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# Proof-Theoretic versus Model-Theoretic Consequence

Peter Schroeder-Heister

## 1. The model-theoretic view

By 'model-theoretic consequence' we mean logical consequence explained in terms of models. According to the standard reading given to it by Tarski, which is related to ideas already developed by Bolzano, a sentence $A$ follows logically from a set of sentences $M$, iff every model of $M$ is a model of $A$, symbolically

$$M \vDash A \ \Leftrightarrow_{df} \ (\forall \mathfrak{M}) \left[\, (\forall B \in M)(\mathfrak{M} \vDash B) \ \Rightarrow \ \mathfrak{M} \vDash A \,\right].$$

Thus consequence is *transmission of truth* from the premisses to the conclusion, where 'transmission' is understood in the simple declarative sense of classical implication: *if* the premisses are true (in a model-theoretic structure), *then* so is the conclusion. This means in particular that truth is conceptually prior to consequence, as the latter is explained in terms of the former. Using a more traditional terminology, we might say that the *categorical* concept of truth is conceptually prior to the *hypothetical* concept of consequence.

Proof-theoretic consequence is normally understood as derivability in a formal system. A sentence $A$ is derivable from a set of sentences $M$ in a formal system $K$ if it can be generated from elements of $M$ using the axioms and inference rules of $K$, formally

$$M \vdash_K A.$$

The justification of inference or deduction in $K$ is then achieved by showing that the primitive rules of $K$ are correct, so that a derivation in $K$ establishes a valid consequence, formally

$$M \vdash_K A \ \Rightarrow \ M \vDash A.$$

If also the converse (i.e., completeness) holds, we can be sure that the proof-theoretic consequence relation matches the model-theoretic one. Therefore, from the model-theoretic point of view, inference as the activity of drawing conclusions according to certain rules is justified in terms of model-theoretic consequence.

Looking in more detail at the notion of truth in a model-theoretic structure and realizing that it is explained with reference to individual constants as denoting objects in the considered domain, and to predicates as denoting $n$-ary relations over that domain, we might claim that *denotation* is even more fundamental: We use the concept of denotation to explain truth, and we use the concept of truth to explain logical consequence. However, truth can be alternatively explained in terms of valuations of atomic sentences, leading to a substitutional interpretation of quantifiers. Nonetheless, even in such a 'nominalistic' approach, truth is the fundamental building stone of consequence and therefore the basis of the justification of inference.

## 2. Proof-theoretic consequence

There has always been a different view, according to which *inference* is the basic concept on which semantics should be based. In the philosophy of language this is the central *tenet* of the 'meaning-as-use'-appraoch that emanated especially from the philosophy of the later Wittgenstein (although it is by no means tied to 'Wittgensteinianism'). In modern philosophy it has become an ingredient of Brandom's inferentialism (Brandom, 2000). An inferential approach cannot be based on derivability in a formal system, since there are no *a priori* grounds as to which formal system to choose. Thus proof-theoretic approaches to logical consequence do not invert the relationship between formal derivability and semantic entailment. This relationship remains as it was, with correctness and completeness being desirable features of formal systems. They rather craft the semantic concept of consequence in terms of proofs, where proofs are no longer understood exclusively as formal derivations, but as defining the meaning of logical constants and providing evidence for assertions. As examples we consider the Brouwer-Heyting-Kolmogorov (BHK) explanation of the logical constants and proof-theoretic semantics in the Dummett-Prawitz tradition.

### *2.1. BHK semantics*

We present a simplified picture of a complex and by no means uniform field of ideas (see (Troelstra & Dalen, 1988)). The BHK explanation of the meaning of the logical constants is given in terms of constructions or proofs. Given a notion of proof for atomic formulas, which provides the *atomic base* of the definition,

- a proof of $A \wedge B$ is a pair consisting of a proof of $A$ and a proof of $B$,
- a proof of $A \vee B$ is a pair $(0, a)$, where $a$ is a proof of $A$, or a pair $(1, a)$, where $a$ is a proof of $B$,
- a proof of $A \rightarrow B$ is a construction which converts each proof of $A$ into a proof of $B$,
- a proof of $\forall x A(x)$ is a construction which for each object (number) $n$ constructs a proof of $A(n)$,
- a proof of $\exists x A(x)$ is a pair $(n, a)$, where $a$ is a proof of $A(n)$,
- nothing is a proof of $\perp$.

A formula $A$ is then valid (with respect to an atomic base), if a proof can be given for it, and $A$ follows logically from $B$, if $B \rightarrow A$ is valid for any atomic base. Here, as usual in constructive theories, we only consider finite sets of premisses, which can be represented by a conjunction of formulas. The difference to the model-theoretic notion of consequence is at least threefold:

1. Instead of classical structures, atomic bases consisting of *proofs* are considered. (This may be related to a classical valuation base for substitutional quantification.)
2. The meaning explanation of the logical constants is *constructivist*. Especially the meaning of implication and universal quantification deviates from the truth-conditional perspective by using, at the metalinguistic level, the notion of a constructive procedure ('construction') which generates a proof either from another proof (in the case of implication) or from a term (in the case of universal quantification). (If this is interpreted using recursive functions, it leads to notions of *recursive realizability*.)
3. Logical consequence refers to this notion of a *constructive procedure*, whereas in the model-theoretic case, just the classical 'if... then' is employed in the metalanguage.

Nonetheless, however important the difference between the classical and the constructivist approach may be, this is only a *partial* change of perspective. As in the classical case we are still working with an *abstract* notion of structure without any *real inference* being involved. The constructions considered are entities built up by certain operations such as pairing and function abstraction, which form the context with respect to which formulas are evaluated, quite in analogy to the model-theoretic notion of truth. They may at best be viewed as *proof objects*, i.e., entities that verify propositions. Proponents of constructive semantics are aware of this, as the following quotation from (Kreisel, 1962) shows:

> '... we give a formal semantic foundation for intuitionistic formal systems in terms of the abstract theory of constructions. This is analogous to the semantic foundation for classical systems [reference to Tarski] in terms of abstract set theory.' (198f.)

With respect to formal derivations, this sort of semantics leads to the same questions as classical semantics, especially to the question of completeness (which is a subtle point in constructive semantics, see (Artёmov, 2001) and the references therein).

### *2.2. Proof-theoretic semantics in the Dummett-Prawitz tradition*

The approach pursued by Dummett and Prawitz (Dummett, 1991; Prawitz, 1973, 2006) in what they call 'theory of meaning' (we prefer the term 'proof-theoretic semantics', as it seems to us to capture exactly what is intended [see (Kahle & Schroeder-Heister, 2006)]) promises to be explicitly inferentialist as it refers to basic inferences as defining the meaning of logical constants. Following Gentzen's claim that the introduction inferences in natural deduction may be viewed as definitions, and the elimination inferences as a sort of consequences thereof (Gentzen, 1934/35), they consider introduction rules for logical constants as basic meaning giving inferences which are 'self-justifying', whereas all other inferences are justified as valid by reference to them. This is achieved by philosophically re-interpreting and generalizing certain proof-theoretic results, which were originally developed in the context of theories of (weak and strong) normalization. The proof-theoretic *result* that a closed proof reduces to a proof in introduction form is interpreted as a philosophical *condition* for a proof to be valid (called the 'fundamental assumption' by Dummett). A closed proof in introduction form would then be a direct proof, whereas a closed proof not in introduction form would be a proof by indirect means which is justified if it can be reduced (transformed) to a direct proof. This yields a taxonomy of direct and indirect proofs following the philosophical idea that a proposition can be either verified directly, or established indirectly by relying on certain transformation procedures. Technically this means that we must distinguish between a proof structure $\mathcal{D}$, which is a tree-like arrangement of propositions which looks like a proof but is not generated by specific rules, and a justification $\mathcal{J}$, which is a *proof reduction system* in the sense that $\mathcal{J}$ can be applied to proof structures yielding new proof structures. A proof structure $\mathcal{D}$ is then valid, i.e., represents a proof, if it either uses definitional means to derive its conclusion (introduction rules) or can be reduced to this form using the justification $\mathcal{J}$.

This leads to a definition of *validity of proofs*, which in the propositional case runs as follows, where $\mathcal{J}$ is a justification of the indicated kind, $S$ is an *atomic base* of proofs of atomic formulas (which may be given by an atomic production system), a 'canonical' proof is a proof structure using an introduction rule in the last step, and an 'open'/'closed' proof is a proof depending/not depending on assumptions, respectively.

**Definition (Validity of proofs with respect to $\mathcal{J}$ and $S$, in short $\langle\mathcal{J}, S\rangle$-validity).**

(i) *Every closed proof in $S$ is $\langle\mathcal{J}, S\rangle$-valid.*

(ii) *A* closed canonical proof *is $\langle\mathcal{J}, S\rangle$-valid, if its* immediate subproofs *are $\langle\mathcal{J}, S\rangle$-valid.*

(iii) *A* closed noncanonical proof *is $\langle\mathcal{J}, S\rangle$-valid, if it* reduces by means of $\mathcal{J}$ *to a $\langle\mathcal{J}, S\rangle$-valid canonical proof.*

(iv) *An* open proof $\begin{array}{c} A_1 \ldots A_n \\ \mathcal{D} \\ B \end{array}$ *is $\langle\mathcal{J}, S\rangle$-valid, if for every list of closed $\langle\mathcal{J}, S\rangle$-valid proofs* $\left(\begin{array}{c}\mathcal{D}_1 \\ A_1\end{array}, \ldots, \begin{array}{c}\mathcal{D}_n \\ A_n\end{array}\right)$*, the proof* $\begin{array}{c} \mathcal{D}_1 \qquad \mathcal{D}_n \\ A_1 \ldots A_n \\ \mathcal{D} \\ B \end{array}$ *is $\langle\mathcal{J}, S\rangle$-valid.*

Now a notion of consequence with respect to $\mathcal{J}$ and $S$ can be defined as follows:

$A_1, \ldots, A_n \vDash_{\langle \mathcal{J}, S \rangle} B$ *holds if there is an open proof* $\begin{matrix} A_1 \ldots A_n \\ \mathcal{D} \\ B \end{matrix}$ *such that for all* $S$ *and for each list of valid closed proofs* $\left( \begin{matrix} \mathcal{D}_1 \\ A_1 \end{matrix}, \ldots, \begin{matrix} \mathcal{D}_n \\ A_n \end{matrix} \right)$*,*

*the proof* $\begin{matrix} \mathcal{D}_1 \quad\quad \mathcal{D}_n \\ A_1 \ldots A_n \\ \mathcal{D} \\ B \end{matrix}$ *is* $\langle \mathcal{J}, S \rangle$*-valid.*

We obtain *logical* consequence $A_1, \ldots, A_n \vDash B$, if *there is* a $\mathcal{J}$ such that the relation $A_1, \ldots, A_n \vDash_{\langle \mathcal{J}, S \rangle} B$ holds *for every* $S$. (For further details see (Schroeder-Heister, 2006).)

At first sight, this looks like a genuine improvement over the BHK way of defining validity, as there now seems to be a relationship to actual inference. We are dealing with real proof structures and transformations on them. However, looking more carefully at this notion, we realize that it does not go beyond constructive semantics in the BHK tradition. This is due to what may be called the *trivialization problem*. Every proof structure

$$\begin{matrix} A_1 \ldots A_n \\ \mathcal{D} \\ B \end{matrix}$$

which is valid with respect to some justification $\mathcal{J}$, can be replaced with the one-step proof structure

$$\frac{A_1 \quad \ldots \quad A_n}{A}$$

which is valid with respect to a justification $\mathcal{J}'$, if $\mathcal{J}'$ is defined in such a way that for this one-step proof, $\mathcal{J}'$ generates exactly the result which $\mathcal{J}$ generates for $\mathcal{D}$. This means that every inferential content present in the proof structure $\mathcal{D}$ can be put into the justification considered, so that for (logical) consequence only one-step proofs need to be taken into account. Moreover, the proof reduction that operates on the premiss proofs $\begin{matrix} \mathcal{D}_i \\ A_i \end{matrix}$ is not necessarily a reduction in the 'natural' sense, which would construct a new proof structure by rearranging parts of the given proof structures, but simply a function that produces a proof of the conclusion without necessarily referring to the premiss proofs. Suppose there is a proof $\frac{\mathcal{D}}{A}$ which is valid with respect to $\mathcal{J}$, then any proof $\frac{\mathcal{D}'}{A}$ is valid with respect

to $\mathcal{J}'$, where $\mathcal{J}'$ just replaces $\mathcal{D}$ with $\mathcal{D}'$ and is otherwise like $\mathcal{J}$. So, in principle, a reduction may 'invent' an appropriate proof structure. This trivializes the idea of a proof *reduction* system. What remains is just the notion of a (constructive) procedure which delivers a proof structure of the conclusion given one of the premisses, but not '*constructing*' one from those in any natural sense of the word. A justification $\mathcal{J}$ is a sort of abstract 'realizer' closely related to the structures considered in BHK semantics. If by $\langle \mathcal{J}, S\rangle \vDash A$ we denote that $\mathcal{J}$ generates a valid closed proof of $A$ (with respect to $S$), then the proof-theoretic notion of logical consequence amounts to the following:
$A_1, \ldots, A_n \vDash B$ *iff there is a* $\mathcal{J}$ *such that for every* $S$ *and all* $\mathcal{J}_1, \ldots, \mathcal{J}_n$ *the following holds:*

$$\textit{If} \quad \langle \mathcal{J}_1, S\rangle \vDash A_1, \ldots, \langle \mathcal{J}_n, S\rangle \vDash A_n, \quad \textit{then} \quad \langle \mathcal{J}, S\rangle \vDash B.$$

From this point of view, proof-theoretic semantics in the Dummett-Prawitz tradition is nothing but a variant of BHK semantics. (It should be noted that Prawitz (Prawitz, 1985) is fully aware of the trivialization problem and the problem of demarcating his proof-theoretic semantics from constructive semantics in general.)

## 3. The dogma of standard semantics

In spite of the fundamental differences between the model-theoretic and the constructive and proof-theoretic approaches to logical consequence, they have two ideas in common:

1. the assumption that a categorical concept is primary to the hypothetical concept of consequence: in the classical case this is the notion of truth in a model-theoretic structure, while in the proof-theoretic case this is validity with respect to a construction or justification;

2. the transformational view of consequence: in the classical case this is the transmission of truth in a structure, while in the constructive or proof-theoretic case this is the transmission of validity from the premisses to the conclusion of an inference.

We call these (interrelated) assumptions the *dogma of standard semantics*, as it underlies both standard model-theoretic and standard proof-theoretic semantics (Schroeder-Heister & Contu, 2005). If we denote constructive or proof-theoretic structures by $\mathfrak{C}, \ldots$ and validity with respect to such a structure by $\mathfrak{C} \vDash A$, then constructive and proof-theoretic consequence is modelled as

$$A_1, \ldots, A_n \vDash A \;\Leftrightarrow_{df}$$
$$(\forall \mathfrak{C})[(\mathfrak{C}_1 \vDash A_1) \;\&\; \ldots \;\&\; (\mathfrak{C}_n \models A_n) \;\Rightarrow\; f(\mathfrak{C}_1, \ldots, \mathfrak{C}_n) \vDash B]$$

where $f$ is a constructive transformation generating a structure that validates the conclusion from structures that validate the premisses. Although there are important differences to the classical picture, not only in the way constructive structures are defined, but especially in the form of the transformation $f$, we would like to emphasize the unifying feature which is the definition of hypothetical consequence by means of the transmission of a categorical concept of validity. This is not bad in itself. However, the fixation on consequence as truth-transmission blocks the way towards a concept which is really based on inference and deserves the name 'inferential semantics'. Along with this fixation a particular view of deduction is associated, namely the emphasis on forward reasoning. If special emphasis is put on introduction rules, then the way by means of which a conclusion is established is highlighted, not the assumptions from which they are obtained. In Dummett-Prawitz-style proof-theoretic semantics closed (= assumption-free) proofs are primary and assumptions are considered to be *placeholders* for closed proofs. An inference *from* $A$ is valid if the proof obtained by replacing $A$ with a closed proof of $A$ is valid. This is a striking asymmetry: Whereas conclusions are distinguished according to their logical form and specific introduction rules are given to introduce them, assumptions are nothing but open places.

This view is intimately related to the choice of natural deduction as the basic model of reasoning. To natural deduction the bias towards forward reasoning is inherent. There are attempts at dualizing proof-theoretic semantics by taking elimination rules as a starting point. However, apart from the fact that these approaches have not been sufficiently worked out so far (they are problematic in the case of 'indirect' elimination rules such as those for disjunction or existential quantification), they would finally arrive at a dual problem, with conclusions (rather than assumptions) not being given appropriate attention.

## 4. Way out: Definitional reasoning in sequent style

If we want to give up the dogma of standard semantics and with it the placeholder view of assumptions and the primacy of closed over open proofs, we have to choose a different model of reasoning. Fortunately, such a model is at hand with Gentzen's sequent calculus. Philosophically interpreted, the sequent calculus with its symmetric treatment of the left and right hand sides overcomes the fixation towards forward reasoning. An assertion of a sequent $\Gamma \vdash A$ may be viewed as an assertion of a proposition $A$ with respect to assumptions $\Gamma$, so from its very beginning it is built on the parity of assumptions and assertion. The sequent calculus is often viewed as a metacalculus of natural deduction. This is, however, a misleading characterization, as the rules operating on the left side have no direct analogue in natural deduction. One might, of course, change the concept of natural deduction in such a way that it gains basic features of the sequent calculus (which might then be called sequent calculus in natural-deduction style). This would lead to a system in which major premisses of elimination rules only occur in top position. It would be very much in the spirit of our enterprise here, but it must be clear that this is not natural deduction in the standard sense (Schroeder-Heister, 2004).

So our idea is to consider reasoning to be something that starts with simple consequence statements $A \vdash A$ and then refines such statements either to the left or to the right side by means of certain inference principles. Therefore, in a sense, our proposal towards a proper proof-theoretic semantics is to proclaim the idea of direct access to the consequence relation (in the form of a sequent) from the very beginning. At this point it should be mentioned that cut

$$\frac{\Gamma \vdash A \quad A, \Delta \vdash B}{\Gamma, \Delta \vdash B}$$

is not a rule that must be eliminable at any price, or perhaps even a rule that one has to acknowledge as primitive. It is intimately connected to the placeholder view of assumptions which corresponds to cuts of the form

$$\frac{\vdash A_1 \quad \ldots \quad \vdash A_n \quad A_1, \ldots, A_n \vdash B}{\vdash B}\ .$$

Now where do the sequent-style inference rules come from? Our idea is to consider the content of reasoning to be given by an *external definition* and the appropriate reasoning rules to be based on it. Inspired by logic programming, we assume that such a definition is a list of clauses of the form

$$\mathcal{D}\begin{cases} a_1 \Leftarrow B_{11} \\ \vdots \\ a_1 \Leftarrow B_{1m_1} \\ \vdots \\ a_n \Leftarrow B_{n1} \\ \vdots \\ a_n \Leftarrow B_{nm_n} \end{cases}$$

where the $a_i$ are atomic formulas ('atoms') and the $B_{ij}$ are lists of atoms. As such a system of clauses is nothing but an inductive definition, we may view our approach as a theory of inductive definitions. If we allow $B_{ij}$ to contain implications, we enter the area of nomonotonic inductive definitions. In the general case, the $B_{ij}$ may be arbitrary first-order formulas.

These definitions are then put into action by certain inference rules, notably right introduction and left introduction rules for formulas being defined. The whole approach is called 'definitional reasoning', with the right and left rules being called 'definitional closure' and 'definitional reflection', respectively. (This terminology as well as the basic idea of definitional reflection is due to Hallnäs (Hallnäs, 1991, 2006).) *Definitional closure* is reasoning along the definitional clauses according to the principle

$$\frac{\Gamma \vdash B_{ij}}{\Gamma \vdash a_i}(\vdash a_i)$$

whereas *definitional reflection* proceeds according to the schema

$$\frac{\Gamma, B_{i1} \vdash C \quad \dots \quad \Gamma, B_{im_i} \vdash C}{\Gamma, a_i \vdash C}(a_i \vdash)$$

This rule says that everything that follows from all defining conditions of $a_i$, follows from $a_i$ itself. (Here we are just considering the propositional case without individual variables.) It interprets the extremal clause sometimes given in inductive definitions: 'There is no further clause defining $a_i$'. It is called 'definitional reflection' as it involves a step of reflection upon the definition as a whole. Obviously, it is non-monotonic in the sense that extending the definition may alter the rule (since premisses may have to be added), whereas definitional closure is monotone in the sense that adding a definitional clause leads to further derivation rules, but leaves the existing rules intact (for a further discussion see (Schroeder-Heister, 1993)).

So our general picture of proof-theoretic consequence is that of a philosophically interpreted sequent system which describes the reasoning with respect to a given definition. Reasoning is two-sided from the very beginning: It affects both assertions and assumptions in the form of closure and reflection rules, respectively.

## 5. Further topics

We mention a few points indicating in which direction these ideas can be extended:

*Reasoning with individual variables.* If we consider definitional systems in which individual variables may occur (which is indispensable for significant applications), we are lead to principles of definitional reflection of various strengths. This leads into the area of inversion principles for rule-based systems, a field initiated by Lorenzen in the 1950s (Schroeder-Heister, 2007).

*Functions and functionals.* More technical applications of the idea of definitional reflection result in general principles for the definition of recursive functions and definitions of functions of higher types (functionals) (Hallnäs, 2006).

*Computational interpretation.* Using appropriate principles for the reasoning with variables, we may develop systems wich answer hypothetical queries of the form

$$(?\theta)\ \Gamma\theta \vdash A\theta$$

for a given sequence $\Gamma \vdash A$ by computing bindings for variables. This gives rise to logic programming systems with hypothetical queries and definitional reflection (Hallnäs & Schroeder-Heister, 1990/91).

*Substructural issues.* As our approach is based on the sequent calculus, questions of structuring assumptions are a natural topic. We may distinguish between different ways of associating the premisses of definitional clauses, yielding substructurally different rules of definitional closure and definitional reflection (Schroeder-Heister, 1991).

*Assumption and denial.* The idea of definitional reflection is not confined to the reasoning with assumptions. It has a natural interpretation when we consider ‘direct’ negation in the form of a denial operator. If explicit denial clauses are allowed to occur in the definitional base along with assertion clauses, we may use definitional reflection to express that denying all defining conditions of an assertion leads to a denial, and denying all defining conditions of a denial leads to an assertion. This results in systems with various forms of negation and might even be used in extensions of logic programming (Schroeder-Heister, 2008).

## 6. Conclusion

Our proof-theoretic notion of consequence, which does not depend on notions like 'truth' or 'construction' or 'validity', is based on a change of perspective: We do not primarily reason *towards a conclusion*, nor do we primarily reason *from certain premisses*, but always focus on the *full consequence relation*. We assert something while at the same time assuming something, and, in a step of reasoning, can extend the consequence statement we have already established either in the direction of a new assertion or in the direction of a new assumption, both with respect to a given definition.

Peter Schroeder-Heister
Universität Tübingen
Wilhelm-Schickard-Institut für Informatik
Sand 13
727 06 Tübingen, Germany
psh@informatik.uni-tuebingen.de
http://www-ls.informatik.uni-tuebingen.de/psh

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