Mathematical Methods in Economics and Business

Winter Semester 2020/2021

Preparatory Course – Part 1

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General Course Information

This preparatory course for the lecture „Mathematical Methods in Economics and Business“ is adressed to first-semester students of the economics and business study programs. It represents a repetition of mathematical basics which can be expected from students after reaching their university entrance qualification. The course aims to build a common basis for students with different mathematical background knowledge in order to facilitate the start of their studies. Since the contents consist of repetition of the Abitur material, attending the tutorials is not mandatory. Understanding the lecture is also possible without the preparatory course.

General Course Information

As supplementary literature to the slides the following book is recommended.

> Sydsæter, K., Hammond, P.: *Mathematik für Wirtschaftswissenschaftler,* 2011 bzw. 2013, Pearson
> Lehrbuchsammlung UB, Signatur wiwi Q 104 Auflage 3 bzw. Auflage 4

Alternatively,

> Cramer, E., Nešlehova, J.: *Vorkurs Mathematik,* 2008, Springer
> elektronische Ressource im Uni-Netzwerk http://dx.doi.org/10.1007/978-3-642-01833-6

can be used which includes a large collection of exercises and solutions.

Contents Part 1

The following topics are covered:

1. Set Theory
2. Types of Numbers
3. Rules of Algebra
4. Powers and Roots
5. Inequalities
6. Intervals and Absolute Values
7. Quadratic Equations
8. Propositional Logic

Elementary Basics of Mathematics: Set Theory

1.1 Set Theory

Starting point of the consideration is the notion of a **set** of objects.

Examples

The following descriptions define sets of objects:

- Students of all universities in Germany,
- Types of fish in German waters,
- Number of car accidents in Tübingen during a certain period.

More abstract examples of sets are real and natural numbers.
In the following, the notion of a set is specified in greater detail.

1.1 Set Theory

Definition: *Set, Element*

*A set is a collection of objects viewed as a whole. The objects are called **elements** or **members** of a set.*

Sets are usually represented by capital letters (e.g. A) and their elements by small letters (e.g. a1, a2, a3).
In order to describe a set a rule is needed which defines its elements unambiguously. The different possibilities are:

- Enumeration of elements: The elements of the set are listed and displayed in curly brackets. Each element is listed **exactly once**.
  $M = \{a, b, c, d\}$
- Specification of properties indicating whether an element belongs to the set or not: $M = \{x \mid x \text{ has the property } E\}$

  Alternative notation: $M = \{x : x \text{ has the property } E\}$

1.1 Set Theory

- **Universal Set** Ω: All elements of a certain consideration.
- **Empty Set** $\{\}$ (alternative symbol: $\emptyset$) has no elements.
- **Equality of Sets:** Two sets are considered equal ($A = B$), if each element of A is an element of B and each element of B is simultaneously an element of A.

  Note: The order in which the elements are listed has no significance!

  $$\{a, b, c, d\} = \{c, a, d, b\}$$

- **Subset:** A is a subset of B if it is true that every member of A is also a member of B $\quad (A \subset B)$.
  For all sets it holds: $\{\} \subset A, \qquad A \subset A$.
- **Complement:** $\overline{M} = \{x \mid x \notin M \ \wedge x \in \Omega\}$ is the complement of $M \subset \Omega$. *Alternative notation:* M^c.

  Note: $\overline{\overline{M}} = M, \qquad \overline{\emptyset} = \Omega, \qquad \overline{\Omega} = \emptyset$.

1.1 Set Theory

Consider a universal set Ω with subsets A, B and C.
Set Operations:

- **Intersection**: $A \cap B = \{x \mid x \in A \land x \in B\}$
  The elements that belong to both, A and B.

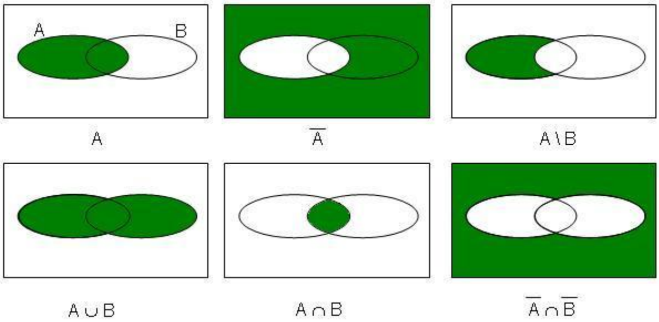
  Disjoint Sets: If two sets A and B have no elements in common, i.e. if their intersection is empty ($A \cap B = \emptyset$), they are said to be disjoint.

- **Union:** $A \cup B = \{x \mid x \in A \lor x \in B\}$
  The elements that belong to at least one of the sets A and B.

- **Difference:** $A \setminus B = \{x \mid x \in A \land x \notin B\}$
  Elements that belong to A, but not to B.

1.1 Set Theory

Graphical representation of sets and set operations using **Venn diagrams**:



1.1 Set Theory

- Identity Laws:

$$A \cup \emptyset = A, \quad A \cap \emptyset = \emptyset, \quad A \cup \Omega = \Omega, \quad A \cap \Omega = A.$$

- Commutative Laws:

$$A \cup B = B \cup A, \quad A \cap B = B \cap A.$$

- Associative Laws:

$$A \cup (B \cup C) = (A \cup B) \cup C, \quad A \cap (B \cap C) = (A \cap B) \cap C.$$

In general, however, this does not apply $A \cup (B \cap C) = (A \cup B) \cap C$!

- Distributive Laws:

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C), \quad A \cap (B \cup C) = (A \cap B) \cup (A \cap C).$$

- De Morgan's Laws:

$$\overline{A \cap B} = \overline{A} \cup \overline{B}, \quad \overline{A \cup B} = \overline{A} \cap \overline{B}.$$

Elementary Basics of Mathematics: Types of Numbers

1.2 Types of Numbers

Natural Numbers $\mathbb{N}$: $\mathbb{N} = \{1, 2, 3, 4, \dots\}$. Sometimes 0 is included as well, in this case some textbooks write $\mathbb{N}_0$. Adding and multiplying numbers from $\mathbb{N}$ again results in an element of $\mathbb{N}$.

Integers $\mathbb{Z}$: Expand natural numbers by negative integers, i.e. $\mathbb{Z} = \{\dots, -3, -2, -1, 0, 1, 2, 3, \dots\}$. Adding, multiplying and subtracting numbers from $\mathbb{Z}$ again results in elements of $\mathbb{Z}$.

Rational Numbers $\mathbb{Q}$: *Rational numbers* include the quantity of all quotients of two integers, with the restriction that the denominator cannot be 0. Rational numbers are repeating or terminating decimal numbers. With the extension to rational numbers all four basic arithmetic operations can be performed.

Irrational Numbers: All nonperiodic, nonterminating decimal fractions are called *irrational numbers*. For example $\sqrt{2}$ (see digression).

1.2 Types of Numbers

Real Numbers $\mathbb{R}$ **:** *Rational* and *irrational* numbers together are called *real numbers*.

An unbroken and endless straight line with an origin and a positive unit of length is a satisfactory model for the real numbers. We frequently state that there is a *one-to-one correspondence* between the real numbers and the points on the number line. Often, too, one speaks of the „real line“ rather than the „number line“.

The set of rational numbers as well as the set of irrational numbers are said to be „dense“ on the number line. This means that between any two different real numbers, irrespective of how close they are to each other, we can always find both, a rational and an irrational number - in fact, we can always find infinitely many of each.

When applied to the real numbers, the four basic arithmetic operations always result in a real number. The expectation is that we cannot divide by 0 (see Sydsaeter *et.al.* S. 24).

Digression: The Pythagoreer and the Irrationality of $\sqrt{2}$.

The stories about the community of Pythagoreans reach far into the past and are accordingly very uncertain. Even the biographical information about the founder of the community, Pythagoras of Samos (c. 540-500 BC), son of a gemcutter, are sparse. He left Samos in the reign of Polycrates, received most of his education in Ancient Egypt and Mesopotamia and settled in Croton, southern Italy, around 525 BC. There he founded a secrete order that for some time had considerable influence.

After the triumph of the democratic party over the aristocracy, the order was persecuted and disappeared at the beginning of the 4th century BC. But there was a revival in the 1st century BC, giving rise to Neopythagoreanism. Most of the information and legends about the original alliance of the Pythagoreans stems from that time.

The bond of the pythagoreans is characterized by typical characteristics of a religious community: conspiracy, strict rules about food, clothes, funeral ceremonies, trial period for newcomers and the teaching of the transmigration of souls.

Digression: The Pythagoreer and the Irrationality of $\sqrt{2}$.

But what made the Pythagoreans stand out from many similar mystery cults and relevant for the history of science and mathematics, is the fact that the union with the divine should be attainable by the immersion in the wonderful laws of the world of numbers. For them, the essence of the world was the harmony of numbers. The turn to mathematics, astronomy and music was in fact a rational by-product of the main religious interest.
Within the Pythagorean school, however, they discovered that some quantities cannot be expressed rationally. This discovery is traced back to Hippasos of Metapont (c. 450 BC): there are distances that do not measure each other (they are incommensurable). This is the case when the length of the distances cannot be measured as the integer multiple of the length of a third distance, which is understood to be the standard route.

In modern terms: there are routs whose ratio of their lengths is an irrational number.

Digression: The Pythagoreer and the Irrationality of $\sqrt{2}$.

The discovery of Hippasos destroyed the basic understanding of „arithmetica universalis“ (so-called crisis of the Greek mathematics) and thus the foundations of the Pythagorean view of the world.
The legend says that Hippasos made this catastrophic discovery public and thus violated the obligation of secrecy. When Hippasos was killed in a shipwreck, this was - as the Neopythagorean Iamblichos felt - a necessary and just punishment and the metaphysical reason for the fact that . . . „*everything unspoken and invisible loves to hide. But when a soul encounters such a form of life and makes it accessible and manifest, it is transported into the sea of becoming and washed around by the iniquitous floods.*“ (Becker/Hofmann 1951, p. 57).

Digression: The Pythagoreer and the Irrationality of $\sqrt{2}$.

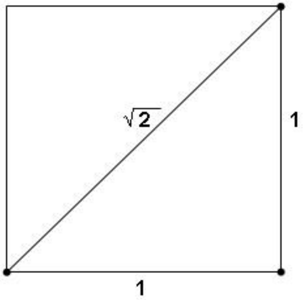


Abbildung: $\sqrt{2}$ as geometrical length.

Digression: The Pythagoreer and the Irrationality of $\sqrt{2}$.

The discovery of incommensurability may have been made, according to more recent research results, by alternation on the pentagram (see figure below) and not on the square, where the diagonal is incommensurable to the square side. The collapse of the „arithmetica universalis“ had far-reaching consequences for the further development of Greek mathematics and thus for the course of Western mathematics as a whole: Greek mathematics took on a specific form known as geometric algebra.

Since, for example, $\sqrt{2}$ can be represented geometrically as a diagonal of a square (see illustration on p. 18), but cannot be represented as a number according to its definition at that time (neither as an integer nor as a ratio of integers), the treatment of irrational and algebraic problems was shifted to geometry, i.e. algebraic operations were performed in the form of geometric constructions whose existence was assured. (Wußing, 6000 Jahre Mathematik, 2008).

Abbildung: The pentagram (Source: Wikipedia).

Digression: The Pythagoreer and the Irrationality of $\sqrt{2}$.

Proof of Irrationality of $\sqrt{2}$:
The statement that $\sqrt{2}$ is irrational means that there are no numbers $p, q \in \mathbb{Z}$ with the property $\frac{p}{q} = \sqrt{2}$.
Proof by Opposition: It is assumed that

$$\frac{p}{q} = \sqrt{2}.$$

Further it is assumed that p and q have no common factors. This assumption is not restrictive as the common factor can always be truncated and the resulting numbers can be set as p and q. Then it holds

$$\left(\frac{p}{q}\right)^2 = 2$$

$$p^2 = 2q^2$$

Digression: The Pythagoreer and the Irrationality of $\sqrt{2}$.

p^2 thus is even, since it's dividible by 2. Further, p is even, otherwise p^2 would be odd. From p being even it follows $p = 2r, r \in \mathbb{Z}$. Thus, it holds

$$(2r)^2 = 2q^2$$

$$4r^2 = 2q^2$$

$$2r^2 = q^2.$$

The contradiction is obvious. From the considerations above it follows that q must be even. However, this means that p and q have a common factor.

Elementary Basics of Mathematics: Rules of Algebra

1.3 Rules of Algebra

The most common Rules of Algebra

If a, b and c are arbitrary numbers, then:

(a) $a + b = b + a$

(b) $(a + b) + c = a + (b + c)$

(c) $a + 0 = a$

(d) $a + (-a) = 0$

(e) $ab = ba$

(f) $(ab)c = a(bc)$

(g) $1 \cdot a = a$

(h) $aa^{-1} = 1$ für $a \neq 0$

(i) $(-a)b = a(-b) = -ab$

(j) $(-a)(-b) = ab$

(k) $a(b + c) = ab + ac$

(l) $(a + b)c = ac + bc$

1.3 Rules of Algebra

Definition: *Algebraic Expressions*

Expressions involving letters such as $3xy - 5x^2y^3 + 2xy + 6y^3x^2 - 3x + 5yx + 8$ are called *algebraic expressions*. We call

- $3xy; -5x^2y^3; 2xy : 6y^3x^2; -3x; 5yx; 8$ *terms* and
- $3, -5, 2, 6, -3, 5, 8$ *numerical coefficients*.

Note: *Terms of the same type* where only the numerical coefficients are different, can – and should – be collected:

$$3xy - 5x^2y^3 + 2xy + 6y^3x^2 - 3x + 5yx + 8 = x^2y^3 + 10xy - 3x + 8$$

1.3 Rules of Algebra

Factoring of Algebraic Expressions

The terms of algebraic expressions can often be factored, for example:

- $6y^3x^2 = 2 \cdot 3 \cdot y \cdot y \cdot y \cdot x \cdot x$
- $15x^4 = 3 \cdot 5 \cdot x \cdot x \cdot x \cdot x$

To factor an expression means to express it as a *product* of simpler factors:

$$9x^2 - 25y^2 = (3x - 5y) \cdot (3x + 5y)$$

1.3 Rules of Algebra

Fractions: Basics

For $b, c, d \neq 0$ it holds that:

1. $\frac{a\cdot\cancel{c}}{b\cdot\cancel{c}} = \frac{a}{b}$
2. $\frac{-a}{-b} = \frac{a\cdot(-1)}{b\cdot(-1)} = \frac{a}{b}$
3. $-\frac{a}{b} = \frac{-a}{b}$
4. $\frac{a}{c} + \frac{b}{c} = \frac{a+b}{c}$
5. $\frac{a}{b} + \frac{c}{d} = \frac{a\cdot d + c\cdot b}{b\cdot d}$
6. $a + \frac{b}{c} = \frac{a\cdot c + b}{c}$
7. $a \cdot \frac{b}{c} = \frac{a\cdot b}{c}$
8. $\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \cdot \frac{d}{c} = \frac{a\cdot d}{b\cdot c}$

Wrong:

- $\frac{5\cancel{a}+3b}{\cancel{a}b} = \frac{5+3\cancel{b}}{\cancel{b}} = \frac{5+3}{1} = 8$
- $\frac{a}{a^2+2a} = \frac{a}{a^2} + \frac{a}{2a} = \frac{1}{a} + \frac{1}{2}$

Elementary Basics of Mathematics: Powers and Roots

1.4 Powers and Roots

Definition: *Integer Powers*

If $a \in \mathbb{R} \setminus \{0\}$ (**base**) and $n \in \mathbb{N}$ (**exponent**) then: $a^n = \underbrace{a \cdot a \cdot \ldots \cdot a}_{\text{n factors}}$.

- We define further: $a^0 \equiv 1$
- We do not assign a numerical value to 0^0.
- For powers with negative exponent we define: $a^{-n} = \frac{1}{a^n}$.

1.4 Powers and Roots

Calculation Rules of Powers:

(i) $a^r a^s = a^{r+s}$

(ii) $(a^r)^s = a^{rs}$

(iii) $\dfrac{a^r}{a^s} = a^{r-s}$

(iv) $a^r b^r = (ab)^r$

(v) $\dfrac{a^r}{b^r} = \left(\dfrac{a}{b}\right)^r$

1.4 Powers and Roots

Remarks:

The above mentioned rules can be extended to cases where there are several factors.

Note: With different bases and different exponents, no power rule is applicable!

For any a, b, r and s it holds that:

$$a^r \pm a^s \neq a^{r+s} \qquad a^r \pm b^r \neq (a \pm b)^r$$

$$a^r b^s \neq (ab)^{r+s} \qquad \frac{a^r}{b^s} \neq \left(\frac{a}{b}\right)^{r+s}$$

If a is any real number and n is any natural number, then a^n is called the nth power of a; here a is the base, and n is the exponent.

1.4 Powers and Roots

Example: Compound Interest

We define the following variables: Deposit K and interest rate $p\%$ per unit of time (e.g. one year).

After t time units the original investment of K will have grown to an amount

$$K\left(1+\frac{p}{100}\right)^t .$$

where $1+\dfrac{p}{100}$ is called the **growth factor**.

The **present value** of an investment can be obtained by $K\left(1+\dfrac{p}{100}\right)^{-t}$.

1.4 Powers and Roots

Definition: *Fractional Powers*

$\sqrt[n]{a} := a^{1/n}\,,\ a \geq 0\,,\ n \in \mathbb{N}$. a is called radical and n root exponent.

The special case for $n = 2$ denotes the **square root** of a.
For $a \in \mathbb{R}^+$, $n \in \mathbb{N}$ and $m \in \mathbb{Z}$ it holds that: $\sqrt[n]{a^m} := a^{\frac{m}{n}}$. In general, for any $a, b > 0$, $n, m \in \mathbb{N}$ and $p, q \in \mathbb{Q}$ the following calculation rules apply.

Calcuation Rules for Fractional Powers:

$$\sqrt[n]{a^p} \cdot \sqrt[m]{a^q} = a^{\frac{pm+qn}{mn}} = \sqrt[nm]{a^{pm+qn}} \qquad \sqrt[n]{a^p} : \sqrt[m]{a^q} = \sqrt[nm]{a^{pm-qn}}$$

$$\sqrt[m]{(\sqrt[n]{a^p})^q} = \sqrt[nm]{a^{pq}} \qquad \sqrt[n]{a^p} \cdot \sqrt[n]{b^p} = \sqrt[n]{(ab)^p}$$

$$\frac{\sqrt[n]{a^p}}{\sqrt[n]{b^p}} = \sqrt[n]{\left(\frac{a}{b}\right)^p}$$

1.4 Powers and Roots

Remarks:

- The most common error when calculating with roots is:
  $\sqrt{a+b} = \sqrt{a} + \sqrt{b}$. This is **wrong!**
- $(-2)^2 = 2^2 = 4$ thus both $x = 2$ and $x = -2$ are solutions of the **quadratic equation** $x^2 = 4$.
  Therefore we have $x^2 = 4$ if and only if: $x = \pm\sqrt{4} = \pm 2$.
- Note, however, that the symbol $\sqrt{4}$ means only 2, not -2.
- For $n \in \mathbb{N}$ being an **odd number** it is possible to sensibly define $a^{\frac{m}{n}}$ for $a < 0$.
  For example: $(-8)^{1/3} = \sqrt[3]{-8} = -2$.

Elementary Basics of Mathematics: Inequalities

1.5 Inequalities

- Definition: *Order of real numbers*: If a and b are real numbers it holds that: $a = b$ or $a < b$ or $a > b$.
- Graphical illustration using number lines.
- We call $>$ or $<$ **strict inequalities**.
- Whereas $\leq$ or $\geq$ are **weak inequalities**.

1.5 Inequalitites

Important Calculation Rules:

- Addition or substraction of $d \in \mathbb{R}$:
  $a > b \quad \Leftrightarrow \quad a \pm d > b \pm d$
- Multiplication or division with **positive** $d \quad (d \in \mathbb{R}^+)$:
  $a > b \quad \Leftrightarrow \quad a \cdot d > b \cdot d$ bzw. $a/d > b/d$.
- Multiplication or division with **negative** $d \quad (d \in \mathbb{R}^-)$:
  $a > b \quad \Leftrightarrow \quad a \cdot d < b \cdot d$ bzw. $a/d < b/d$
  (Inversion of the order!)

1.5 Inequalities

- The calculation rules are also valid for the case of weak inequalities.
- The solutions of an inequality are summarized in a **solution set** $\mathbb{L}$.
- Two inequalities, which should be valid at the same time, are called **double inequality**.

  Example: *If the following should apply simultaneously:* $a \leq z$ *and* $z < b$*, then one can write compactly:* $a \leq z < b$.
- The solution of an inequality often requires the performance of a **case distinction**.
- A structured help for this is provided by the so-called **sign diagram**. See Sydsæter and Hammond (2.A., 3.A.), Section 1.6.
- Application of inequalities especially in curve sketching and linear optimization.

Elementary Basics of Mathematics: Intervals and Absolute Values

1.6 Intervals and Absolute Values

Definition: *Intervals*

The set of numbers $x \in \mathbb{R}$, which satisfy an inequality $a \leq x \leq b$, is called an interval.

- Intervals can be represented as complete sections of the number line.
- We distinguish between different **types of intervals**:

Bounded intervals:

$$\{x|a \leq x \leq b\} \quad =: \quad [a,b] \quad \text{closed interval}$$

$$\left.\begin{array}{lllll} \{x|a < x \leq b\} & =: &]a,b] & \text{or} & (a,b] \\ \{x|a \leq x < b\} & =: & [a,b[& \text{or} & [a,b) \end{array}\right\} \quad \text{half-open intervals}$$

$$\{x|a < x < b\} \quad =: \quad]a,b[\quad \text{or} \quad (a,b) \quad \text{open interval}$$

Unbounded intervals:

$$\begin{array}{lllll} \{x|a \leq x\} & =: & [a,+\infty[& \text{or} & [a,\infty) \\ \{x|x < b\} & =: &]-\infty,b) & \text{or} & (\infty,b) \end{array}$$

1.6 Intervals and Absolute Values

Definition: *Absolute Values*

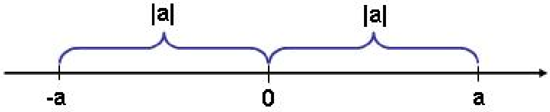
$$|a| = \begin{cases} a & \text{if} \quad a > 0 \\ 0 & \text{if} \quad a = 0 \\ -a & \text{if} \quad a < 0 \end{cases}$$

Some Calculation Rules for Absolute Values:

$|-a| = |a|$ $\qquad$ $|a \cdot b| = |a| \cdot |b|$

$\left|\frac{a}{b}\right| = \frac{|a|}{|b|}$ $\qquad$ $|a + b| \leq |a| + |b|$ $\quad$ (triangle inequality $\star$)

Visually, the absolute value is the distance between a and 0 on the number line:



1.6 Intervals and Absolute Values

($\star$) About the triangle inequality:
This harmless looking inequality is of great importance. That it really concerns a triangle is shown in the following.
Given are three real numbers a, b and c.

$$\begin{aligned} |a-b| &= |a-b+0| \\ &= |a-b+c-c| \\ &= |(a-c)+(c-b)| \\ &\leq |a-c|+|c-b| \text{ due to the triangle inequality.} \end{aligned}$$

It is therefore valid

$$|a-b| \leq |a-c|+|c-b| \tag{1}$$

As mentioned above, the expression $|a-b|$ is equal to the distance between the points a and b on the number line. Now let a, b and c be points in the plane. Then inequality (1) makes the plausible statement that the distance from a to b is less than or equal the distance from a to c plus the distance from c to b. Geometrically, it is a triangle.

Elementary Basics of Mathematics: Quadratic Equations

1.7 Quadratic Equations

- Solving equations plays an important role in economics.
- The principle of **equivalent transformation** always applies: The set of solutions remains unchanged if on both sides the same arithmetic operation is performed with the same number. Not allowed is the multiplication with zero and the generally forbidden division by zero.
- We distinguish between a large number of equation types.
- Hints regarding the solution of linear equations with an unknown can be found for example in Sydsæter and Hammond (2.A., 3.A.), Section 2.1.
- The solution of **quadratic equations** is of particular importance.

1.7 Quadratic Equations

- The solution for the **general form** of a quadratic equation

  $ax^2 + bx + c = 0$ is $$x_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

- The solution of the **normal form** of a quadratic equation

  $x^2 + px + q = 0$ is $$x_{1,2} = -\frac{p}{2} \pm \sqrt{\frac{p^2}{4} - q}$$

- A quadratic equation has *no, one* or *two real solutions* if the expression $b^2 - 4ac$ or $p^2/4 - q$ (*discriminant*) is *smaller, equal* or *bigger* than zero.

- A quadratic equation has no real solution if the discriminant is negative. But it has two (conjugate) complex solutions in the range of complex numbers. *See the digression below.*

- The solutions of a (quadratic) equation are often called **roots** of the equation.

1.7 Quadratic Equations

Vieta's Formulas:

If x_1 and x_2 are the solutions of the quadratic formula $ax^2 + bx + c = 0$, then it holds:

$$-a(x_1 + x_2) = b \quad \text{and} \quad ax_1x_2 = c \ .$$

For quadratic equations in the normal form $(x^2 + px + q = 0)$ it applies accordingly:

$$-(x_1 + x_2) = p \quad \text{and} \quad x_1x_2 = q \ .$$

Thus, quadratic equations $ax^2 + bx + c = 0$ can also be written in the form $a(x - x_1)(x - x_2) = 0$. Analogously, you can write quadratic equations in the normal form $(x^2 + px + q = 0)$ and also as $(x - x_1)(x - x_2) = 0$

Digression: Complex Numbers

So far, we have solutions for $x^2 = a$ with $a > 0$, which are located on the real number line.

For $x^2 = -1$ there exists no real solution. Therefore, it is defined:

$$i^2 = -1 \quad or \quad i = \sqrt{-1}.$$

Definition: *Complex Numbers*

$$\begin{aligned} Z &= a + bi && a, b \in \mathbb{R} \\ a &= Re(Z) && \text{real part of } Z \\ b &= Im(Z) && \text{imaginary part of } Z. \end{aligned}$$

For $Z = a + bi$ is called $\bar{Z} = a - bi$ <u>complex conjugate</u>.

Literature: Chiang and Wainwright (2005), Sec. 16.2 or Opitz, Section 1.6

Digression: Equation Systems

- Regularly in economic sciences, it is necessary to solve equation systems for various unknowns.
- If all unknowns occur in linear form in the equations, we speak of **linear equation systems**.
- In a later chapter, efficient solution methods are discussed for such equations systems.
- Here: simple procedures from school mathematics:
  - Substitution method
  - Addition/substraction method

    $\longrightarrow$ Reference to the Gaussian elimination method!
  - Equation method.

Elementary Basics of Mathematics: Propositional Logic

1.8 Propositional Logic

Logic is the doctrine of logically consistent thinking, i.e. drawing correct conclusions based on given propositions. Its aim is the development of a simple formalism to convert (colloquial) statements into an accessible form for mathematics.

The instruments of propositional logic are (also) necessary for mathematical reasoning. In the following slides, the basic notations of binary (bivalent) proposition logic are demonstrated.

The central concept of mathematical propositional logic is the proposition:

Definition: Proposition

A proposition (A) is a statement, which is either true (t) or false (f).

1.8 Propositional Logic

Therefore:

- For a proposition there are no other truth values allowed except from t and f: law of excluded middle.
- A proposition does not get the values t and f simultaneously: law of non-contradiction.

Alternative notation: a proposition is true / valid / fulfilled.
Or: a proposition is not true / not valid / not fulfilled.

Definition: Negation

$\overline{A}$ (negation of the proposition A) is true if A is false, and $\overline{A}$ is false, if A is true.

Alternative notation: $\neg A$.

1.8 Propositional Logic

Besides the consideration of elementary propositions, also linked propositions can be considered. Thereby, conjunction and disjunction of propositions matter.

Definition: Conjunction

The conjunction $A \wedge B$ is true if both, A and B are true.
$A \wedge B$ is false if at least one of the two propositions is false (logical AND).

Definition: Disjunction

The disjunction $A \vee B$ is true if at least one of the two propositions is true. $A \vee B$ is false if both propositions are false. (This corresponds to the colloquial: either A or B or both; inclusive-or).

1.8 Propositional Logic

In mathematical logic, conclusions of the truth value of a proposition to that of another proposition are of particular importance.

Definition: Implication

The implication or conclusion $A \Rightarrow B$ is false if A is true and B is false. Otherwise, $A \Rightarrow B$ is true.
The proposition A is called **implication** or **premise**.

The proposition B is called **conclusion**.

A is a sufficient condition for B, B is a necessary condition for A.

Definition: Equivalence

If $A \Rightarrow B$ and $B \Rightarrow A$ are valid, A and B are called equivalent ($A \Leftrightarrow B$).

1.8 Propositional Logic

Definition: *Deductive* and *Inductive* Reasoning

Deductive reasoning is based on consistent rules of logic, whereas *inductiven* reasoning is build upon only a few (or even many) observations.

Conclusions based upon inductive reasoning can be never absolute certain and therefore, inductive reasoning is not recognized as a form of proof in mathematics!

1.8 Propositional Logic

Mathematical Induction

Let $A(n)$ be a proposition for all natural numbers n and it holds:

(a) $A(1)$ is true.

(b) If the induction hypothesis $A(k)$ is true, then $A(k+1)$ is also true (for every natural number k).

With this procedure one can show by means of *mathematical induction* that $A(n)$ is true for all natural numbers n.