 1, \dots, n$ have the property

- ① $\tilde{\mathbf{x}}_j' \tilde{\mathbf{x}}_j = 1$ for all j and
- ② $\tilde{\mathbf{x}}_i' \tilde{\mathbf{x}}_j = 0$ for all $i \neq j$.

1.8 Eigenvalues and eigenvectors

A und **B** (quadratic matrices of order n) are similar if a regular $(n \times n)$ - matrix **C** exists, such that

$$\mathbf{B} = \mathbf{C}^{-1} \mathbf{A} \mathbf{C} .$$

Special case: **symmetric matrices**

For a symmetric $(n \times n)$ -matrix **A** it holds that the normalized eigenvectors $\tilde{\mathbf{x}}_j$ with $j = 1, \dots, n$ have the property

- ① $\tilde{\mathbf{x}}_j' \tilde{\mathbf{x}}_j = 1$ for all j and
- ② $\tilde{\mathbf{x}}_i' \tilde{\mathbf{x}}_j = 0$ for all $i \neq j$.

1.8 Eigenvalues and eigenvectors

Principle axis theorem

collecting the normalized eigenvectors $\tilde{\mathbf{x}}_j$ ($j = 1, \dots, n$) in a new matrix $\mathbf{T} = [\tilde{\mathbf{x}}_1 \cdots \tilde{\mathbf{x}}_n]$ with the property $\mathbf{T}^{-1} = \mathbf{T}'$ yields the diagonalization of \mathbf{A} as follows:

$$\mathbf{D} = \mathbf{T}'\mathbf{A}\mathbf{T} = \mathbf{T}^{-1}\mathbf{A}\mathbf{T} = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & \lambda_n \end{bmatrix}.$$

1.8 Eigenvalues and eigenvectors

Properties of eigenvalues

- 1) The product of the eigenvalues of a $n \times n$ matrix yields its determinant: $|\mathbf{A}| = \prod_{i=1}^n \lambda_i$.
- 2) From 1.) it follows that a singular matrix must have at least one eigenvalue $\lambda_i = 0$.
- 3) The matrices \mathbf{A} and \mathbf{A}' have the same eigenvalues.
- 4) For a non-singular matrix \mathbf{A} with eigenvalues λ we have:
 $|\mathbf{A}^{-1} - \frac{1}{\lambda} \mathbf{I}| = 0$.
- 5) Symmetric matrices have only real eigenvalues and orthogonal eigenvectors.

1.8 Eigenvalues and eigenvectors

Properties of eigenvalues

- 1) The product of the eigenvalues of a $n \times n$ matrix yields its determinant: $|\mathbf{A}| = \prod_{i=1}^n \lambda_i$.
- 2) From 1.) it follows that a singular matrix must have at least one eigenvalue $\lambda_i = 0$.
- 3) The matrices \mathbf{A} and \mathbf{A}' have the same eigenvalues.
- 4) For a non-singular matrix \mathbf{A} with eigenvalues λ we have:
 $|\mathbf{A}^{-1} - \frac{1}{\lambda} \mathbf{I}| = 0$.
- 5) Symmetric matrices have only real eigenvalues and orthogonal eigenvectors.

1.8 Eigenvalues and eigenvectors

Properties of eigenvalues

- 1) The product of the eigenvalues of a $n \times n$ matrix yields its determinant: $|\mathbf{A}| = \prod_{i=1}^n \lambda_i$.
- 2) From 1.) it follows that a singular matrix must have at least one eigenvalue $\lambda_i = 0$.
- 3) The matrices \mathbf{A} and \mathbf{A}' have the same eigenvalues.
- 4) For a non-singular matrix \mathbf{A} with eigenvalues λ we have:
 $|\mathbf{A}^{-1} - \frac{1}{\lambda} \mathbf{I}| = 0$.
- 5) Symmetric matrices have only real eigenvalues and orthogonal eigenvectors.

1.8 Eigenvalues and eigenvectors

Properties of eigenvalues

- 1) The product of the eigenvalues of a $n \times n$ matrix yields its determinant: $|\mathbf{A}| = \prod_{i=1}^n \lambda_i$.
- 2) From 1.) it follows that a singular matrix must have at least one eigenvalue $\lambda_i = 0$.
- 3) The matrices \mathbf{A} and \mathbf{A}' have the same eigenvalues.
- 4) For a non-singular matrix \mathbf{A} with eigenvalues λ we have:
 $|\mathbf{A}^{-1} - \frac{1}{\lambda} \mathbf{I}| = 0$.
- 5) Symmetric matrices have only real eigenvalues and orthogonal eigenvectors.

1.8 Eigenvalues and eigenvectors

Properties of eigenvalues

- 1) The product of the eigenvalues of a $n \times n$ matrix yields its determinant: $|\mathbf{A}| = \prod_{i=1}^n \lambda_i$.
- 2) From 1.) it follows that a singular matrix must have at least one eigenvalue $\lambda_i = 0$.
- 3) The matrices \mathbf{A} and \mathbf{A}' have the same eigenvalues.
- 4) For a non-singular matrix \mathbf{A} with eigenvalues λ we have:
 $|\mathbf{A}^{-1} - \frac{1}{\lambda} \mathbf{I}| = 0$.
- 5) Symmetric matrices have only real eigenvalues and orthogonal eigenvectors.

1.8 Eigenvalues and eigenvectors

Properties of eigenvalues

- 6) The rank of a symmetric matrix \mathbf{A} is equal to the number of eigenvalues different from zero.
- 7) The sum of the eigenvalues is equal to the trace:
$$\text{tr}(\mathbf{A}) = \sum_{i=1}^n \lambda_i.$$
- 8) It holds that the eigenvalues of \mathbf{A}^k are λ_i^k for all $i = 1, \dots, n$ as $\mathbf{A}^k = \mathbf{T} \mathbf{\Lambda}^k \mathbf{T}^{-1}$.
- 9) \mathbf{A} has n independent eigenvectors and is diagonalizable if all eigenvalues λ_i are distinct.

1.8 Eigenvalues and eigenvectors

Properties of eigenvalues

- 6) The rank of a symmetric matrix \mathbf{A} is equal to the number of eigenvalues different from zero.
- 7) The sum of the eigenvalues is equal to the trace:
$$\text{tr}(\mathbf{A}) = \sum_{i=1}^n \lambda_i.$$
- 8) It holds that the eigenvalues of \mathbf{A}^k are λ_i^k for all $i = 1, \dots, n$ as $\mathbf{A}^k = \mathbf{T} \mathbf{\Lambda}^k \mathbf{T}^{-1}$.
- 9) \mathbf{A} has n independent eigenvectors and is diagonalizable if all eigenvalues λ_i are distinct.

1.8 Eigenvalues and eigenvectors

Properties of eigenvalues

- 6) The rank of a symmetric matrix \mathbf{A} is equal to the number of eigenvalues different from zero.
- 7) The sum of the eigenvalues is equal to the trace:
$$\text{tr}(\mathbf{A}) = \sum_{i=1}^n \lambda_i.$$
- 8) It holds that the eigenvalues of \mathbf{A}^k are λ_i^k for all $i = 1, \dots, n$ as $\mathbf{A}^k = \mathbf{T} \mathbf{\Lambda}^k \mathbf{T}^{-1}$.
- 9) \mathbf{A} has n independent eigenvectors and is diagonalizable if all eigenvalues λ_i are distinct.

1.8 Eigenvalues and eigenvectors

Properties of eigenvalues

- 6) The rank of a symmetric matrix \mathbf{A} is equal to the number of eigenvalues different from zero.
- 7) The sum of the eigenvalues is equal to the trace:
$$\text{tr}(\mathbf{A}) = \sum_{i=1}^n \lambda_i.$$
- 8) It holds that the eigenvalues of \mathbf{A}^k are λ_i^k for all $i = 1, \dots, n$ as $\mathbf{A}^k = \mathbf{T}\mathbf{\Lambda}^k\mathbf{T}^{-1}$.
- 9) \mathbf{A} has n independent eigenvectors and is diagonalizable if all eigenvalues λ_i are distinct.

1.8 Eigenvalues and eigenvectors

Properties of eigenvalues

- 6) The rank of a symmetric matrix \mathbf{A} is equal to the number of eigenvalues different from zero.
- 7) The sum of the eigenvalues is equal to the trace:
$$\text{tr}(\mathbf{A}) = \sum_{i=1}^n \lambda_i.$$
- 8) It holds that the eigenvalues of \mathbf{A}^k are λ_i^k for all $i = 1, \dots, n$ as $\mathbf{A}^k = \mathbf{T}\mathbf{\Lambda}^k\mathbf{T}^{-1}$.
- 9) \mathbf{A} has n independent eigenvectors and is diagonalizable if all eigenvalues λ_i are distinct.

1.9 Quadratic forms and sign definiteness

Definitions

- Degree of a polynomial
- Form of n th degree
- special case: quadratic form

$$Q(x_1, x_2) = a_{11}x_1^2 + 2a_{12}x_1x_2 + a_{22}x_2^2$$

1.9 Quadratic forms and sign definiteness

A quadratic form $Q(x_1, x_2)$ for two variables x_1 and x_2 is defined as

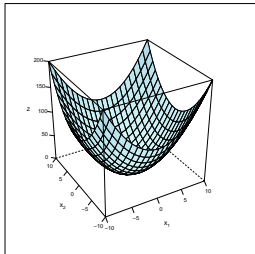
$$Q(x_1, x_2) = \underset{(1 \times 2)}{x'} \underset{(2 \times 2)}{A} \underset{(2 \times 1)}{x} = \sum_{i=1}^2 \sum_{j=1}^2 a_{ij} x_i x_j$$

where $a_{ij} = a_{ji}$ and, thus,

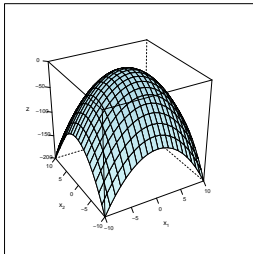
with the symmetric coefficient matrix $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{bmatrix}$.

1.9 Quadratic forms and sign definiteness

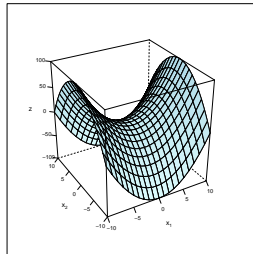
Graph of the positive definite form $Q(x_1, x_2) = x_1^2 + x_2^2$



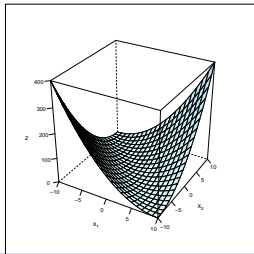
Graph of the negative definite form $Q(x_1, x_2) = -x_1^2 - x_2^2$



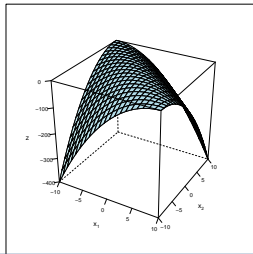
Graph of the indefinite form $Q(x_1, x_2) = x_1^2 - x_2^2$



Graph of the positive semidefinite form $Q(x_1, x_2) = (x_1 + x_2)^2$



Graph of the negative semidefinite form $Q(x_1, x_2) = -(x_1 + x_2)^2$



1.9 Quadratic forms and sign definiteness

The quadratic form associated with the matrix A (and thus the matrix A itself) is said to be

positive definite,	if $Q = x'Ax > 0$	for all $x \neq 0$
positive semi-definite,	if $Q = x'Ax \geq 0$	for all x
negative definite,	if $Q = x'Ax < 0$	for all $x \neq 0$
negative semi-definite,	if $Q = x'Ax \leq 0$	for all x

Otherwise the quadratic form is **indefinite**.

Note: For any quadratic matrix A it holds that $x'Ax = x'Bx$ with $B = 0,5 \cdot (A + A')$, a symmetric matrix.

1.9 Quadratic forms and sign definiteness

The quadratic form associated with the matrix A (and thus the matrix A itself) is said to be

positive definite,	if $Q = x'Ax > 0$	for all $x \neq 0$
positive semi-definite,	if $Q = x'Ax \geq 0$	for all x
negative definite,	if $Q = x'Ax < 0$	for all $x \neq 0$
negative semi-definite,	if $Q = x'Ax \leq 0$	for all x

Otherwise the quadratic form is **indefinite**.

Note: For any quadratic matrix A it holds that $x'Ax = x'Bx$ with $B = 0,5 \cdot (A + A')$, a symmetric matrix.

1.9 Quadratic forms and sign definiteness

The quadratic form $Q(\mathbf{x})$ is

- positive (negative) definite, if **all** eigenvalues of the matrix A are positive (negative): $\lambda_j > 0$ ($\lambda_j < 0$) $\forall j = 1, 2, \dots, n$;
- positive (negative) semi-definite, if **all** eigenvalues of the matrix A are non-negative (non-positive): $\lambda_j \geq 0$ ($\lambda_j \leq 0$) $\forall j = 1, 2, \dots, n$ and **at least one** eigenvalue is equal to zero;
- indefinite, if two eigenvalues have different signs.

1.9 Quadratic forms and sign definiteness

Properties of positive definite and positive semi-definite matrices

- 1) Diagonal elements of a positive definite matrix are strictly positive. Diagonal elements of a positive semi-definite matrix are nonnegative.
- 2) If A is positive definite, then A^{-1} exists and is positive definite.
- 3) If X is $n \times k$, then $X'X$ and XX' are positive semi-definite.
- 4) If X is $n \times k$ and $\text{rk}(X) = k$, then $X'X$ is positive definite (and therefore non-singular).

1.9 Quadratic forms and sign definiteness

Properties of positive definite and positive semi-definite matrices

- 1) Diagonal elements of a positive definite matrix are strictly positive. Diagonal elements of a positive semi-definite matrix are nonnegative.
- 2) If A is positive definite, then A^{-1} exists and is positive definite.
- 3) If X is $n \times k$, then $X'X$ and XX' are positive semi-definite.
- 4) If X is $n \times k$ and $\text{rk}(X) = k$, then $X'X$ is positive definite (and therefore non-singular).

1.9 Quadratic forms and sign definiteness

Properties of positive definite and positive semi-definite matrices

- 1) Diagonal elements of a positive definite matrix are strictly positive. Diagonal elements of a positive semi-definite matrix are nonnegative.
- 2) If A is positive definite, then A^{-1} exists and is positive definite.
- 3) If X is $n \times k$, then $X'X$ and XX' are positive semi-definite.
- 4) If X is $n \times k$ and $\text{rk}(X) = k$, then $X'X$ is positive definite (and therefore non-singular).

1.9 Quadratic forms and sign definiteness

Properties of positive definite and positive semi-definite matrices

- 1) Diagonal elements of a positive definite matrix are strictly positive. Diagonal elements of a positive semi-definite matrix are nonnegative.
- 2) If A is positive definite, then A^{-1} exists and is positive definite.
- 3) If X is $n \times k$, then $X'X$ and XX' are positive semi-definite.
- 4) If X is $n \times k$ and $\text{rk}(X) = k$, then $X'X$ is positive definite (and therefore non-singular).