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S414

Advanced Mathematical Methods

Exercises

Parameter Estimation

Exercise 1 Properties of an estimator

(a) Illustrate graphically a biased and an unbiased estimator for the parameter θ.

(b) Illustrate graphically a consistent estimator for the parameter θ.

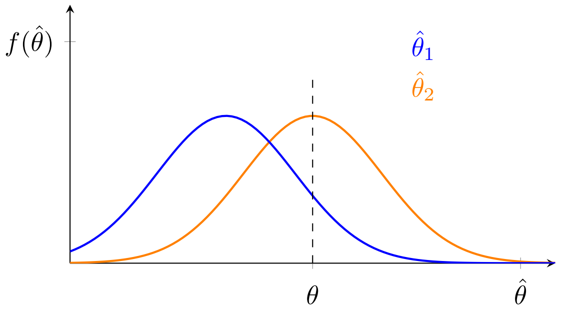
(c) Assume there are two alternative estimators $\hat{\theta}_1$ and $\hat{\theta}_2$ for the parameter θ. Both estimators are unbiased, however, $\hat{\theta}_1$ is more efficient than $\hat{\theta}_2$. Illustrate this in a graph.

(d) Decompose the mean squared error (MSE) of an estimator into its components bias and variance.

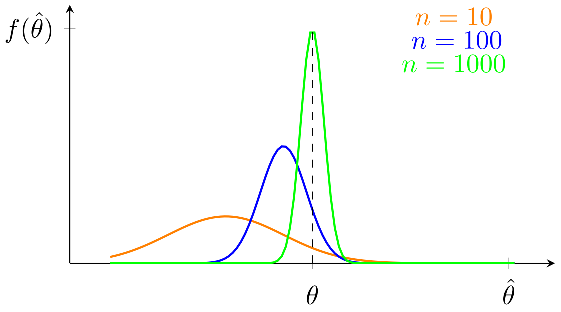
(e) Using a graph, explain the possible bias variance trade-off of an estimator (draw two estimators).
Comprehension question: Why does an estimator have a variance? Why is the estimator a random variable?

Solution:

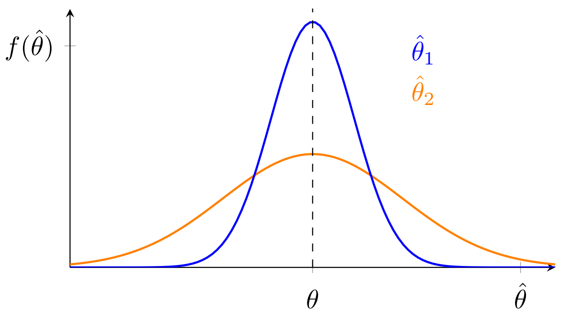
(a)



(b)



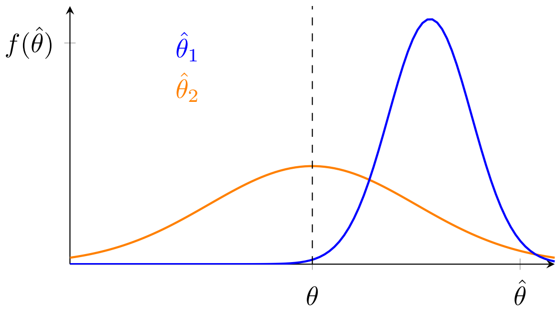
(c)



(d)

$$
\begin{aligned}
MSE &= \mathbb{E}[(\hat{\theta}-\theta)^2] \\
&= \mathbb{E}\left\{[(\hat{\theta}-\mathbb{E}(\hat{\theta})) + (\mathbb{E}(\hat{\theta})-\theta)]^2\right\} \\
&= \mathbb{E}\left\{(\hat{\theta}-\mathbb{E}(\hat{\theta}))^2 + 2(\hat{\theta}-\mathbb{E}(\hat{\theta}))(\mathbb{E}(\hat{\theta})-\theta) + (\mathbb{E}(\hat{\theta})-\theta)^2\right\} \\
&= \mathbb{E}\left[(\hat{\theta}-\mathbb{E}(\hat{\theta}))^2\right] + 2\,\mathbb{E}\left[(\hat{\theta}-\mathbb{E}(\hat{\theta}))(\mathbb{E}(\hat{\theta})-\theta)\right] + \mathbb{E}\left[(\mathbb{E}(\hat{\theta})-\theta)^2\right] \\
&= \mathbb{E}\left[(\hat{\theta}-\mathbb{E}(\hat{\theta}))^2\right] + 2\,\mathbb{E}\left[\hat{\theta}\,\mathbb{E}(\hat{\theta}) - \hat{\theta}\theta - \mathbb{E}(\hat{\theta})^2 + E(\hat{\theta})\theta\right] + (\mathbb{E}(\hat{\theta})-\theta)^2 \\
&= \mathbb{E}\left[(\hat{\theta}-\mathbb{E}(\hat{\theta}))^2\right] + 2\Big[\mathbb{E}(\hat{\theta})^2 - \mathbb{E}(\hat{\theta})\theta - \mathbb{E}(\hat{\theta})^2 + \mathbb{E}(\hat{\theta})\theta\Big] + (\mathbb{E}(\hat{\theta})-\theta)^2 \\
&= \mathbb{E}\left[(\hat{\theta}-\mathbb{E}(\hat{\theta}))^2\right] + \Big[\mathbb{E}(\hat{\theta})-\theta\Big]^2 \\
&= Var(\hat{\theta}) + Bias(\hat{\theta})^2
\end{aligned}
$$

(e)



Estimators are measurable functions of random variables and thus, random variables themselves.

Exercise 2 **Method of Moments and Maximum Likelihood**

(a) Assume that in the population the random variable X follows a Poisson distribution with parameter λ. Propose a method of moment (MM) estimator for the parameter λ. What do you need for the MM estimator? Are there any further possible estimators?

(b) Assume that in the population the random variable X follows an Exponential distribution with parameter λ. Propose a MM estimator for the parameter λ. Justify your choice.

(c) Assumptions as in (a): Derive the maximum likelihood (ML) estimator for the parameter λ of the Poisson distribution.

(d) Assumptions as in (b): Derive the ML estimator for the parameter λ of the Exponential distribution.

Solution:

(a) $X \sim Po(\lambda)$, $\mathbb{E}[X] = \lambda$, $Var[X] = \lambda$
Idea: Set theoretical moments equal to empirical moments to obtain the MM estimator
$\hat{\lambda}_1 = \frac{1}{n}\sum_{i=1}^{n} x_i$
$\hat{\lambda}_2 = \frac{1}{n}\sum_{i=1}^{n} x_i^2 - \left(\frac{1}{n}\sum_{i=1}^{n} x_i\right)^2$

(b) $X \sim Ex(\lambda)$, theoretical moments: $\mathbb{E}[X] = \frac{1}{\lambda}$, $Var[X] = \frac{1}{\lambda^2}$
Solve for λ:
$\lambda = \frac{1}{\mathbb{E}(X)}$
$\lambda = \frac{1}{\sqrt{Var(X)}} = \frac{1}{\sqrt{\mathbb{E}(X^2) - \mathbb{E}(X)^2}}$
Replace theoretical by empirical moments:
$\hat{\lambda}_1 = \frac{1}{\frac{1}{n}\sum_{i=1}^{n} x_i}$
$\hat{\lambda}_2 = \frac{1}{\sqrt{\frac{1}{n}\sum_{i=1}^{n} x_i^2 - (\frac{1}{n}\sum_{i=1}^{n} x_i)^2}}$

(c)

$$\begin{aligned}
\mathcal{L}(\tilde{\lambda}) = \mathcal{L}(x_1, ..., x_n; \tilde{\lambda}) &= \prod_{i=1}^{n} \frac{\tilde{\lambda}^{x_i}}{x_i!} e^{-\tilde{\lambda}} \\
ln(\mathcal{L}(x_1, ..., x_n; \tilde{\lambda})) &= ln\left[\prod_{i=1}^{n} \frac{\tilde{\lambda}^{x_i}}{x_i!} e^{-\tilde{\lambda}}\right] \\
&= \sum_{i=1}^{n} ln\left[\frac{\tilde{\lambda}^{x_i}}{x_i!} e^{-\tilde{\lambda}}\right] \\
&= \sum_{i=1}^{n} \left[ln(\tilde{\lambda}^{x_i}) - ln(x_i!) - \tilde{\lambda}\right] \\
&= \sum_{i=1}^{n} \left[x_i \cdot ln(\tilde{\lambda}) - ln(x_i!) - \tilde{\lambda}\right] \\
FOC: & \\
\frac{\partial\, ln\mathcal{L}(\tilde{\lambda})}{\partial \tilde{\lambda}} &= \sum_{i=1}^{n} \left[x_i \frac{1}{\tilde{\lambda}} - 1\right] \overset{!}{=} 0 \\
\hat{\lambda} &= \frac{1}{n}\sum_{i=1}^{n} x_i
\end{aligned}$$

(d)

$$\begin{aligned}
\mathcal{L}(x_1, ..., x_n; \tilde{\lambda}) &= \prod_{i=1}^{n} \tilde{\lambda} e^{-\tilde{\lambda} x_i} \\
ln(\mathcal{L}(x_1, ..., x_n; \tilde{\lambda})) &= \sum_{i=1}^{n} \left[ln\big(\tilde{\lambda} e^{-\tilde{\lambda} x_i}\big) \right] \\
&= n \cdot ln(\tilde{\lambda}) - \tilde{\lambda} \sum_{i=1}^{n} x_i \\
FOC : & \\
\frac{\partial \, ln\mathcal{L}(\tilde{\lambda})}{\partial \tilde{\lambda}} &= \frac{n}{\tilde{\lambda}} - \sum_{i=1}^{n} x_i \stackrel{!}{=} 0 \\
\hat{\lambda} &= \frac{1}{\frac{1}{n} \sum_{i=1}^{n} x_i}
\end{aligned}$$