Advanced Econometrics - 2^{nd} assignment sheet

Task 1

Show that for the model: $Y_i = m(X_i) + \sigma(X_i)U_i$, with $\mathsf{E}[U_i] = 0$ and U_i and X_i being independent, the following holds:

$$\mathsf{Var}[Y_i|X_i] = \sigma^2(X_i)\mathsf{Var}(U_i)$$

Solution

$$\begin{aligned}\mathsf{Var}[Y_i \mid X_i] &= \mathsf{Var}[m(X_i) + \sigma(X_i)U_i \mid X_i] \\ &= \mathsf{Var}[m(X_i) \mid X_i] + \mathsf{Var}[\sigma(X_i) \cdot U_i \mid X_i] \\ &= \sigma^2(X_i)\mathsf{Var}[U_i \mid X_i] = \sigma^2(X_i)\end{aligned}$$

Thereby we used that $\mathsf{Var}[m(X_i) \mid X_i] = 0$ respectively assumed that $\mathsf{Var}[U_i \mid X_i] = 1$.

Task 2

Let

$$\hat{f}_{Y,X}(y,x) = \frac{1}{nh^2}\sum_i^n K\left(\frac{x_i - x}{h}\right) K\left(\frac{y_i - y}{h}\right)$$

and

$$\hat{f}_X(x) = \frac{1}{nh}\sum_i^n K\left(\frac{x_i - x}{h}\right).$$

Show then, that

$$\hat{m}(x) = \int y \frac{\hat{f}_{Y,X}(y,x)}{\hat{f}_X(x)}\,\mathrm{d}y$$

is equal to

$$\hat{m}(x) = \frac{\sum_i K\left(\frac{x_i - x}{h}\right) y_i}{\sum_i K\left(\frac{x_i - x}{h}\right)}$$

Solution

$$
\begin{aligned}
\hat{m}(x) &= \int_{-\infty}^{\infty} y \frac{\hat{f}_{Y,X}(y,x)}{\hat{f}_X(x)} \,\mathrm{d}y \\
&= \int_{-\infty}^{\infty} y \frac{\frac{1}{nh^2}\sum_{i=1}^n K\left(\frac{x_i-x}{h}\right) K\left(\frac{y_i-y}{h}\right)}{\frac{1}{nh}\sum_{i=1}^n K\left(\frac{x_i-x}{h}\right)} \,\mathrm{d}y \\
&= \frac{1}{h}\frac{1}{\sum_{i=1}^n K\left(\frac{x_i-x}{h}\right)} \int_{-\infty}^{\infty} y \sum_{i=1}^n K\left(\frac{x_i-x}{h}\right) K\left(\frac{y_i-y}{h}\right) \,\mathrm{d}y \\
&= \frac{1}{h}\frac{1}{\sum_{i=1}^n K\left(\frac{x_i-x}{h}\right)} \int_{+\infty}^{-\infty} (y_i - h\psi) \sum_{i=1}^n K\left(\frac{x_i-x}{h}\right) K(\psi)\,(-h) \,\mathrm{d}\psi \\
&= \frac{1}{\sum_{i=1}^n K\left(\frac{x_i-x}{h}\right)} \int_{-\infty}^{\infty} \sum_{i=1}^n (y_i - h\psi) K\left(\frac{x_i-x}{h}\right) K(\psi) \,\mathrm{d}\psi \\
&= \frac{1}{\sum_{i=1}^n K\left(\frac{x_i-x}{h}\right)} \int_{-\infty}^{\infty} \left(\sum_{i=1}^n y_i\, K\left(\frac{x_i-x}{h}\right) K(\psi)\right) - \left(\sum_{i=1}^n h\psi K\left(\frac{x_i-x}{h}\right) K(\psi)\right) \mathrm{d}\psi \\
&= \frac{1}{\sum_{i=1}^n K\left(\frac{x_i-x}{h}\right)} \left(\sum_{i=1}^n y_i\, K\left(\frac{x_i-x}{h}\right) \int_{-\infty}^{\infty} K(\psi) \,\mathrm{d}\psi - h\sum_{i=1}^n K\left(\frac{x_i-x}{h}\right) \int_{-\infty}^{\infty} \psi K(\psi) \,\mathrm{d}\psi\right) \\
&= \frac{1}{\sum_{i=1}^n K\left(\frac{x_i-x}{h}\right)} \left(\sum_{i=1}^n y_i\, K\left(\frac{x_i-x}{h}\right) \cdot 1 - h\sum_{i=1}^n K\left(\frac{x_i-x}{h}\right) \cdot 0\right) \\
&= \frac{\sum_{i=1}^n y_i\, K\left(\frac{x_i-x}{h}\right)}{\sum_{i=1}^n K\left(\frac{x_i-x}{h}\right)}
\end{aligned}
$$

Note: The change of variables in line 4 is done using $\psi = \frac{y_i-y}{h}$ respectively $y = y_i - h\psi$. Therefore it holds: $\mathrm{d}\psi = -h\,\mathrm{d}y$. Because of this issues the bounds of the integral change their signs.

Task 3

Show that

$$
\begin{aligned}
\frac{1}{nh^2}\mathsf{E}\left[K\left(\frac{x_i-x}{h}\right)^2 \left(\frac{x_i-x}{h}\right)^2\right] &= \frac{1}{nh^2}\int K\left(\frac{\xi-x}{h}\right)^2 \left(\frac{\xi-x}{h}\right)^2 f_x(\xi)\,\mathrm{d}\xi \\
&= \dots \\
&= O((nh)^{-1})
\end{aligned}
$$

Solution

$$
\begin{aligned}
&\frac{1}{nh^2}\mathsf{E}\left[K\left(\frac{x_i - x}{h}\right)^2\left(\frac{x_i - x}{h}\right)^2\right] \\
&= \frac{1}{nh^2}\int_{-\infty}^{\infty} K\left(\frac{\xi - x}{h}\right)^2\left(\frac{\xi - x}{h}\right)^2 f_x(\xi)\,\mathrm{d}\xi \\
&= \frac{1}{nh^2}\int_{-\infty}^{\infty} K(\psi)^2\psi^2 f_x(\psi h + x) h\,\mathrm{d}\psi \\
&= \frac{1}{nh} f_x(x)\int_{-\infty}^{\infty} K(\psi)^2\psi^2\,\mathrm{d}\psi + \frac{1}{n}\partial_x f_x(x)\int_{-\infty}^{\infty} K(\psi)^2\psi^3\,\mathrm{d}\psi \\
&\quad + \frac{h}{2n}\partial_x^2 f_x(x)\int_{-\infty}^{\infty} K(\psi)^2\psi^4\,\mathrm{d}\psi + \frac{h^2}{6n}\int_{-\infty}^{\infty} K(\psi)^2\psi^5\partial_x^3 f_x(x_r)\,\mathrm{d}\psi \\
&= \frac{1}{nh} f_x(x)\kappa_1 + \frac{1}{n}\partial_x f_x(x)\kappa_2 \\
&\quad + \frac{h}{2n}\partial_x^2 f_x(x)\kappa_3 + \frac{h^2}{6n}\int_{-\infty}^{\infty} K(\psi)^2\psi^5\partial_x^3 f_x(x_r)\,\mathrm{d}\psi \\
&= \frac{1}{nh} f_x(x)\kappa_1 + \frac{1}{n}\partial_x f_x(x)\kappa_2 \\
&\quad + \frac{h}{2n}\partial_x^2 f_x(x)\kappa_3 + O(h^2/n)
\end{aligned}
$$

The last equation holds because:

$$
\begin{aligned}
\left|\frac{h^2}{6n}\int_{-\infty}^{\infty} K(\psi)^2\psi^5\partial_x^3 f_x(x_r)\,\mathrm{d}\psi\right| &\leq \frac{h^2}{6n}\int_{-\infty}^{\infty} |K(\psi)^2\psi^5\partial_x^3 f_x(x_r)|\,\mathrm{d}\psi \\
&\leq \frac{h^2}{6n}\int_{-\infty}^{\infty} K(\psi)^2|\psi^5||\partial_x^3 f_x(x_r)|\,\mathrm{d}\psi \\
&\leq \frac{h^2}{6n} b\int_{-\infty}^{\infty} K(\psi)^2|\psi^5|\,\mathrm{d}\psi \\
&\leq \frac{h^2}{6n} b\kappa_4
\end{aligned}
$$

And therefore we can state that:

$$
\begin{aligned}
\frac{1}{nh^2}\mathsf{E}\left[K\left(\frac{x_i - x}{h}\right)^2\left(\frac{x_i - x}{h}\right)^2\right] &= \frac{1}{nh} f_x(x)\kappa_1 + \frac{1}{n}\partial_x f_x(x)\kappa_2 \\
&\quad + \frac{h}{2n}\partial_x^2 f_x(x)\kappa_3 + O(h^2/n) \\
&= O((nh)^{-1}) + O(n^{-1}) + O(h/n) + O(h^2/n) \\
&= O((nh)^{-1})
\end{aligned}
$$

Task 4

Show that (by L_2 convergence)

$$\begin{aligned} a_{2,2,n} &= \frac{1}{nh}\sum_i K\left(\frac{x_i - x}{h}\right)\left(\frac{x_i - x}{h}\right)^2 \\ &= \mu_2 f_x(x) + o_p(h) \end{aligned}$$

Solution

$$\begin{aligned} \mathsf{E}\left[a_{2,2,n}\right] &= \left[\frac{1}{nh}\sum_i K\left(\frac{x_i - x}{h}\right)\left(\frac{x_i - x}{h}\right)^2\right] \\ &= \frac{1}{h}\left[K\left(\frac{x_i - x}{h}\right)\left(\frac{x_i - x}{h}\right)^2\right] \\ &= \frac{1}{h}\int_{-\infty}^{\infty} K\left(\frac{\xi - x}{h}\right)\left(\frac{\xi - x}{h}\right)^2 f_x(\xi)\,\mathrm{d}\xi \\ &= \frac{1}{h}\int_{-\infty}^{\infty} K\left(\psi\right)\psi^2 f_x(\psi h + x) h\,\mathrm{d}\psi \\ &= f_x(x)\int_{-\infty}^{\infty} K(\psi)\psi^2\,\mathrm{d}\psi + h\partial_x f_x(x)\int_{-\infty}^{\infty} K(\psi)\psi^3\,\mathrm{d}\psi \\ + &\quad \frac{h^2}{2}\partial_x^2 f_x(x)\int_{-\infty}^{\infty} K(\psi)\psi^4\,\mathrm{d}\psi + \frac{h^3}{6}\partial_x^3 f_x(x_r)\int_{-\infty}^{\infty} K(\psi)\psi^5\,\mathrm{d}\psi \\ &= f_x(x)\mu_2 + \frac{h^2}{2}\partial_x^2 f_x(x)\mu_4 + \frac{h^3}{6}\partial_x^3 f_x(x_r)\int_{-\infty}^{\infty} K(\psi)\psi^5\,\mathrm{d}\psi \\ &= f_x(x)\mu_2 + \frac{h^3}{2}\partial_x^2 f_x(x)\mu_4 + O(h^4) \\ &= f_x(x)\mu_2 + O(h^3) + O(h^4) = f_x(x)\mu_2 + O(h^3) \end{aligned}$$

Showing the variance term:

$$\begin{aligned} \mathsf{Var}\left[a_{2,2,n}\right] &= \mathsf{Var}\left[\frac{1}{nh}\sum_{i=1}^{n} K\left(\frac{x_i - x}{h}\right)\left(\frac{x_i - x}{h}\right)^2\right] \\ &= \frac{1}{n^2h^2}\mathsf{Var}\left[\sum_{i=1}^{n} K\left(\frac{x_i - x}{h}\right)\left(\frac{x_i - x}{h}\right)^2\right] \\ &= \frac{1}{n^2h^2} n\mathsf{Var}\left[K\left(\frac{x_i - x}{h}\right)\left(\frac{x_i - x}{h}\right)^2\right] \\ &= \frac{1}{nh^2}\left(\mathsf{E}\left[\left(K\left(\frac{x_i - x}{h}\right)\left(\frac{x_i - x}{h}\right)^2\right)^2\right] - \mathsf{E}\left[K\left(\frac{x_i - x}{h}\right)\left(\frac{x_i - x}{h}\right)^2\right]^2\right) \\ &= \frac{1}{nh^2}\left(\mathsf{E}\left[K\left(\frac{x_i - x}{h}\right)^2\left(\frac{x_i - x}{h}\right)^4\right] - \mathsf{E}\left[K\left(\frac{x_i - x}{h}\right)\left(\frac{x_i - x}{h}\right)^2\right]^2\right) \end{aligned}$$

Therefore we have to show what $\mathsf{E}\left[K\left(\frac{x_i-x}{h}\right)^2\left(\frac{x_i-x}{h}\right)^4\right]$ is:

$$\begin{aligned}
\mathsf{E}\left[K\left(\frac{x_i-x}{h}\right)^2\left(\frac{x_i-x}{h}\right)^4\right] &= \int_{-\infty}^{\infty} K\left(\frac{\xi-x}{h}\right)^2\left(\frac{\xi-x}{h}\right)^4 f_x(\xi)\,\mathrm{d}\xi \\
&= \int_{-\infty}^{\infty} K(\psi)^2\psi^4 f_x(\psi h+x)h\,\mathrm{d}\psi \\
&= h\int_{-\infty}^{\infty} K(\psi)^2\psi^4(f_x(x)+\psi h\partial_x f_x(x_r))\,\mathrm{d}\psi \\
&= hf_x(x)\int_{-\infty}^{\infty} K(\psi)^2\psi^4 + h^2\int_{-\infty}^{\infty} K(\psi)^2\psi^5\partial_x f_x(x_r))\,\mathrm{d}\psi \\
&= hf_x(x)\kappa_{24}+O(h^2)
\end{aligned}$$

Put together:

$$\begin{aligned}
\mathsf{Var}\left[a_{2,2,n}\right] &= \frac{1}{nh^2}\left(\mathsf{E}\left[K\left(\frac{x_i-x}{h}\right)^2\left(\frac{x_i-x}{h}\right)^4\right]-\mathsf{E}\left[K\left(\frac{x_i-x}{h}\right)\left(\frac{x_i-x}{h}\right)^2\right]^2\right) \\
&= \frac{1}{nh^2}\left(hf_x(x)\kappa_{24}+O(h^2)-(hf_x(x)\mu_2+O(h^4))^2\right) \\
&= (hn)^{-1}f_x(x)\kappa_{24}+O(n^{-1})-\frac{1}{nh^2}(h^2f_x(x)^2\mu_2^2+2hf_x(x)\mu_2O(h^4)+O(h^8)) \\
&= (hn)^{-1}f_x(x)\kappa_{24}+O(n^{-1})-(n^{-1})f_x(x)^2\mu_2^2-O(h^3/n)-O(h^6/n) \\
&= O((hn)^{-1})+O(n^{-1})-O(n^{-1})-O(h^3/n)-O(h^6/n)=O((hn)^{-1})=o(1)
\end{aligned}$$