1. Mathematical Techniques of Time Series Analysis

Necessary Techniques:

Complex Numbers, the Unit circle, working with Difference- and Lag-Operators, Solutions of stochastic difference equations

Unit circle: Working with complex numbers.

In order to solve stochastic difference equations it is necessary to know the calculation rules of complex numbers.

Basics:

The algebraic equation in x

$$x^2 - 2ax + \left(a^2 + b^2\right) = 0$$

e.g. has the formal solution

$$x = a \pm b\sqrt{-1}$$

But those solutions are only defined for $b = 0$ (for the set of real numbers).

Solution:

Definition of the set **C** of complex numbers as a superset of **R**

Requirements of **C**

(1) The sum (the product) of real numbers as elements of **C** is equal to the sum (the product) defined for real numbers.

(2) The set **C** contains an element with the property $i^2 = -1$.

(3) For each element z of **C** there exist two real numbers a, b, so that the complex number z can be written as $z = a + ib$. Here, a is called the **real part** of Z and b the **imaginary part** of z.

We render this definition more precisely by defining the 2x2 Matrices:

$$a := \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}, a \in R$$

$$i := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

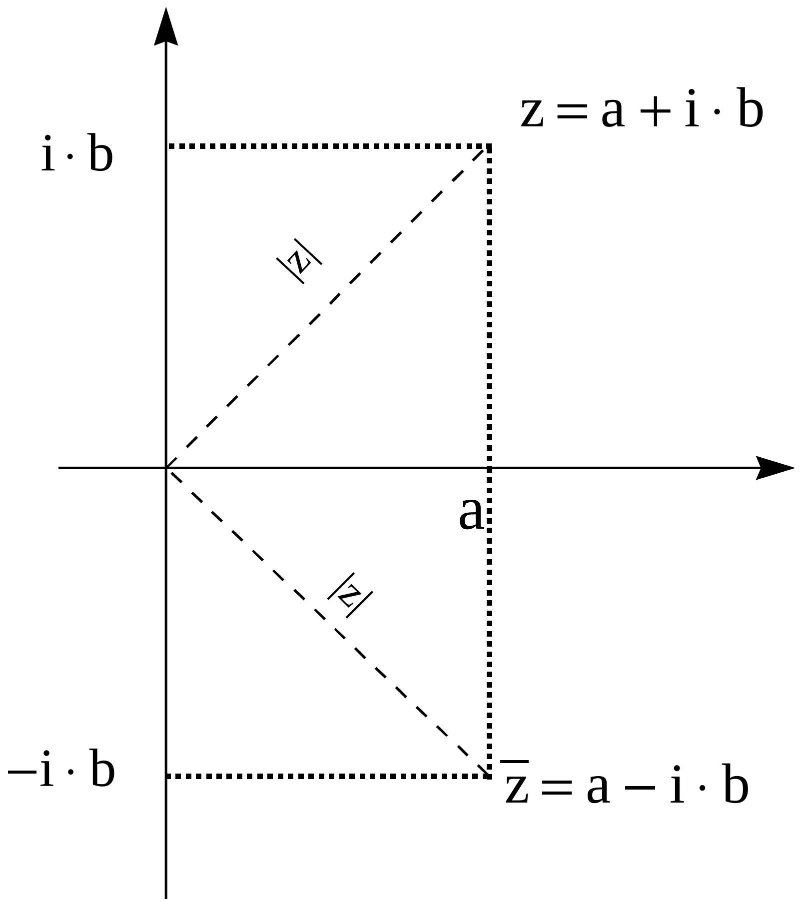
We define the complex number $a + bi$ as

$$a + bi := \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} + \begin{pmatrix} 0 & b \\ -b & 0 \end{pmatrix} = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}, a, b \in R$$

The set of 2X2 Matrices together with basic matrix algebra (addition and multiplication) represents a model for complex numbers. The complex number $z = a + ib$ is called imaginary, if $a = 0$ and $b \neq 0$ and real, if $b = 0$. The complex number $\bar{z} = a - ib$ is the complex conjugate of $z = a + ib$.

Example: The equation $x^2 + c = 0 \quad (c > 0)$ has as solutions the imaginary numbers $z_1 = i\sqrt{c}$ and $z_2 = -i\sqrt{c}$, since $z_1^2 = z_2^2 = -c$. The numbers z_1 and z_2 are complex conjugates.

For Illustration of complex numbers the **complex plane** is used:



The horizontal axis depicts the real numbers while on the vertical axis we find the imaginary numbers. Each point in the plane represents exactly one complex number.

Die real number $|z| = \sqrt{a^2 + b^2}$ is the **absolute value** of $z = a + i \cdot b$

$|z|$ is the distance to the point of origin.

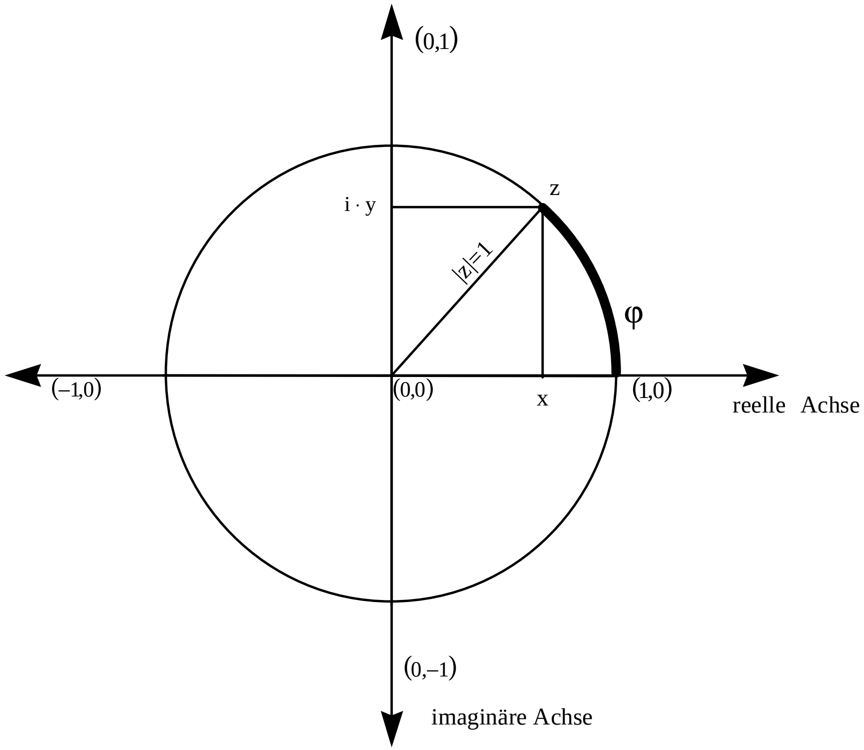
It is therefore identical to the usual absolute value of real numbers.

Important results:

$$(a + i \cdot b) + (c + i \cdot d) = (a + c) + i(b + d)$$
$$(a + i \cdot b) - (c + i \cdot d) = (a - c) + i(b - d)$$
$$(a + i \cdot b) \cdot (c + i \cdot d) = (a \cdot c - b \cdot d + i(a \cdot d + b \cdot c))$$

Trigonometric Representation of complex numbers

A complex number $z = x + iy$ with an absolute value of 1 satisfies $x^2 + y^2 = 1$. We say z lies on the unit circle in the complex plane.



The circumference of the unit circle is 2π. The arc length from (1,0) to (0,1), (-1,0),(0,-1) equals $\frac{\pi}{2}, \pi, \frac{3\pi}{2}$.

φ is the arc length from (1,0) to z

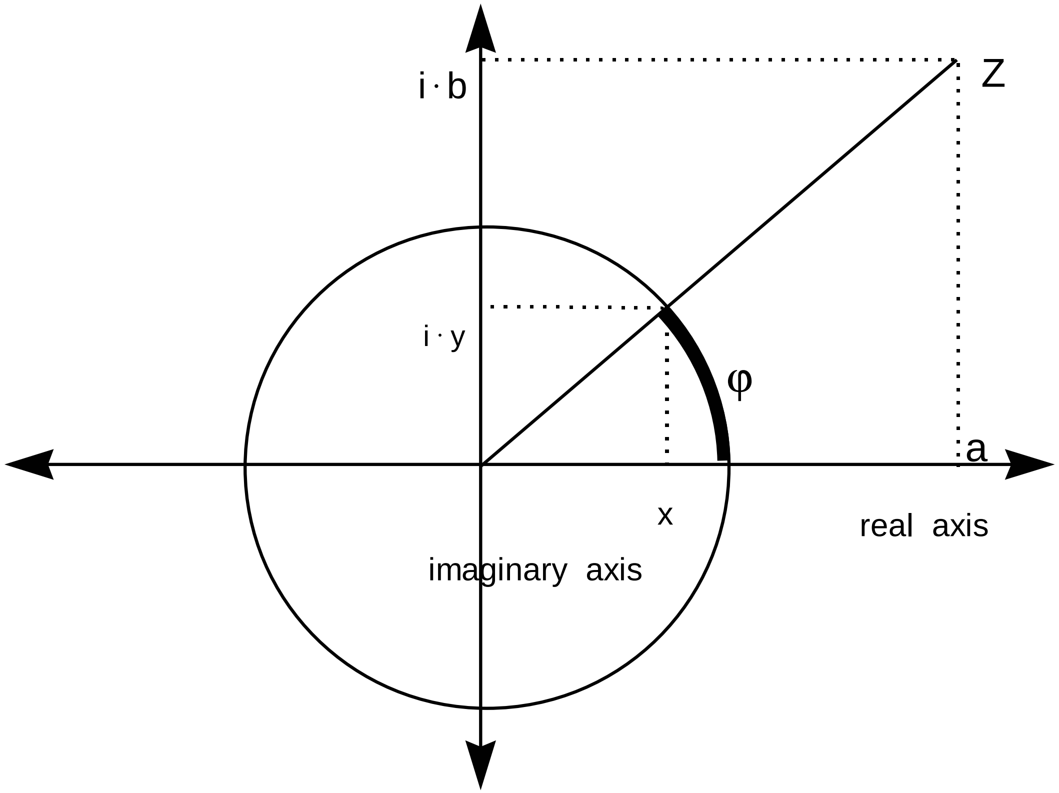
$$\cos(\varphi) := x$$
$$\sin(\varphi) := y \quad if \;\; y \neq 0$$
$$\tan(\varphi) := \frac{y}{x} \quad if \;\; x \neq 0$$

If the complex number z lies on the unit circle it can be represented as:

$$z = \cos(\varphi) + i \cdot \sin(\varphi)$$

Any complex number has an absolute value of $R = \sqrt{a^2 + b^2}$,

it can be represented as $z = R(x + i \cdot y)$ with $x = \frac{a}{R}$, $y = \frac{b}{R}$ and (x,y) on the unit circle.



Hence, z has the trigonometric form $z = R \cdot \left(\cos(\varphi) + i \cdot \sin(\varphi)\right)$ $\Rightarrow$ Polarcoordinate representation of z

Theorem of de Moivre: For each complex number $z \neq 0$ and each rational number q :
$z^q = R^q\left[\cos(q \cdot \varphi) + i \cdot \sin(q \cdot \varphi)\right]$

Exponential representation of complex numbers

$$e^x = 1 + x + \frac{x^2}{2!} + \ldots\ldots \frac{x^5}{5!} + \ldots \text{ (power series)}$$

with $x = i \cdot \varphi$ we can write, using $i^2 = -1, i^3 = -i, i^4 = 1, i^5 = i$

$$e^{i\varphi} = 1 + i \cdot \varphi - \frac{\varphi^2}{2!} - i\frac{\varphi^3}{3!} + i\frac{\varphi^4}{4!} + i\frac{\varphi^5}{5!} - \frac{\varphi^6}{6!} - i\frac{\varphi 7}{7!} \ldots$$

$$= \underbrace{\left(1 - \frac{\varphi^2}{2!} + \frac{\varphi^4}{4!} - \frac{\varphi^6}{6!} + \ldots\right)}_{\text{Potenzreihe cosinus}} + i\underbrace{\left(\varphi - \frac{\varphi^3}{3!} + \frac{\varphi^5}{5!} - \frac{\varphi 7}{7!} + \ldots\right)}_{\text{Potenzreihe sinus}}$$

$$= \cos\varphi + i \cdot \sin\varphi$$

The representation of a complex number $z = a + i \cdot b$ according to $z = R \cdot e^{i \cdot \varphi}$ with $R = |z|, \tan(\varphi) = \frac{b}{a}$ is called its Exponential form.

2. Difference equations

(The presentation follows Hamilton (1994), chapter 1)

Difference equations of 1st order

Dynamic properties of

$$y_t = \phi y_{t-1} + w_t \tag{1}$$

w_t can be a random variable. Then: (stochastic difference equation of 1st order)

Example: money demand equation Goldfeld (1973) for the USA

m_j (log real money demand) as a function of log GDP (real) I_t, the logarithm of the interest rate on deposits r_{Gt} and the interest rate for bonds r_{Ct}.

$$m_t = 0,27 + 0,72 m_{t-1} + 0,19 I_t - 0,045 r_{Gt} - 0,019 r_{Ct} \tag{2}$$

This is a special case of equation 1 with

$$w_t = 0{,}27 + 0{,}19 I_t - 0{,}045 r_{Gt} - 0{,}019 r_{Ct}$$

$$y_t = m_t$$

$$\phi = 0{,}72$$

Objective: understanding the dynamic behavior of y if w changes.

Period	Equation
0	$y_0 = \phi y_{-1} + w_0$
1	$y_1 = \phi y_0 + w_1$
2	
.	....
t	$y_t = \phi y_{t-1} + w_t$

$\Rightarrow$ If the starting value y_{-1} for $t = -1$ and w_t for 0,1,...,t is known, the sequence of y_t can be calculated via recursive substitution:

$$y_t = \phi^{t+1} y_{-1} + \phi^t w_0 + \phi^{t-1} w_1 + \phi^{t-2} w_2 + \ldots + \phi w_{t-1} + w_t \qquad (3)$$

Dynamic behavior

If w_0 changes and $w_1 \ldots w_t$ are unaffected, the effect on y_t: $\frac{\partial y_t}{\partial w_0} = \phi^t$

Dynamic multiplier = (impulse response function)

How strong the effect of the dynamic multiplier is depends on the time span from 0 - t and the parameter ϕ.

If the dynamic simulation begins in t:

$$y_{t+j} = \phi^{j+1} y_{t-1} + \phi^j w_t + \phi^{j-1} w_{t+1} \ldots + w_{t+j}\,.$$

The size and the sign of ϕ determine the sequence of the dynamic multipliers.

The effect of w_t on y_{t+j} is given by

$$\frac{\partial y_{t+j}}{\partial w_t} = \phi^j$$

Thus, the dynamic multiplier depends only on j, the time span between w_t and y_{t+j}.

Possible dynamics: exponential increase $(\phi > 1)$, a geometric decay ($0 < \phi < 1$), an oscillating decay ($-1 < \phi < 0$), an explosive oscillating behavior ($\phi < -1$)

Difference equations of higher order

As a generalization consider a difference equation of order p

$$y_t = \phi_1 y_{t-1} + \phi_2 y_{t-2} + \ldots + \phi_p y_{t-p} + w_t \qquad (4)$$

Objective: Explaining the dynamic behavior of (4)

First, we transfer the difference equation of order p in a vector difference equation of order 1. Therefore, we need the following notation:

$$\xi_t \equiv \begin{bmatrix} y_t \\ y_{t-1} \\ \vdots \\ y_{t-p+1} \end{bmatrix} \qquad (p \times 1) - \text{Vektor}$$

$$F \equiv \begin{bmatrix} \phi_1 & \phi_2 & \cdots & \phi_{p-1} & \phi_p \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix} \qquad (pxp)\text{ - Matrix}$$

$$v_t = \begin{bmatrix} w_t \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad (p\,x\,1) - \text{Vektor}$$

For $p = 1$ (difference equation of order 1) the matrix F becomes $F = \phi$ (scalar)

Now, we can define the vector difference equation of order 1:

$$\xi_t = F\xi_{t-1} + v_t \tag{5}$$

Recursion, as in the case of difference equations of order 1:

For period 0: $\xi_0 = F\xi_{-1} + v_0$

For period 1: $\xi_1 = F\xi_0 + v_1 = F\left(F\xi_{-1} + v_0\right) + v_1 = F^2\xi_{-1} + Fv_0 + v_1$

For period t: $\xi_t = F^{t+1}\xi_{-1} + F^t v_0 + F^{t-1}v_1 \cdots Fv_{t-1} + v_t$ (6)

Of special importance concerning the dynamics of the system: 1. row of the system given in (6) for period t:

Define $f_{11}^{(t)}$: as the (1,1) element of F^t, $f_{12}^{(t)}$ as the (1,2) element of F^t.

For the **first row** of $\xi_t = \ldots$ we can write

$$y_t = f_{11}^{(t+1)} y_{-1} + f_{12}^{(t+1)} y_{-2} + \ldots + f_{1p}^{(t+1)} y_{-p} + f_{11}^{(t)} w_0 + f_{11}^{(t-1)} w_1 + \ldots + f_{11}^{(1)} w_{t-1} + w_t \qquad (7)$$

$\Rightarrow y_t$ is a function of **p initial values** of y and the **complete History** of w.

Beginning the dynamic simulation in t:

$$\xi_{t+j} = F^{j+1}\xi_{t-1} + F^j v_t + F^{j-1} v_{t+1} + \ldots + F v_{t+j-1} + v_{t+j} \qquad (8)$$

For a difference equation of order p the impulse-response-function is $\frac{\partial y_{t+j}}{\partial w_t} = f_{11}^{(j)}$ (9)

For j=1 this is the (1,1) element of F, or the parameter ϕ_1 !

For each system of order p the effect of a one unit increase of w_t on y_{t+1} is given by:

$$\frac{\partial y_{t+1}}{\partial w_t} = \phi_1$$

Via direct multiplication of the matrix F we obtain F^2:

$$\frac{\partial y_{t+2}}{\partial w_t} = \phi_1^2 + \phi_2$$

This is the (1,1) element of F^2. In order to describe the dynamic behavior of difference equations of higher order analytically (when is the system explosive?) we can analyze the **eigenvalues of the matrix F**.

⇒ **Matrix-Algebra** (see e.g. Greene (1993))

Eigenvalues (characteristic roots) of the matrix F are the solutions λ of:

$$\left| F - \lambda \cdot I_p \right| = 0$$

with I_p as the identity matrix of order p. For a system of difference equatons of order 2 , i.e. $p = 2$

$$\left| \begin{bmatrix} \phi_1 & \phi_2 \\ 1 & 0 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \right| = \begin{vmatrix} (\phi_1 - \lambda) & \phi_2 \\ 1 & -\lambda \end{vmatrix} = \lambda^2 - \phi_1 \lambda - \phi_2 = 0$$

⇒ characteristic equation

The two eigenvalues are then:

$$\lambda_1 = \frac{\phi_1 + \sqrt{\phi_1^2 + 4\phi_2}}{2} \text{ and } \lambda_2 = \frac{\phi_1 - \sqrt{\phi_1^2 + 4\phi_2}}{2}$$

⇒ Eigenvalues can be **complex numbers**

For difference equations of order p a general result is, that the eigenvalues of F can be obtained as a solution of the charakteristic equation

$$\lambda^{p} - \phi_1 \cdot \lambda^{p-1} - \phi_2 \cdot \lambda^{p-2}, \ldots \phi_{p-1} \cdot \lambda - \phi_p = 0$$

Proposition from matrix algebra (see e.g. Hamilton (1994), S. 729-731)

If the eigenvalues of a (p x p) -Matrix F are distinct, there exists a non-singular matrix T, so that

$$F = T\Lambda T^{-1}$$

where Λ is a (p x p)-diagonal matrix with the eigenvalues of F as the diagonal elements:

$$\Lambda = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & \cdots & \lambda_p \end{bmatrix}$$

We can write:

$$F^2 = T\Lambda T^{-1} \cdot T\Lambda T^{-1} = T\Lambda^2 T^{-1}$$

Because of the diagonal structure of Λ we can write:

$$\Lambda^2 = \begin{bmatrix} \lambda_1^2 & 0 & \cdots & 0 \\ 0 & \lambda_2^2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & \cdots & \lambda_p^2 \end{bmatrix}$$

In general, $F^j = T\Lambda^j T^{-1}$ (10). The diagonal structure of Λ^j is retained:

$$\Lambda^j = \begin{bmatrix} \lambda_1^j & 0 & \cdots & 0 \\ 0 & \lambda_2^j & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & & \lambda_p^j \end{bmatrix}$$

Denote with t_{ij} the element of row i, column j of T and t^{ij} the element of row i, column j of T^{-1}, then we can obtain through simple matrix multiplication the (1,1) element of F^j :

$$f_{11}^{(j)} = \left[t_{11} \cdot t^{11}\right] \cdot \lambda_1^j + \left[t_{12} \cdot t^{21}\right] \cdot \lambda_2^j + \ldots + \left[t_{1p} \cdot t^{p1}\right] \cdot \lambda_p^j = c_1 \cdot \lambda_1^j + c_2 \cdot \lambda_2^j + \ldots + c_p \cdot \lambda_p^j$$

$$c_i = \left[t_{1i} \cdot t^{i1}\right]$$

(To show this, write (10) extensively)

$c_1 + c_2 + \ldots + c_p$ is the (1,1) element of $T \cdot T^{-1} = I_p$, so that $c_1 + c_2 + \ldots + c_p = 1$

Plugging this into (9) leads to:

$$\frac{\partial y_{t+j}}{\partial w_t} = c_1 \cdot \lambda_1^j + c_2 \cdot \lambda_2^j + \ldots + c_p \cdot \lambda_p^j$$

The impulse-response-function of order j is a weighted average of the p eigenvalues raised to the jth power.

For p=1 the charakteristic equation is

$$\lambda_1 - \phi_1 = 0 \Rightarrow \lambda_1 = \phi_1$$

It follows for the dynamic multiplier:

$$\frac{\partial y_{t+j}}{\partial w_t} = c_1 \cdot \lambda_1^j = \phi_1^j$$ da $c_1 = 1$ (see above)

If at least one eigenvalue of F has an absolute value > 1, the system is **explosive**, since:

The eigenvalue with the largest absolute value dominates the dynamic multiplier in an **exponential function**. For real eigenvalues with an absolute value <1 the dynamic multiplier converges either geometrically or oscillating towards zero.

(Caculate the dynamic multiplier of the equation $y_t = 0{,}6y_{t-1} + 0.2y_{t-2} + w_t$)

Complex eigenvalues for p=2:

Eigenvalues of F are complex, if $\phi_1^2 + 4 \cdot \phi_2 < 0$. Writing the solutions of the characteristic polynom as **complex numbers:**

$$\lambda_1 = a + bi$$
$$\lambda_2 = a - bi$$

with $a = \frac{\phi_1}{2}, b = 0.5\sqrt{-\phi_1^2 - 4 \cdot \phi_2}$

To illustrate the dynamics of the system of difference equations, we write the eigenvalues in their polar coordinate representation:

$$\lambda_1 = \mathrm{R}\left[\cos(\varphi) + \mathrm{i}\cdot\sin(\varphi)\right]$$

$$\mathrm{R} = \sqrt{\mathrm{a}^2 + \mathrm{b}^2}\ ,\ \cos(\varphi) = \frac{a}{R}, \sin(\varphi) = \frac{b}{R}$$

Or in their exponential representation:

$$\lambda_1 = \mathrm{R}\left[\mathrm{e}^{\mathrm{i}\varphi}\right]$$

$$\lambda_1^{\mathrm{j}} = \mathrm{R}^{\mathrm{j}}\left[\mathrm{e}^{\mathrm{i}\varphi\mathrm{j}}\right] = \mathrm{R}^{\mathrm{j}}\left[\cos(\varphi\mathrm{j}) + \mathrm{i}\sin(\varphi\mathrm{j})\right]$$

For the complex conjugate of λ_1, which is λ_2 we can write:

$$\lambda_2^{\mathrm{j}} = \mathrm{R}^{\mathrm{j}}\cdot\left[\mathrm{e}^{-\mathrm{i}\varphi\mathrm{j}}\right] = \mathrm{R}^{\mathrm{j}}\cdot\left[\cos(\varphi\mathrm{j}) - \mathrm{i}\cdot\sin(\varphi\mathrm{j})\right]$$

Substitution results in:

$$\frac{\partial y_{t+j}}{\partial w_t} = c_1 \cdot \lambda_1^j + c_2 \cdot \lambda_2^j = c_1 \cdot R^j \cdot \left[\cos(\varphi j) + i \cdot \sin(\varphi j)\right] + c_2 \cdot R^j \cdot \left[\cos(\varphi j) - i \cdot \sin(\varphi j)\right]$$

$$= \left[c_1 + c_2\right] \cdot R^j \cdot \cos(\varphi j) + i \cdot \left[c_1 - c_2\right] \cdot R^j \cdot \sin(\varphi j)$$

It can be shown that c_1, c_2 are also complex conjugates (For a prove: see Hamilton (1994) p. 15):

$$c_1 = \alpha + \beta \cdot i,$$
$$c_2 = \alpha - \beta \cdot i$$

Plugging this result in we obtain real multipliers:

$$c_1 \cdot \lambda_1^j + c_2 \lambda_2^j = 2 \cdot \alpha \cdot R^j \cdot \cos(\varphi j) - 2 \cdot \beta \cdot R^j \cdot \sin(\varphi j)$$

⇒ If the modulus of the eigenvalues is larger than 1 the system explodes at the rate R^j. For R=1 (the eigenvalues lie on the unit circle) the mulpliers are periodic sine and cosine functions of j. Only for R<1 („the eigenvalues lie inside the unit circle“) the amplitude of the multipliers decays with the rate R^j.

Because of the enormous importance of difference equation systems of order 2 we present the stationarity triangle of Sargent (1981). For an easy derivation, see Hamilton (1994) p. 17f.)