Advanced Mathematical Methods

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5 Parameter Estimation

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Online References

Applied Econometrics Lecture (by Prof. Grammig, available on TIMMS)

- Lecture 37: Law of Large Numbers
- Lecture 38: Central Limit Theorem

Implications of a random sample

$\{X_1, X_2, ..., X_n\}$ is a random sample if all draws of the random variable are **independently** and **identically distributed (iid)**.

Implications:

$$F_{X_1,X_2,...,X_n}(x_1, x_2, ..., x_n) \overset{\text{independent}}{=} F_{X_1}(x_1) \cdot F_{X_2}(x_2) \cdot ... \cdot F_{X_n}(x_n)$$

$$\overset{\text{identical}}{=} F_X(x_1) \cdot F_X(x_2) \cdot ... \cdot F_X(x_n)$$

$$f_{X_1,X_2,...,X_n}(x_1, x_2, ..., x_n) \overset{\text{independent}}{=} f_{X_1}(x_1) \cdot f_{X_2}(x_2) \cdot ... \cdot f_{X_n}(x_n)$$

$$\overset{\text{identical}}{=} f_X(x_1) \cdot f_X(x_2) \cdot ... \cdot f_X(x_n)$$

Asymptotic ($n \to \infty$) results

For a random sample $\{X_1, X_2, \ldots, X_n\}$ with finite $\mathbb{E}(X_i)$ and $Var(X_i)$ and an appropriately large n, following concepts apply:

1. Law of Large Numbers (LLN)
2. Central Limit Theorem (CLT)

Law of Large Numbers

Law of Large Numbers

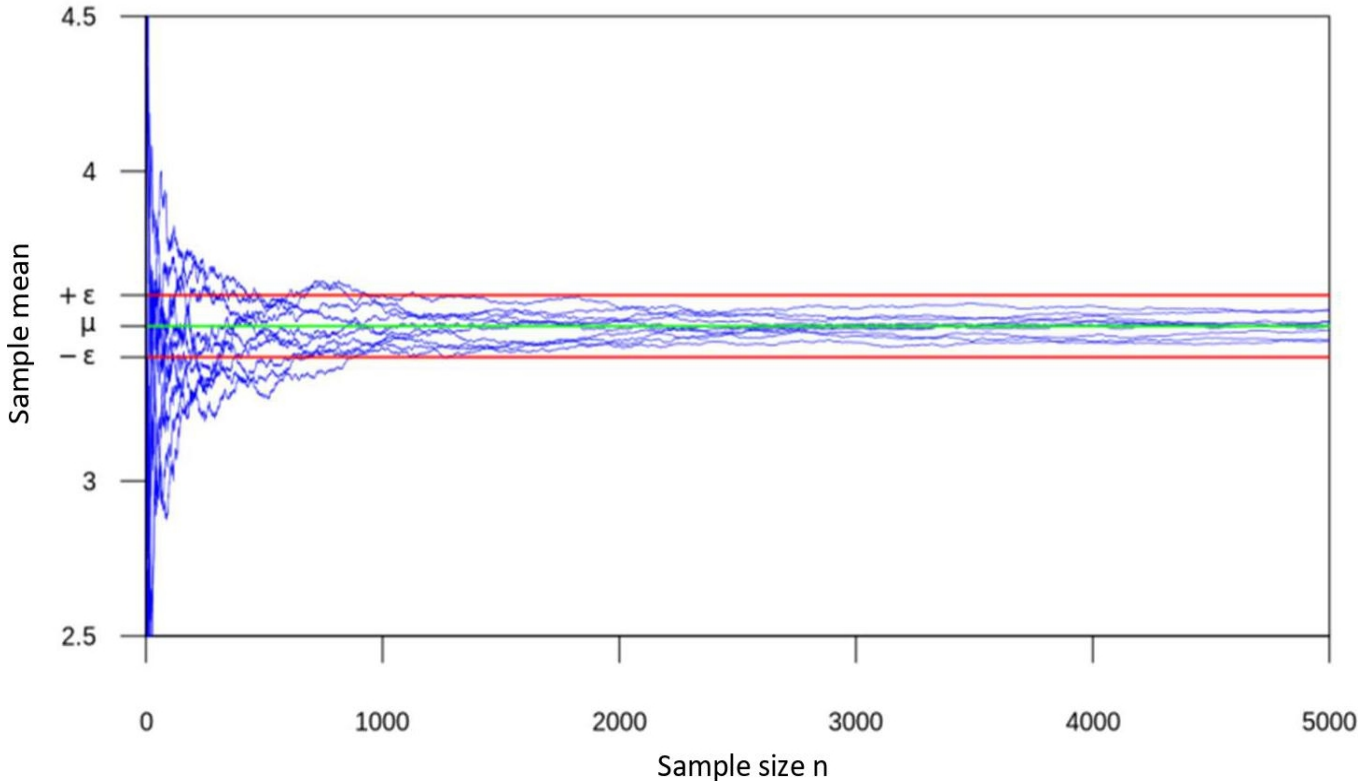
$$\lim_{n\to\infty} P\left[\left|\frac{1}{n}\sum_{i=1}^{n} X_i - \mathbb{E}(X_i)\right| > \epsilon\right] = 0 \quad \text{for any } \epsilon > 0.$$

When the LLN holds, moments of the distribution of a population can be consistently estimated by moments of a random sample. I.e., we can write:

- $\text{plim}\frac{1}{n}\sum_{i=1}^{n} X_i = \mathbb{E}(X)$
- $\frac{1}{n}\sum_{i=1}^{n} X_i \xrightarrow{p} \mathbb{E}(X)$

LLN and convergence in probability

Illustration of convergence in probability using LLN:



$\Rightarrow$ A sequence $\{X_i\}$ converges in probability to a constant μ

Central Limit Theorem

Central Limit Theorem

$$\sqrt{n}\left[\frac{1}{n}\sum_{i=1}^{n} X_i - \mathbb{E}(X_i)\right] \overset{a}{\sim} \mathcal{N}(0, Var(X_i))$$

E.g. if $\{z_i\}$ is **iid** with $\mathbb{E}(z_i) = \mu$ and $Var(z_i) = \sigma^2$ and $\bar{z}_n = \frac{1}{n}\sum_{i=1}^{n} z_i \xrightarrow{p} \mu$ then,

$$\sqrt{n}(\bar{z}_n - \mu) \xrightarrow{d} y \sim \mathcal{N}(0, \sigma^2)$$

$$\text{or} \quad \bar{z}_n - \mu \overset{a}{\sim} \mathcal{N}(0, \frac{\sigma^2}{n})$$

$$\text{or} \quad \bar{z}_n \overset{a}{\sim} \mathcal{N}(\mu, \frac{\sigma^2}{n})$$

Parameter estimation

Underlying principle:

1. Assumption about the distribution of a random variable in the population: $\mathbb{E}(X) = \mu$, $Var(X) = \sigma^2 < \infty$
2. Draw of a random sample
3. Use estimation functions to estimate the unknown parameters of the distribution: μ, σ^2
4. Estimation functions (short: estimators) are measurable functions of random variables $\Rightarrow \widehat{\boldsymbol{\theta}}_n = \widehat{\theta}_n(X_1, X_2, ..., X_n)$

Quality of estimators

Finite sample properties of estimators:

- Bias
- Variance
- Efficiency (MSE)

Asymptotic concepts ($n \to \infty$):

- Consistency
- Asymptotic normality

Quality of estimators

Bias of $\widehat{\theta}_n$:

- If $\mathbb{E}(\widehat{\theta}_n) = \theta \Rightarrow$ unbiased estimator
- If $\mathbb{E}(\widehat{\theta}_n) \neq \theta \Rightarrow$ biased estimator

Variance of $\widehat{\theta}_n$:

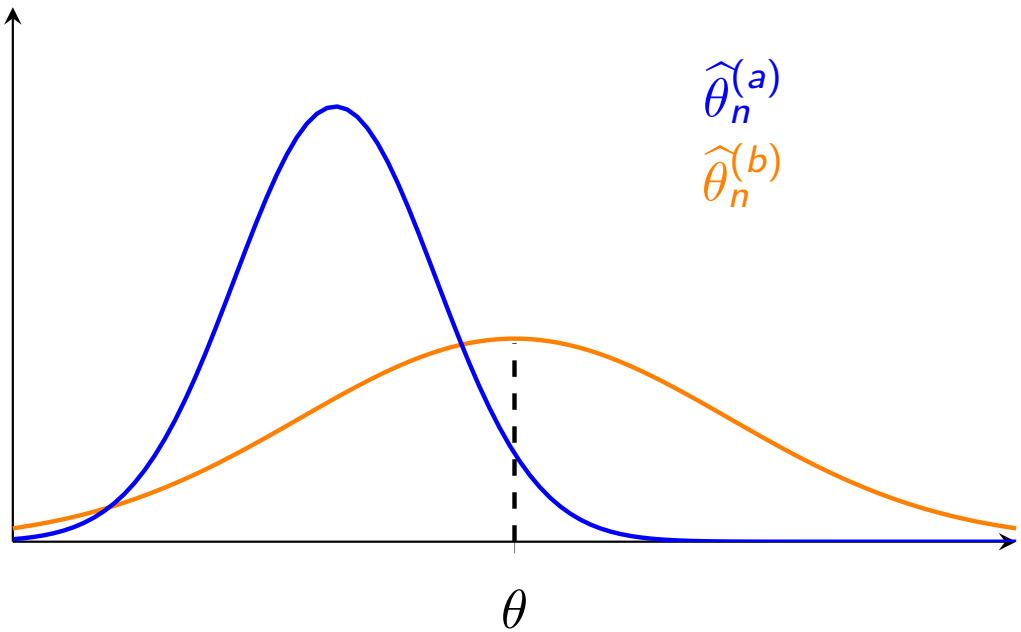
- Preferable: small variance of the estimation function $\Rightarrow$ small $Var[\widehat{\theta}_n(X_1, X_2, ..., X_n)]$

Quality of estimators

Mean Squared Error (MSE) of $\widehat{\theta}_n$:

- Preferable: an unbiased estimator with a small variance

  $\Rightarrow$ Efficiency
- $\mathrm{MSE}(\widehat{\theta}_n) \equiv \mathbb{E}[(\widehat{\theta}_n - \theta_n)^2] = \mathit{Var}(\widehat{\theta}_n) + \mathit{Bias}(\widehat{\theta}_n)^2$
- Trade-off: variance vs. bias of an estimator



Quality of estimators

Consistency of $\widehat{\theta}_n$:

- The bigger the sample size gets, the smaller the sampling error $|\widehat{\theta}_n - \theta|$ should become
- Formally $\widehat{\theta}_n$ is consistent if

$$\lim_{n \to \infty} P(|\widehat{\theta} - \theta| > \varepsilon) = 0$$

- Short-hand notation: $\widehat{\theta}_n \xrightarrow{p} \theta$ or $\operatorname{plim}_{n \to \infty} \widehat{\theta}_n = \theta$
- Thus, for an infinitely large sample the estimator must be equal to the population parameter

Quality of estimators

Convergence in distribution and asymptotic normality of $\widehat{\theta}_n$

Convergence in distribution: $z_n \xrightarrow{d} z$

If the c.d.f. of z_n converges to the c.d.f. of z at each point of continuity, then z_n converges in distribution to z.

Asymptotic normality of $\widehat{\theta}_n$:

The distribution of an estimator $\widehat{\theta}_n$ converges to the distribution of another random variable Z, which is normally distributed.
Generally:

$x_N \xrightarrow{d} Z \sim \mathcal{N}(0, \sigma^2)$

Applied to the Central Limit Theorem:

$\sqrt{n}(\widehat{\theta}_n - \theta) \xrightarrow{d} Z \sim \mathcal{N}(0, Var(\widehat{\theta}_n))$

Estimation techniques

1. Method of Moments
2. Maximum Likelihood Method
3. Ordinary Least Squares (OLS)

Method of Moments

Idea:

- Use the known relationship between parameters and theoretical moments
- Replace theoretical by empirical moments
- Method of Moment estimators are consistent

Method of Moments

For example, Exponential distribution $f_x(x;\lambda) = \lambda e^{-\lambda x}$:

- theoretical moments:
  1. $\mathbb{E}(X) = \frac{1}{\lambda} \Longrightarrow \lambda = \frac{1}{\mathbb{E}(X)}$
  2. $Var(X) = \frac{1}{\lambda^2} \Longrightarrow \lambda = \frac{1}{\sqrt{Var(X)}} = \frac{1}{\sqrt{\mathbb{E}(X^2) - \mathbb{E}(X)^2}}$
- replace by empirical moments:
  1. $\widehat{\lambda}_1 = \frac{1}{\frac{1}{n}\sum_{i=1}^{n} x_i}$
  2. $\widehat{\lambda}_2 = \frac{1}{\sqrt{\frac{1}{n}\sum_{i=1}^{n} x_i^2 - (\frac{1}{n}\sum_{i=1}^{n} x_i)^2}}$

Maximum Likelihood Method

Idea:

- Chose $\widehat{\boldsymbol{\theta}} = (\widehat{\theta}_1, \widehat{\theta}_2, ..., \widehat{\theta}_K)'$ such that the likelihood function $\mathcal{L}(\tilde{\boldsymbol{\theta}})$ is maximized
- The likelihood function $\mathcal{L}(\tilde{\boldsymbol{\theta}})$ is the joint density function of $X_1, ..., X_n$ with parameters $\tilde{\boldsymbol{\theta}}$

$$\mathcal{L}(\tilde{\boldsymbol{\theta}}) = f_{X_1, X_2, ..., X_n}(x_1, x_2, ...x_n; \tilde{\boldsymbol{\theta}})$$

Intuition:

1. We decide on the distribution of $X \sim (\boldsymbol{\theta})$ in the population
2. Draw a random sample of X
3. Keep the sample fixed and choose the parameter such that $\mathcal{L}(\tilde{\boldsymbol{\theta}})$ is maximized $\Rightarrow$ "Maximize the probability to observe the realization of our sample"

Maximum Likelihood Method

Recipe:

1. Set up the likelihood function and exploit the **iid** sample implications
2. Take the *ln* of the likelihood function
3. Maximize the log-likelihood function w.r.t. the parameter $\tilde{\theta}$

ML - 1. Set up the likelihood function

Use the **iid** property of a random sample:

$$\mathcal{L}(\tilde{\boldsymbol{\theta}}) = f_{X_1,X_2,\ldots,X_n}(x_1, x_2, \ldots x_n; \tilde{\boldsymbol{\theta}})$$

$$\overset{\text{independent}}{=} f_{X_1}(x_1; \tilde{\boldsymbol{\theta}}) \cdot f_{X_2}(x_2; \tilde{\boldsymbol{\theta}}) \cdot \ldots \cdot f_{X_n}(x_n; \tilde{\boldsymbol{\theta}})$$

$$\overset{\text{identical}}{=} f_X(x_1; \tilde{\boldsymbol{\theta}}) \cdot f_X(x_2; \tilde{\boldsymbol{\theta}}) \cdot \ldots \cdot f_X(x_n; \tilde{\boldsymbol{\theta}})$$

$$= \prod_{i=1}^{n} f_X(x_i; \tilde{\boldsymbol{\theta}})$$

ML - 2. Log-Transformation

Logarithmic rules

1. $ln(a \cdot b) \Leftrightarrow ln(a) + ln(b)$
2. $ln(\frac{a}{b}) \Leftrightarrow ln(a) - ln(b)$
3. $ln(a)^b \Leftrightarrow \ b \cdot ln(a)$
4. $ln(e^x) \Leftrightarrow e^{ln(x)} \Leftrightarrow x$

$$\mathcal{L}(\tilde{\theta}) = \prod_{i=1}^{n} f_X(x_i; \tilde{\theta}) \qquad | \ ln$$

$$ln\mathcal{L}(\tilde{\theta}) = \sum_{i=1}^{n} ln\, f_X(x_i; \tilde{\theta})$$

ML - 3. Log-likelihood maximization

Maximize the log-likelihood function:

$$\widehat{\boldsymbol{\theta}}_{ML} = \underset{\tilde{\boldsymbol{\theta}}}{\operatorname{argmax}}\, ln\mathcal{L}(\tilde{\boldsymbol{\theta}}) = \sum_{i=1}^{n} ln\, f_X(x_i; \tilde{\boldsymbol{\theta}})$$

Determine $\widehat{\boldsymbol{\theta}}_{ML}$ via the F.O.C. $\frac{\partial ln\mathcal{L}(\tilde{\boldsymbol{\theta}})}{\partial \tilde{\boldsymbol{\theta}}} \stackrel{!}{=} 0$:

$$\sum_{i=1}^{n} \frac{\partial ln\, f_X(x_i; \widehat{\boldsymbol{\theta}})}{\partial \tilde{\theta}_1} \stackrel{!}{=} 0$$

$$\vdots$$

$$\sum_{i=1}^{n} \frac{\partial ln\, f_X(x_i; \widehat{\boldsymbol{\theta}})}{\partial \tilde{\theta}_K} \stackrel{!}{=} 0$$

Properties of the ML estimator

If the likelihood function is correctly specified, the ML estimator has following properties:

- Consistency
- Asymptotic efficiency (for large n this estimator has the smallest MSE)
- Asymptotic normality

Conditional Maximum Likelihood (CML)

Procedure:

1. Specification of the conditional distribution $Y|X = x$
2. Specification of conditional moments (functions of X)
3. Insert conditional moments into the likelihood function
4. Maximize the (log-) likelihood function w.r.t. the unknown parameter

The marginal, joint and conditional distribution

Relationship of the marginal, joint, and conditional distribution:

$$f_{X_1|X_2} = \frac{f_{X_1,X_2}}{f_{X_2}}$$

$$\Leftrightarrow \quad f_{X_1,X_2} = f_{X_1|X_2} \cdot f_{X_2}$$

$$\Leftrightarrow \quad f_{X_2} = \frac{f_{X_1,X_2}}{f_{X_1|X_2}}$$

The marginal, joint and conditional distribution

Example: Exploiting this relationship allows us to write the joint density $f_{X_1,\ldots,X_5}$ as product of four conditional and one marginal density:

$$f_{X_1,X_2} = f_{X_2|X_1} \cdot f_{X_1} \tag{1}$$

$$f_{X_1,X_2,X_3} = f_{X_3|X_1,X_2} \cdot f_{X_1,X_2} \tag{2}$$

$$f_{X_1,X_2,X_3,X_4} = f_{X_4|X_1,X_2,X_3} \cdot f_{X_1,X_2,X_3} \tag{3}$$

$$f_{X_1,X_2,X_3,X_4,X_5} = f_{X_5|X_1,X_2,X_3,X_4} \cdot f_{X_1,X_2,X_3,X_4} \tag{4}$$

Plugging in (1)-(4) $f_{X_1,\ldots,X_5}$ can be expressed as:

$$= f_{X_5|X_1,X_2,X_3,X_4} \cdot f_{X_4|X_1,X_2,X_3} \cdot f_{X_3|X_1,X_2} \cdot f_{X_2|X_1} \cdot f_{X_1}$$

Example CML: Probit Model

- A binary response model $\rightarrow$ outcome y is either 0 or 1
- $Y|X = x \sim Be(p(x))$
- Thus, not a linear probability model $\rightarrow$ probability for a certain event doesn't change linear in x

$$p(x) = E[Y|X = x] = \underbrace{F(\beta_0 + \beta_1 x) = \int_{-\infty}^{\beta_0+\beta_1 x} \frac{1}{\sqrt{2\pi}} e^{\frac{z^2}{2}} dz}_{c.d.f.\ of\ a\ standard\ normal\ distribution}$$

Example CML: Probit Model

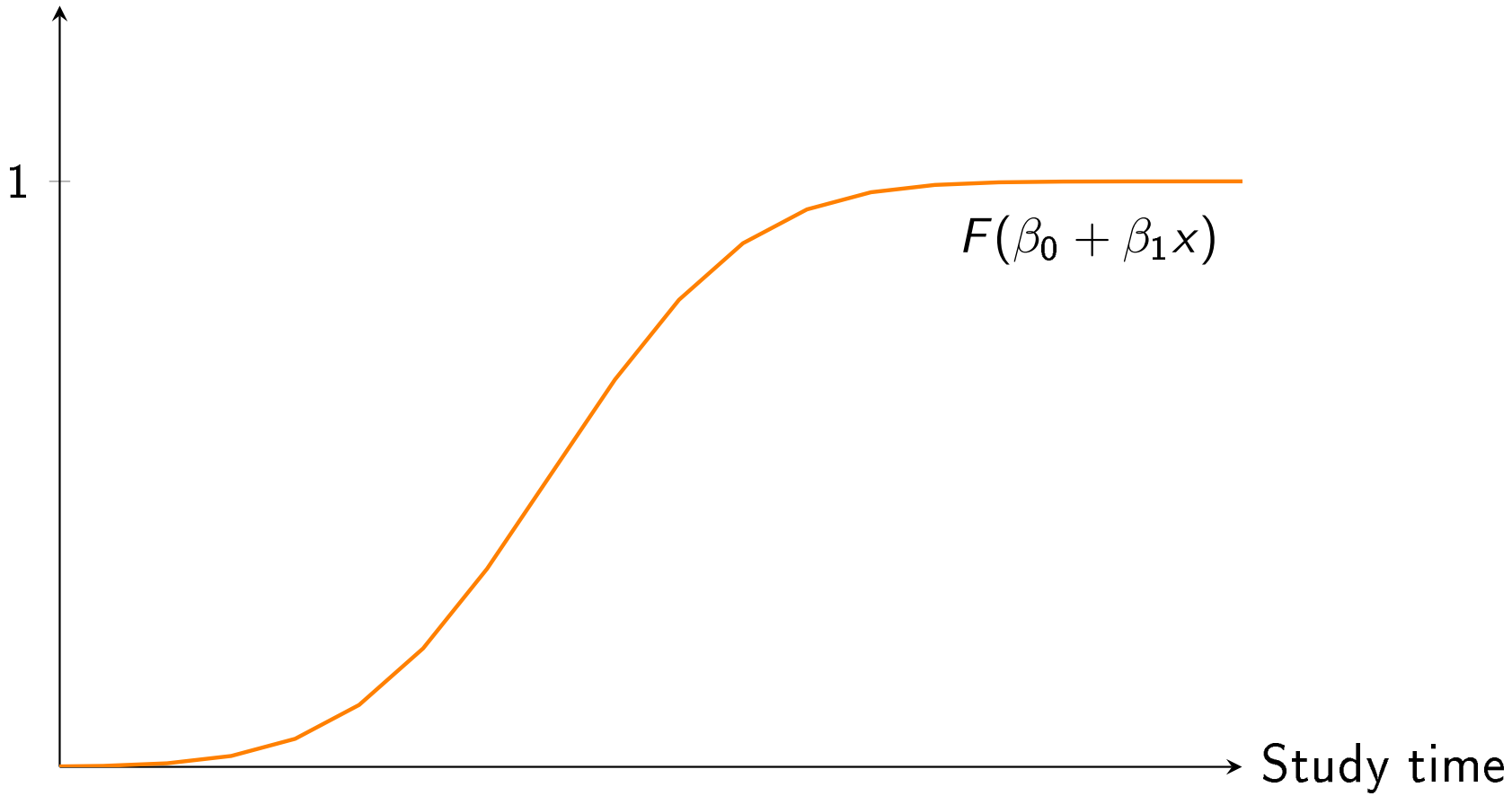
- Pass or fail in an exam given the amount of study time
- X: Study time
  y = 1: Pass
  y = 0: Fail
- Probability to pass the exam doesn't change linearly with increasing study time
- $F(\beta_0 + \beta_1 x) = P(Y = 1|X) = p(x) \rightarrow$ probability Pass

  $1 - F(\beta_0 + \beta_1 x) = P(Y = 0|X) = 1 - p(x) \rightarrow$ probability Fail

  $\Rightarrow$ Bernoulli distribution

Example CML: Probit Model

Probability to pass the exam as a function of study time:



Example CML: Probit Model

Estimating the parameters β_0 and β_1:

1. Set up conditional likelihood function using a random sample

$$\mathcal{L}(\tilde{\beta}_0, \tilde{\beta}_1) = \prod_{i=1}^{n} \underbrace{p(x_i)^{y_i}(1 - p(x_i))^{1-y_i}}_{\textit{Bernoulli prob func}}$$

2. Maximize the log-likelihood w.r.t. β_0 and β_1

$$\frac{\partial ln\mathcal{L}}{\partial \tilde{\beta}_0} \stackrel{!}{=} 0 \qquad \frac{\partial ln\mathcal{L}}{\partial \tilde{\beta}_1} \stackrel{!}{=} 0$$

3. Solve the F.O.C. for $\widehat{\beta}_0$ and $\widehat{\beta}_1$

Ordinary Least Squares (OLS)

Basic idea:

- Explain a variable Y (dependent variable) as linear combination of other variables $X_1, X_2, ..., X_K$ (explanatory variables/regressors) and the residual

$$Y = \tilde{\beta}_1 X_1 + \tilde{\beta}_2 X_2 + ... + \tilde{\beta}_K X_K + \tilde{u}$$

- While Y and X are directly observable, $\tilde{u}$ is not $\Rightarrow$ $\tilde{u}$ depends on how we choose $\tilde{\beta}_1, \tilde{\beta}_2, ..., \tilde{\beta}_K$

$$\tilde{u}_i = \tilde{u}_i(\tilde{\beta}_1, \tilde{\beta}_2, ..., \tilde{\beta}_K) = y_i - \tilde{\beta}_1 X_{i1} - \tilde{\beta}_2 X_{i2} - ... - \tilde{\beta}_K X_{iK}$$

- However, it is clearly defined how $\tilde{\beta}_1, \tilde{\beta}_2, ..., \tilde{\beta}_K$ must be chosen

OLS - Choosing $\tilde{\beta}_1, \tilde{\beta}_2, ..., \tilde{\beta}_K$

Choosing $\tilde{\beta}_1, \tilde{\beta}_2, ..., \tilde{\beta}_K$ either by:

1. Minimization of the least squares function:

$$\widehat{\beta} = \underset{\tilde{\beta}}{argmin} = \sum_{i=1}^{n} (y_i - \tilde{\beta}' x_i)^2$$

2. Moment restrictions: $\widehat{u} = y_i - \widehat{\beta}_1 - \widehat{\beta}_2 x_{i2} - ... - \widehat{\beta}_K x_{iK}$

$$\frac{1}{n}\sum_{i=1}^{n} \widehat{u}_i x_{i1} \stackrel{!}{=} 0\,, \quad \frac{1}{n}\sum_{i=1}^{n} \widehat{u}_i x_{i2} \stackrel{!}{=} 0\,, \ \ldots\,, \frac{1}{n}\sum_{i=1}^{n} \widehat{u}_i x_{iK} \stackrel{!}{=} 0$$

$\Rightarrow$ Both approaches lead to the same result

$$\widehat{\boldsymbol{\beta}} = \left(\frac{1}{n}\sum_{i=1}^{n} \boldsymbol{x_i' x_i}\right)^{-1} \left(\frac{1}{n}\sum_{i=1}^{n} \boldsymbol{x_i' y_i}\right) = (\boldsymbol{X'X})^{-1}(\boldsymbol{X'y})$$

OLS - classical assumptions

1. Linearity
   $Y = \beta_1 + \beta_2 X_2 + ... + \beta_K X_K$
2. Strict exogeneity
   $\mathbb{E}(u|X_1, X_2, ..., X_K) = 0$
3. Conditional homoscedasticity
   $Var(u|X_1, X_2, ..., X_K) = \sigma^2$
4. Distribution assumption
   $u|X_1, X_2, ..., X_K \sim \mathcal{N}(0, \sigma^2)$

OLS Example

We want to estimate the effect of more education on wage (β_2: return to schooling)

$$wage = \beta_1 + \beta_2 education + \beta_3 experience + \beta_4 experience^2 + u$$

Estimate $\widehat{\boldsymbol{\beta}} = (\boldsymbol{X'X})^{-1}(\boldsymbol{X'y})$

$\Rightarrow$ Under the above mentioned assumptions $\widehat{\beta}_2$ gives the estimated marginal effect (ceteris paribus) of schooling on wage