Abstracts

Photon sphere uniqueness and the static n-body problem

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Generalizing a phenomenon well-know in Schwarzschild and other spherically symmetric spacetimes, we give a geometric definition of *photon spheres* in static asymptotically flat spacetimes [2]. Photon spheres are relevant in the analysis of black hole stability and in gravitational lensing. We then use this definition to prove that static vacuum asymptotically flat spacetimes possessing a single [2] or multiple photon spheres – together with Gregory J. Galloway [3] – must be isometric to the Schwarzschild spacetime in the exterior region of the photon sphere. In particular, multiple photon spheres cannot occur in the same static vacuum asympt. flat spacetime.

The two methods used in these two approaches can be extended to the electrostatic electro-vacuum setting, which has been done by Yazadjiev and Lazov [5] for a single and in joint work with Gregory J. Galloway [4] for multiple photon spheres, respectively. Here, the unique electro-vacuum asymptotically flat spacetime possessing an electrically charged photon sphere is the Reissner-Nordström spacetime, which is again spherically symmetric.

The uniqueness proofs in [2, 5] adapt and generalize the single (electro-)static black hole uniqueness proofs going back to Israel and will not be further discussed here. The proofs in [3, 4] modify and generalize arguments given for static black hole uniqueness by Bunting and Masood-ul-Alam [1] in the vacuum and by Masood-ul-Alam [**?**] in the electro-vacuum case, see below.

The uniqueness results for multiple photon spheres [3, 4] can easily be extended to include additional non-degenerate Killing black hole horizons. They can be re-interpreted as saying that there are no (electro-)static configurations of $k \in \mathbb{N}$ black holes and $n \in \mathbb{N}$ 'very compact' bodies with $k + n > 1$. Here, a body is considered 'very compact' if it is surrounded by a photon sphere; a property that astrophysicists expect to hold for suitably compact bodies.

In the following, we will restrict our attention to the non-charged case for simplicity of the exposition. We define a *photon sphere* $\mathfrak{P}^3$ in a (standard) static spacetime $(\mathbb{R} \times M^3, -N^2 dt^2 + g)$ to be a timelike umbilic hypersurface on which the static *lapse function* N – the length of the static Killing vector field – is constant. Here, umbilicity captures that any null geodesic initially tangent to $\mathfrak{P}^3$ is tangent to $\mathfrak{P}^3$ throughout. Constancy of N ensures that every null geodesic tangent to $\mathfrak{P}^3$ has constant potential energy $\log N$, or, equivalently, that its energy E – and color/frequency ν – as observed by the static observers is constant. The latter property is essential to characterize photon spheres as will be shown in joint work with Gregory J. Galloway elsewhere.

From this definition and the vacuum Einstein equations, we then derive quasi-local geometric properties of a photon sphere in a static vacuum spacetime:

Proposition 1 (Cederbaum [2])**.** *Let $(\mathbb{R}\times M^3, -N^2dt^2+g)$ be a static vacuum spacetime and let $(\mathfrak{P}^3,p)\hookrightarrow(\mathbb{R}\times M^3,-N^2dt^2+g)$ be a photon sphere. Write*

$$(1)\qquad \left(\mathfrak{P}^3,p\right)=\left(\mathbb{R}\times\Sigma^2,-N^2dt^2+\sigma\right)=\bigcup_{i=1}^{I}\left(\mathbb{R}\times\Sigma_i^2,-N_i^2dt^2+\sigma_i\right),$$

where each $\mathfrak{P}_i^3=\mathbb{R}\times\Sigma_i^2$ is a connected component of $\mathfrak{P}^3$. Then the embedding $(\Sigma^2,\sigma)\hookrightarrow(M^3,g)$ is totally umbilic with constant mean curvature H_i on the component Σ_i^2. The scalar curvature of the component (Σ_i^2,σ_i), ${}^{\sigma_i}\mathrm{R}$, is a non-negative constant, namely ${}^{\sigma_i}\mathrm{R}=\frac{3}{2}H_i^2$. Moreover, the normal derivative of the lapse function N in direction of the outward unit normal ν to Σ^2, $\nu(N)$, is also constant on every component (Σ_i^2,σ_i), $\nu(N)_i:=\nu(N)|_{\Sigma_i^2}$. For each $i\in\{1,\dots,I\}$, either $H_i=0$ and Σ_i^2 is a totally geodesic flat torus or Σ_i^2 is an intrinsically and extrinsically round CMC sphere for which the above constants are related via

$$(2)\qquad N_iH_i=2\nu(N)_i,\quad (r_iH_i)^2=\frac{4}{3},$$

where $r_i:=\sqrt{\frac{|\Sigma_i^2|_{\sigma_i}}{4\pi}}$ denotes the area radius *of Σ_i^2.*

Using Proposition 1, we obtain the following theorem:

Theorem 1 (Cederbaum–Galloway [3])**.** *Let $(\mathbb{R}\times M^3,-N^2dt^2+g)$ be a static vacuum asymptotically flat spacetime that possesses a (possibly disconnected) photon sphere $(\mathfrak{P}^3,p)\hookrightarrow(\mathbb{R}\times M^3,-N^2dt^2+g)$, arising as the inner boundary of $\mathbb{R}\times M^3$. Let m denote the ADM-mass of (M^3,g). Then $m>0$ and $(\mathbb{R}\times M^3,-N^2dt^2+g)$ is isometric to the region $\{r\geq 3m\}$ exterior to the photon sphere $\{r=3m\}$ in the Schwarzschild spacetime of mass m. In particular, $(\mathfrak{P}^3,p)$ is connected and a cylinder over a topological sphere.*

Before sketching the proof of Theorem 1, let us very quickly review the proof by Bunting–Masood-ul-Alam [1]. In short, they double the asympt. flat static 3-manifold (M^3,g) across its black hole inner boundary $\cup_{i=1}^I\Sigma_i^2$ to obtain a new manifold $(\bar{M}^3,\bar{g})$ which is smooth away from a finite set of gluing 2-surfaces Σ_i^2, $C^{1,1}$ across them, and has two asympt. flat ends. They then conformally modify the manifold $(\bar{M}^3,\bar{g})$ such that the original asymptotic end transforms to have vanishing ADM-mass and the doubled end can be one-point compactified. By construction, the new manifold $(\tilde{M}^3,\tilde{g})$ has vanishing scalar curvature, is geodesically complete, and is asympt. flat with vanishing ADM-mass. By the rigidity statement of the positive mass theorem – more precisely, a weak version due to Bartnik –, the conformally modified manifold $(\tilde{M}^3,\tilde{g})$ must be isometric to Euclidean space. In other words, the original manifold (M^3,g) is conformally flat. Combining this with the static equations, it follows that $(\mathbb{R}\times M^3,-N^2dt^2+g)$ is necessarily isometric to the Schwarzschild spacetime.

For the proof of Theorem 1, we proceed as follows: Each photon sphere component Σ_i^2 is assigned a *Schwarzschild mass* $\mu_i:=r_i/3>0$ computed from its area radius r_i. We then show via Proposition 1 that the neck $(2r_i,\mu_i]\times\mathbb{S}^2$ of the Schwarzschild spatial slice $(2r_i,\infty)\times\mathbb{S}^2$ with metric $\varphi_i(r)^{-2}dr^2+r^2\Omega$ can be glued to (M^3, g) across Σ_i^2 in a $C^{1,1}$ fashion. Here, Ω is the canonical metric on the unit sphere and $\varphi_i(r) = \sqrt{1 - 2\mu_i/r}$ as usual. In order to glue the lapse function N of (M^3, g) to the Schwarzschild lapse function φ_i across Σ_i^2, more care needs to be taken: We exploit the lapse scaling invariance of the static vacuum equations $\triangle N = 0$, $\nabla^2 N = N\,\mathrm{Ric}$, and glue N to $3m_i\varphi_i/r_i$, with $m_i := \int_{\Sigma_i^2} \nu(N)_i \, d\sigma_i/4\pi$ the pseudo-Newtonian mass of Σ_i^2. In this way, we obtain a new static vacuum asympt. flat 3-manifold $(\hat{M}^3, \hat{g})$ with black hole inner boundary. This manifold $(\hat{M}^3, \hat{g})$ is smooth away from the gluing 2-surfaces Σ_i^2 and $C^{1,1}$ across them. The Bunting–Masood-ul-Alam method can then be applied to $(\hat{M}^3, \hat{g})$ after ensuring that the conformal factor stays positive. The claim of Theorem 1 follows.

In a forthcoming paper, the author will combine the ideas described above with new geometric and PDE arguments, in particular a new class of metrics generalizing the Schwarzschild class of metrics, to prove the following theorem:

Theorem 2 (Cederbaum, to appear). *Let (M^n, g) be a smooth, asymptotically flat Riemannian manifold of non-negative scalar curvature and ADM-mass m and let $N : M^n \to \mathbb{R}^+$ be harmonic function on (M^n, g) that tends to 1 at infinity. Assume that (M^n, g) has an inner boundary $\cup_{i=1}^I \Sigma_i^{n-1}$ such that each $(\Sigma_i^{n-1}, \sigma_i)$ is umbilic, has constant mean curvature H_i and constant scalar curvature ${}^{\sigma_i}\mathrm{R} > 0$. Assume moreover that $N|_{\Sigma_i^{n-1}} =: N_i$ and its normal derivative $\nu(N)|_{\Sigma_i^{n-1}} =: \nu(N)_i$ are constant on Σ_i^{n-1} and that there exist constants $0 \leq c_i < (n-1)/(n-2)$ such that*

$$c_i \nu(N)_i = H_i N_i \left(1 - \frac{n-2}{n-1} c_i\right) \quad \text{and} \quad H_i^2 = c_i \, {}^{\sigma_i}\mathrm{R}, \tag{3}$$

$$\sum_{i=1}^{I} m_i = m, \quad \text{where} \quad m_i := \frac{1}{4\pi} \int_{\Sigma_i^2} \nu(N)_i \, d\sigma_i. \tag{4}$$

Then (M^n, g) is isometric to n-dim. Schwarzschild-Tangherlini of mass m.

Theorem 2 can be applied to re-prove static vacuum black hole uniqueness in $n+1$ spacetime dimensions (reproducing a result by Gibbons, Ida, and Shiromizu), and to prove static vacuum photon sphere uniqueness in $n+1$ spacetime dimensions, generalizing Theorem 1. It does not appeal to the full static vacuum equations but instead to a generalization of the assertions in Proposition 1.

References

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