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WIRTSCHAFTS- UND  
SOZIALWISSENSCHAFTLICHE  
FAKULTÄT

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**S414**  
**Advanced Mathematical Methods**  
Exercises

## LINEAR ALGEBRA

EXERCISE 1 **Eigenvalues**

Derive the characteristic equations for the matrices from exercise a)-c) and determine the eigenvalues.

a)  $\mathbf{B} = \begin{pmatrix} 4 & 1 \\ -2 & -0,5 \end{pmatrix}$

b)  $\mathbf{C} = \begin{pmatrix} 1 & 2 \\ 3 & -4 \end{pmatrix}$       c)  $\mathbf{D} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & -3 \\ 0 & -1 & 3 \end{pmatrix}$

EXERCISE 2 **Eigenvalues and Eigenvectors**

Given the matrix:

$$\mathbf{A} = \begin{bmatrix} -3 & 2 \\ -2 & 2 \end{bmatrix}$$

- a) Calculate the eigenvalues and the respective eigenvectors of  $\mathbf{A}$ .
- b) Use the eigenvalues to calculate the determinant of  $\mathbf{A}$ .

EXERCISE 3 **Eigenvalues**

A  $3 \times 3$  matrix  $\mathbf{A}$  has the eigenvalues  $\lambda_1 = 1$ ,  $\lambda_2 = 3$  and  $\lambda_3 = 4$ . Compute the determinant of  $\mathbf{A}$ ,  $\text{rg}(\mathbf{A})$ , the determinant of  $\mathbf{A}^{-1}$  and the eigenvalues of  $\mathbf{A}^{-1}$ . What can be said about the quadratic form  $\mathbf{x}'\mathbf{A}\mathbf{x}$  of the matrix  $\mathbf{A}$  for any vectors of  $\mathbf{x}$ ?

EXERCISE 4 **Eigenvalues**

Find the characteristic vectors of the matrix  $\begin{pmatrix} 4 & 2 \\ 2 & 1 \end{pmatrix}$ :

**EXERCISE 5 Quadratic Form**

Given the matrix

$$\mathbf{A} = \begin{pmatrix} 2 & -2 \\ 1 & 2 \end{pmatrix}.$$

- a) Determine the definiteness of the quadratic form  $Q = \mathbf{x}'\mathbf{A}\mathbf{x}$ .
- b) Explain in two sentences maximum what this means for the graph  $\{(x_1, x_2, Q) | Q = (x_1; x_2)\mathbf{A}(x_1; x_2)'\}$ .

**EXERCISE 6 Quadratic Form**

Write the quadratic form

$$Q = 4x_1^2 + 4x_1x_2 - x_2^2$$

in matrix notation and determine its definiteness.

**EXERCISE 7 Sign definiteness**

Express each quadratic form below as a matrix product involving a *symmetric* coefficient matrix:

- a)  $q = 3u^2 - 4uv + 7v^2$
- b)  $q = u^2 + 7uv + 3v^2$
- c)  $q = 8uv - u^2 - 31v^2$
- d)  $q = 6xy - 5y^2 - 2x^2$
- e)  $q = 3u_1^2 - 2u_1u_2 + 4u_1u_3 + 5u_2^2 + 4u_3^2 - 2u_2u_3$
- f)  $q = -u^2 + 4uv - 6uw - 4v^2 - 7w^2$

**EXERCISE 8   Sign definiteness**

Given a quadratic form  $u'Du$ , where  $D$  is  $2 \times 2$ , the characteristic equation of  $D$  can be written as:

$$\begin{vmatrix} d_{11} - r & d_{12} \\ d_{21} & d_{22} - r \end{vmatrix} = 0 \quad (d_{12} = d_{21})$$

Expand the determinant; express the roots of this equation by use of the quadratic formula and deduce the following:

- a) No imaginary number (a number involving  $\sqrt{-1}$ ) can occur in  $r_1$  and  $r_2$ .
- b) To have repeated roots, the matrix  $D$  must be in the form of  $\begin{pmatrix} c & 0 \\ 0 & c \end{pmatrix}$
- c) To have either positive or negative smidefiniteness, the determinant of the matrix  $D$  must vanish, i.e.  $|D| = 0$ .

**Solution Exercise 1:**

- a)  $\lambda_1 = 3.5$  and  $\lambda_2 = 0$
- b)  $\lambda_1 = 2$  and  $\lambda_2 = -5$
- c)  $\lambda_1 = 4.30278$ ;  $\lambda_2 = 0.69722$ ;  $\lambda_3 = 1$

**Solution Exercise 2:**

- a) Eigenvector for  $\lambda_1 = 1$ :  
 $\Rightarrow \begin{pmatrix} a \\ 2a \end{pmatrix} \quad \text{for } a \in \mathbb{R} \setminus \{0\}$   
Eigenvector for  $\lambda_2 = -2$ :  
 $\Rightarrow \begin{pmatrix} b \\ \frac{1}{2}b \end{pmatrix} \quad \text{for } b \in \mathbb{R} \setminus \{0\}$

- b)  $\det(\mathbf{A}) = -2$

**Solution Exercise 4:**

$$v_1 = \begin{pmatrix} -\frac{1}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} \end{pmatrix}, \quad v_2 = \begin{pmatrix} \frac{2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \end{pmatrix}$$

**Solution Exercise 5:**

- a) positive definite

**Solution Exercise 6:**

$$\mathbf{Q} = \mathbf{x}'\mathbf{A}\mathbf{x} \text{ with } \mathbf{A} = \begin{pmatrix} 4 & 2 \\ 2 & -1 \end{pmatrix}$$

$\mathbf{A}$  is indefinite

**Solution Exercise 7:**

Quadratic form:  $q = \mathbf{x}'\mathbf{A}\mathbf{x}$

a)

$$q = \begin{pmatrix} u \\ v \end{pmatrix}' \begin{pmatrix} 3 & -2 \\ -2 & 7 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}$$

b)

$$q = \begin{pmatrix} u \\ v \end{pmatrix}' \begin{pmatrix} 1 & 3.5 \\ 3.5 & 3 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}$$

c)

$$q = \begin{pmatrix} u \\ v \end{pmatrix}' \begin{pmatrix} -1 & 4 \\ 4 & -31 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}$$

d)

$$q = \begin{pmatrix} x \\ y \end{pmatrix}' \begin{pmatrix} -2 & 3 \\ 3 & -5 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

e)

$$q = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}' \begin{pmatrix} 3 & -1 & 2 \\ -1 & 5 & -1 \\ 2 & -1 & 4 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}$$

f)

$$q = \begin{pmatrix} u \\ v \\ w \end{pmatrix}' \begin{pmatrix} -1 & 2 & -3 \\ 2 & -4 & 0 \\ -3 & 0 & -7 \end{pmatrix} \begin{pmatrix} u \\ v \\ w \end{pmatrix}$$

**Solution Exercise 8:**

Refer to the video.