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Implications as Rules in Dialogical Semantics

Thomas Piecha Peter Schroeder-Heister*

Abstract

The conception of implications as rules is interpreted in Lorenzen-style dialogical semantics. Implications-as-rules are given attack and defense principles, which are asymmetric between proponent and opponent. Whereas on the proponent's side, these principles have the usual form, on the opponent's side implications function as database entries that can be *used* by the proponent to defend assertions independent of their logical form. The resulting system, which also comprises a principle of cut, is equivalent to the sequent-style system for implications-as-rules. It is argued that the asymmetries arising in the dialogical setting are not deficiencies but reflect the pre-logical 'structural' character of the notion of rule.

Keywords: dialogues, rules, sequent calculus, proof-theoretic semantics, cut

1 Introduction

Various constructive interpretations of implication have been proposed, the most prominent being those based on or related to the Brouwer–Heyting–Kolmogorov (BHK) interpretation[1]. The latter are based on the *transmission view*, according to which a proof of an implication $A \to B$ consists of a constructive procedure which transforms

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[1] Cf. (Heyting, 1971; de Campos Sanz & Piecha, 2011).

any given proof of A into a proof of B. The dialogical or game-theoretical interpretation in Lorenzen-style dialogues[2] can be viewed as a variant of it: An implication $A \to B$ is attacked by claiming A and defended by claiming B. This means that in order to have a winning strategy for $A \to B$, the proponent must be able to generate an argument for B depending on what the opponent can offer in defense of A. In contradistinction to standard constructive interpretations, the attacker need not necessarily spell out a full proof of A. Instead, the proponent may force the opponent to produce certain fragments of a proof of A that are sufficient to successfully defend B. In this sense one may speak of a *partial* or *piecemeal* transmission view as being present in this approach.

2 Implications as rules

There is a more elementary view of implication, which is not based on transmission, but on the view of $A \to B$ being a rule, which allows one to pass over from A to B. This view is particularly supported by the treatment of implication in natural deduction. There modus ponens can be read as the application of $A \to B$ as a rule, which is used to pass from A to B, that is, modus ponens can be read as a schema of rule application. The introduction of an implication $A \to B$ can be read as establishing a rule, namely by deriving its conclusion B from its premiss A. Applications of logic such as logic programming or deductive databases support this perspective. Reading implications as rules motivates an alternative implication-left schema

$$(\to\vdash)^{\circ}\ \frac{\Gamma \vdash A}{\Gamma, A \to B \vdash B}$$

in Gentzen's sequent calculus for intuitionistic logic, yielding what we call the sequent calculus LI°. This schema expresses that by assuming the implication-as-rule $A \to B$ we are entitled to infer B from A. When reading implications as rules, we give implication an elementary meaning which is conceptually prior to the meaning of other operators. In particular, it is explained independent of harmony or symmetry considerations that would normally apply to logical connectives, simply because it is more elementary.

[2]See e.g. (Lorenzen, 1960; Sørensen & Urzyczyn, 2006, Ch. 7; Felscher, 1985, 2002).

The relationship to Gentzen's standard schema is spelled out in (Schroeder-Heister, 2010). Here we just point out that LI° does not have the cut elimination property. The sequent $a, a \to (b \wedge c) \vdash b$ (for atomic and distinct formulas a, b, c) can only be derived by using (Cut):

$$\dfrac{(\to\vdash)^\circ\ \dfrac{(\mathrm{Id})\ \dfrac{}{a \vdash a}}{a, a \to (b \wedge c) \vdash b \wedge c} \qquad (\wedge\vdash)\ \dfrac{(\mathrm{Id})\ \dfrac{}{b \vdash b}}{b \wedge c \vdash b}}{a, a \to (b \wedge c) \vdash b}\ (\mathrm{Cut}) \tag{1}$$

This is the only kind of derivation where (Cut) cannot be eliminated.

Although LI° does not have the cut elimination property, it does have the *weak cut elimination property*. That is, every LI°-derivation containing an application of (Cut) can be transformed into an LI°-derivation of the form

$$\dfrac{(\to\vdash)^\circ\ \dfrac{\vdots}{\Gamma \vdash A} \qquad \begin{array}{c}\vdots\\ \Delta, A \vdash C\end{array}}{\Gamma, \Delta \vdash C}\ (\mathrm{Cut}) \tag{2}$$

where the left premiss of (Cut) is the conclusion of an application of $(\to\vdash)^\circ$. Furthermore, the right premiss of (Cut) can be assumed to be either the conclusion of a derivation of the above form, or it is the endsequent in a derivation such that the cut formula A is the result of an application of a left introduction rule in the last step. As a consequence of the weak cut elimination property, LI° has the subformula property.[3]

3 Dialogical semantics

In what follows, we carry the implications-as-rules approach over to the framework of dialogical semantics. Once an implication $A \to B$ has been claimed by the opponent, it is considered to be a rule in a sort of 'database', which later on can be used by the proponent in order to reduce the justification of its conclusion B to that of A. This is achieved by allowing the proponent to defend an attack on B by asserting A whenever $A \to B$ has been claimed by the opponent before. In case no such claim has been made before (i.e., if no applicable rule

[3]See (Schroeder-Heister, 2010) for these results.

is available in the database), the argument for B continues as usual with an opponent attack on B (which must eventually be defended by the proponent), depending on the respective form of B.

We first recall the standard dialogues and winning strategies for intuitionistic logic.[4] We then introduce dialogues for implications as rules and compare them with the standard dialogues. We also discuss the inference schema of cut and the special role it takes in the implications-as-rules framework. We restrict ourselves to propositional logic throughout.

3.1 Dialogues and strategies

Our *language* consists of propositional *formulas* A, B, C, … that are constructed from *atomic formulas* a, b, c, … with the *logical constants* $\neg$ (negation), $\wedge$ (conjunction), $\vee$ (disjunction) and $\to$ (implication). Furthermore, $\vee$, $\wedge_1$ and $\wedge_2$ are used as *special symbols.* In addition, the letters P ('proponent') and O ('opponent') are used. An *expression* e is either a formula or a special symbol. For each expression e there is a *P-signed expression* $P\,e$ and an *O-signed expression* $O\,e$. A signed expression is called *assertion* if the expression is a formula; it is called *symbolic attack* if the expression is a special symbol. X and Y, where $X \neq Y$, are used as variables for P and O.

For each logical constant the following *argumentation forms* determine how a complex formula (having the respective logical constant in outermost position) that is asserted by X can be attacked by Y and how this attack can be defended (if possible) by X:

AF($\neg$):	assertion:	$X\,\neg A$	
	attack:	$Y\,A$	
	defense:	*no defense*	
AF($\wedge$):	assertion:	$X\,A_1 \wedge A_2$	
	attack:	$Y\,\wedge_i$	(Y chooses $i = 1$ or $i = 2$)
	defense:	$X\,A_i$	
AF($\vee$):	assertion:	$X\,A_1 \vee A_2$	
	attack:	$Y\,\vee$	
	defense:	$X\,A_i$	(X chooses $i = 1$ or $i = 2$)

[4] We follow the presentation of Felscher (1985, 2002), with slight deviations.

AF($\to$): assertion: $X\, A \to B$
attack: $Y\, A$
defense: $X\, B$

Let $\delta(n)$, for $n \geq 0$, be a signed expression and $\eta(n)$ a pair $[m, Z]$, for $0 \leq m < n$, where Z is either A (for 'attack') or D (for 'defense'), and where $\eta(0)$ is empty. The numbers in the domain of $\delta(n)$ are called *positions*, and m in $\eta(n) = [m, Z]$ refers to a position $m < n$. Pairs $\langle \delta(n), \eta(n) \rangle$ are called *moves*. When talking about a move $\langle \delta(n), \eta(n) \rangle$, we write $\langle \delta(n) = X\, e, \eta(n) = [m, Z] \rangle$ to express that $\delta(n)$ has the form $X\, e$ and $\eta(n)$ has the form $[m, Z]$. A move $\langle \delta(n), \eta(n) = [m, A] \rangle$ is called *attack move*, and a move $\langle \delta(n), \eta(n) = [m, D] \rangle$ is called *defense move*. An attack $\langle \delta(n), \eta(n) = [m, A] \rangle$ at position n on an assertion at position m is called *open at position* k for $n < k$ if there is no position n' such that $n < n' \leq k$ and $\langle \delta(n'), \eta(n') = [n, D] \rangle$, that is, if there is no defense at or before position k to an attack at position n.

We now define a *D-dialogue* as a (possibly infinite) sequence of moves $\langle \delta(n), \eta(n) \rangle$ ($n = 0, 1, 2, \dots$) satisfying the following conditions:

($D00$) $\delta(n)$ is a P-signed expression if n is even and an O-signed expression if n is odd. The expression in $\delta(0)$ is a complex formula.

($D01$) If $\eta(n) = [m, A]$, then the expression in $\delta(m)$ is a complex formula and $\delta(n)$ is an attack on this formula as determined by the relevant argumentation form.

($D02$) If $\eta(p) = [n, D]$, then $\eta(n) = [m, A]$ for $m < n < p$ and $\delta(p)$ is the defense of the attack $\delta(n)$ as determined by the relevant argumentation form.

($D10$) If, for an atomic formula a, $\delta(n) = P\, a$, then there is an m such that $m < n$ and $\delta(m) = O\, a$. That is, P may assert an atomic formula only if it has been asserted by O before.

($D11$) If $\eta(p) = [n, D]$, $n < n' < p$, $n' - n$ is even and $\eta(n') = [m, A]$, then there is a p' such that $n' < p' < p$ and $\eta(p') = [n', D]$. That is, if at a position $p - 1$ there are more than one open attacks, then only the last of them may be defended at position p.

($D12$) For every m there is at most one n such that $\eta(n) = [m, D]$. That is, an attack may be defended at most once.

($D13$) If m is even, then there is at most one n such that $\eta(n) = [m, A]$. That is, a P-signed formula may be attacked at most once.

Proponent P and opponent O are not interchangeable due to the asymmetries between P and O introduced in ($D10$) and ($D13$): For atomic formulas a the proponent move $\langle \delta(n) = P\, a, \eta(n) = [m, Z] \rangle$ is possible only after an opponent move $\langle \delta(m) = O\, a, \eta(m) = [k, Z] \rangle$ for $k < m < n$, and O can attack a P-signed formula only once, whereas P can attack O-signed formulas repeatedly. The argumentation forms are symmetric with respect to P and O, however, in the sense that they are independent of whether the assertion is made by P or O.

We say that P *wins a D-dialogue for a formula A* if the D-dialogue is finite, begins with the move $P\, A$ and ends with a move of P such that O cannot make another move.

A *D-dialogue tree* is a tree whose branches contain as paths all possible D-dialogues for a given formula (where a *path* in a branch of a tree with root node n_0 is a sequence $n_0, n_1, \ldots, n_k$ of nodes for $k \geq 0$ where n_i and n_{i+1} are adjacent for $0 \leq i < k$).

We define a *(winning) D-strategy* for a formula A as a subtree S of the D-dialogue tree for A such that S does not branch at even positions, S has as many nodes at odd positions as there are possible moves for O, and all branches of S are D-dialogues for A won by P.

To give an example, the following is a D-strategy for the formula $a \to ((a \to (b \wedge c)) \to b)$:

0.	$P\, a \to ((a \to (b \wedge c)) \to b)$	
1.	$O\, a$	$[0, A]$
2.	$P\, (a \to (b \wedge c)) \to b$	$[1, D]$
3.	$O\, a \to (b \wedge c)$	$[2, A]$
4.	$P\, a$	$[3, A]$
5.	$O\, b \wedge c$	$[4, D]$
6.	$P\, \wedge_1$	$[5, A]$
7.	$O\, b$	$[6, D]$
8.	$P\, b$	$[3, D]$

(In this example the D-strategy consists in only one D-dialogue, which is not necessarily the case in general.)

3.2 Dialogues for implications as rules

Now we introduce dialogues for the implications-as-rules approach. Its guiding idea is the following: When making an assertion A, the proponent P must be prepared to either defend A in the 'standard' way against an attack of the opponent O, or else make the assertion C for some C, for which O has already claimed $C \to A$, that is, for which the implication-as-rule $C \to A$ is sufficient to generate A. This is modelled by saying that every assertion of P is symbolically questioned by O, following which P chooses which of the two ways described P is prepared to take. Contrary to P, O is not given a choice. O's non-implicational assertions are attacked and defended as usual. O's implicational assertions are considered as providing rules which P can *use*, but not question; so there are no attacks and defenses defined for them.

We first define *argumentation forms* for each logical constant that determine how a complex formula that has been asserted by the opponent O can be attacked and how this attack can be defended:

AF($\neg\vdash$):	assertion:	$O\,\neg A$	
	attack:	$P\,A$	
	defense:	*no defense*	
AF($\wedge\vdash$):	assertion:	$O\,A_1 \wedge A_2$	
	attack:	$P\,\wedge_i$	(P chooses $i = 1$ or $i = 2$)
	defense:	$O\,A_i$	
AF($\vee\vdash$):	assertion:	$O\,A_1 \vee A_2$	
	attack:	$P\,\vee$	
	defense:	$O\,A_i$	(O chooses $i = 1$ or $i = 2$)
AF($\to\vdash$)°:	assertion:	$O\,A \to B$	
	attack:	*no attack*	
	defense:	*no defense*	

Except for AF($\to\vdash$)°, these argumentation forms coincide with the standard ones in case of assertions made by the opponent O.

We extend our language by the two special symbols ? and $|\cdot|$. For assertions made by the proponent P there is a pair of argumentation forms for each logical constant (depicted below as trees having two branches which are separated by $|$). An assertion A made by P can be questioned by the opponent with the move $O\,?$. The proponent P can then answer this question either by allowing an attack on the assertion (this is indicated by the special symbol $|\cdot|$; see the argumentation forms on the left side of $|$ below), or by asserting any C for which O has asserted $C \to A$ at an earlier position. We call this the *rule condition* R:

(R) P may answer a question $O\,?$ on a formula A by choosing C provided O has asserted the formula $C \to A$ before.

Then the argumentation forms for assertions made by P are as follows:

$$
\begin{array}{llll}
\text{AF}(\vdash \neg): & \text{assertion:} & & P\,\neg A\\
& \text{question:} & & O\,?\\
& \text{choice:} & P\,|\neg A| & \mid\; P\,C \quad (R)\\
& \text{attack:} & O\,A & \mid\\
& \text{defense:} & \textit{no defense} & \mid
\end{array}
$$

$$
\begin{array}{llll}
\text{AF}(\vdash \wedge): & \text{assertion:} & & P\,A_1 \wedge A_2\\
& \text{question:} & & O\,?\\
& \text{choice:} & P\,|A_1 \wedge A_2| & \mid\; P\,C \quad (R)\\
& \text{attack:} & O\,\wedge_i \quad (i = 1 \text{ or } 2) & \mid\\
& \text{defense:} & P\,A_i & \mid
\end{array}
$$

$$
\begin{array}{llll}
\text{AF}(\vdash \vee): & \text{assertion:} & & P\,A_1 \vee A_2\\
& \text{question:} & & O\,?\\
& \text{choice:} & P\,|A_1 \vee A_2| & \mid\; P\,C \quad (R)\\
& \text{attack:} & O\,\vee & \mid\\
& \text{defense:} & P\,A_i \quad (i = 1 \text{ or } 2) & \mid
\end{array}
$$

$$
\begin{array}{llll}
\text{AF}(\vdash \to): & \text{assertion:} & & P\,A \to B\\
& \text{question:} & & O\,?\\
& \text{choice:} & P\,|A \to B| & \mid\; P\,C \quad (R)\\
& \text{attack:} & O\,A & \mid\\
& \text{defense:} & P\,B & \mid
\end{array}
$$

In the case of an attack $O\,\wedge_i$ according to the argumentation form AF($\vdash \wedge$) the opponent O chooses $i = 1$ or $i = 2$, and in the case of a defense $P\,A_i$ to an attack $O\,\vee$ according to the argumentation form AF($\vdash \vee$) the proponent P chooses $i = 1$ or $i = 2$. The argumentation forms on the left (i.e., the respective left branches) correspond to the argumentation forms of D-dialogues (where the device of question and choice moves is not needed). The argumentation forms on the right (i.e., the respective right branches) reflect the implications-as-rules view.

For assertions of atomic formulas a made by the proponent P an argumentation form is given by the rule condition (R) itself:

AF(R):	assertion:	$P\,a$	
	question:	$O\,?$	
	choice:	$P\,C$	only if O has asserted $C \to a$ before

In addition, we define an argumentation form AF(Cut) such that any expression e (i.e., question, symbolic attack or formula) stated by O can be followed by a move $P\,A$, which can then be followed by the move $O\,A$, for any *cut formula* A:

AF(Cut):	statement:	$O\,e$
	cut:	$P\,A$
	cut:	$O\,A$

This argumentation form differs from the others in that the move $O\,e$ need not be an assertion (i.e. the statement of a formula) but can be the statement of any expression e (i.e., question, symbolic attack or formula). Another difference is that the cut formula is completely independent of the expression e. Calling the P-move an attack and the subsequent O-move a defense as in the other argumentation forms would thus be inadequate. We therefore simply speak of *cut moves* in both cases. The idea behind cut is that at any (even) position, instead of proceeding in the original way, P can introduce an arbitrary formula A as a lemma. P must then later be prepared both to defend this lemma A as an assertion and to defend his original claim *given* this lemma, that is, given the opponent's claim of A.

Formally, we extend the definition of *moves*: For $\delta(n)$ being a signed expression and $\eta(n)$ being a pair $[m, Z]$ for $0 \leq m < n$, Z is now either A (for 'attack'), D (for 'defense'), Q (for 'question'), C (for 'choice') or *Cut*. As before, pairs $\langle \delta(n), \eta(n) \rangle$ are called *moves*, where $\eta(m)$ is empty for $m = 0$ and in case of *Cut*. We have thus the following types of moves:

$$
\begin{array}{rl}
\textit{attack move} & \langle \delta(n) = X\,e, \eta(n) = [m, A] \rangle, \\
\textit{defense move} & \langle \delta(n) = X\,A, \eta(n) = [m, D] \rangle, \\
\textit{question move} & \langle \delta(n) = O\,?, \eta(n) = [m, Q] \rangle, \\
\textit{choice move} & \langle \delta(n) = P\,|A|, \eta(n) = [m, C] \rangle, \\
 & \langle \delta(n) = P\,A, \eta(n) = [m, C] \rangle, \\
\textit{cut move} & \langle \delta(n) = X\,A, \eta(n) = [\textit{Cut}] \rangle.
\end{array}
$$

(A question move can only be made by O and a choice move can only be made by P. The other types of moves are available for both the proponent P and the opponent O.)

A D°-*dialogue*, which is a dialogue based on the implications-as-rules view plus cut, is now defined as a sequence of moves $\langle\delta(n), \eta(n)\rangle$ $(n = 0, 1, 2, \ldots)$ satisfying the following conditions:

($D00^{\circ}$) $\delta(n)$ is a P-signed expression if n is even and an O-signed expression if n is odd. The expression in $\delta(0)$ is a (complex or atomic) formula.

($D01^{\circ}$) If $\eta(n) = [m, A]$, then for $m < n$ the expression in $\delta(m)$ is a complex formula for even n, or, for odd n, the expression is of the form $|B|$ for a complex formula B. In both cases $\delta(n)$ is an attack as determined by the relevant argumentation form.

($D02$) is the same as above.

($D03^{\circ}$) If $\eta(n) = [m, Q]$ (for odd n), then for $m < n$ the expression in $\delta(m)$ is a (complex or atomic) formula, $\eta(m) = [l, Z]$ for $l < m$, $Z = A$, D, C or Cut (where l is empty if $Z = Cut$), and the expression in $\delta(n)$ is the question mark '?'.

($D04^{\circ}$) If $\eta(n) = [m, C]$ (for even n), then $\eta(m) = [l, Q]$ for $l < m < n$ and $\delta(n)$ is the choice answering the question $\delta(m)$ as determined by the relevant argumentation form.

($D05^{\circ}$) If $\eta(n) = [Cut]$ for even n, then $\eta(m) = [l, Z]$ (where l is empty if $Z = Cut$) for $l < m < n$ and $\delta(n)$ is a formula (i.e. the cut formula). If $\eta(n) = [Cut]$ for odd n, then $\eta(n-1) = [Cut]$ and $\delta(n) = O\,A$ for $\delta(n-1) = P\,A$.

($D11$) and ($D12$) are the same as above.

($D13^{\circ}$) If m is even, then there is at most one n such that $\eta(n) = [m, Z]$ for $Z = Q$ or $Z = A$. That is, a P-signed formula, resp. a P-signed expression of the form $|B|$, may be questioned, resp. attacked, at most once.

($D14^{\circ}$) O can question a formula C if and only if (i) C has not yet been asserted by O, or (ii) C has already been attacked by P.

The notions 'dialogue won by P', 'dialogue tree' and 'strategy' as defined for D-dialogues are directly carried over to the corresponding notions for D°-dialogues.

The conditions defining D°-dialogues are similar to those defining D-dialogues. Two important differences are the absence of condition ($D10$) and the additional condition ($D14^{\circ}$) in the former. The absence of ($D10$) is compensated for by allowing O to question assertions of atomic formulas made by P, and by the presence of ($D14°$). Condition ($D00°$) allows $D°$-dialogues to start with the assertion of an atomic formula, contrary to the restriction to complex formulas in D-dialogues. Conditions ($D03°$) and ($D04°$) have been added for the question and choice moves, respectively, and condition ($D05°$) has been added for the cut moves. Note that by ($D05°$) the opponent O can make a cut move only immediately after a cut move made by P.

For example, a $D°$-strategy for the formula $a \to ((a \to (b \wedge c)) \to b)$ is the following (for comparison, see the above D-strategy for this formula):

$$
\begin{array}{rllcll}
0. & \multicolumn{4}{c}{P\, a \to ((a \to (b \wedge c)) \to b)} & \\
1. & \multicolumn{4}{c}{O\, ?} & [0, Q] \\
2. & \multicolumn{4}{c}{P\, |a \to ((a \to (b \wedge c)) \to b)|} & [1, C] \\
3. & \multicolumn{4}{c}{O\, a} & [2, A] \\
4. & \multicolumn{4}{c}{P\, (a \to (b \wedge c)) \to b} & [3, D] \\
5. & \multicolumn{4}{c}{O\, ?} & [4, Q] \\
6. & \multicolumn{4}{c}{P\, |(a \to (b \wedge c)) \to b|} & [5, C] \\
7. & \multicolumn{4}{c}{O\, a \to (b \wedge c)} & [6, A] \\
8. & \multicolumn{4}{c}{P\, b \wedge c} & [Cut] \\
9. & O\, ? & [8, Q] & \vert & O\, b \wedge c & [Cut] \\
10. & P\, a & [9, C] & \vert & P\, \wedge_1 & [9, A] \\
11. & & & \vert & O\, b & [10, D] \\
12. & & & \vert & P\, b & [7, D]
\end{array}
$$

The moves at positions 0–4 and at positions 4–7 + 12 (in the right dialogue) are made according to the argumentation form AF($\vdash \to$). In the choice moves at positions 2 resp. 6 the proponent P can only choose $|a \to ((a \to (b \wedge c)) \to b)|$ resp. $|(a \to (b \wedge c)) \to b|$, since O has not asserted any implications before that could be used as rules by choosing their antecedents. This is different in the choice move at position 10 (in the left dialogue): The opponent O has claimed the implication $a \to (b \wedge c)$ before at position 7, whose succedent is exactly the formula asserted by P at position 8, which is questioned by O at position 9. The proponent P can now use this implication as a rule by answering the question on $b \wedge c$ with the assertion of its antecedent a in the choice move at position 10. This assertion cannot be questioned further due to condition ($D14°$); likewise for the assertion of b at position 12. Hence both dialogues are won by P, and we have a $D°$-strategy.

It can be shown that there is no $D°$-strategy without cut moves for the formula $a \to ((a \to (b \wedge c)) \to b)$. The above D°-strategy corresponds to the LI°-derivation (1). Furthermore, it can be shown that the weak cut elimination property also holds for D°-strategies. That is, every D°-strategy containing cut moves can be transformed into a D°-strategy of the form[5]

$$
\begin{array}{llll|l}
 & & & \vdots & \\
m. & & O\,A \to B & [m-1, Z] & \\
 & & & \vdots & \\
n. & & P\,B & [Cut] & \\
n+1. & O\,? & [n, Q] & & O\,B\ \ [Cut] \\
n+2. & P\,A & [n+1, C] & & \quad s_2 \\
n+3. & O\,? & [n+2, Q] & & \\
 & & s_1 & &
\end{array}
$$

where the O-move at position m is either an attack or a defense (i.e., either $Z = A$ or $Z = D$), and the move $\langle \delta(n+1) = O\,B, \eta(n+1) = [Cut] \rangle$ is the uppermost cut move made by O (i.e., there is no cut move at positions $k < n-1$). The O-move at position $n+3$ might not be possible due to ($D14^{\circ}$). In this case the left dialogue ends with the P-move at position $n+2$. Moreover, the substrategy s_2 is either of the same form as the above D°-strategy, or it depends on a sequence of moves made according to AF($\neg\vdash$), AF($\wedge\vdash$), AF($\vee\vdash$) or AF($\to\vdash$)$^{\circ}$. This corresponds to the properties of LI°-derivations (cf. the LI°-derivation (2) above).

It can be shown that the sequent calculus LI° is sound and complete with respect to the dialogical semantics given by D°-dialogues.

4 Discussion

We have presented a Lorenzen-style dialogue framework for the interpretation of implications as rules which is equivalent to the sequent calculus LI° incorporating this interpretation. The dialogical framework is not as straightforward as LI°, which can be read as the proof-theoretic semantics for implications as rules. Does this speak against the dialogical approach, or perhaps against the idea of implications as rules?

What makes the dialogical presentation difficult to grasp at first

[5] Where the moves at positions m, $n+1$, $n+2$ and $n+3$ can even be assumed to refer to the immediately preceding moves, respectively.

sight is that the usual symmetry between proponent and opponent is lost. Although P and O play different roles in any Lorenzen-style dialogue game, with respect to the attack and defense principles we normally have a perfect symmetry. Just attacks and defenses are defined, not different ways of attacking and defending for P or O. This idea is so deeply rooted in the dialogical paradigm that giving it up may appear as giving up the dialogical setting itself as a foundational approach. The counterargument from the implications-as-rules view would be that implication is different from the other connectives, and that this difference requires an asymmetric treatment. If one wants to formally keep symmetry for implication as a logical connective, one could distinguish between implications $A \to B$ and rules $A \Rightarrow B$, and reduce implications to rules by separate inferences. An attack on an implication $A \to B$ would be defended by claiming a rule $A \Rightarrow B$. Asymmetry would only come in for the rule $A \Rightarrow B$ considered as a 'structural entity', not yet for the implication $A \to B$. This way of proceeding involves, of course, some duplication of notation.

The asymmetry in the treatment of implication brings another asymmetry with it: The proponent can now defend a proposition A by means of the *rule condition* independent of the logical form of A, as an alternative to the 'standard' defense of A which depends on its logical form. This possibility is open only to the proponent and does not fit into the dialogical schema which decomposes formulas according to their logical form.

However, principles of decomposition and symmetry should not be taken as sacrosanct, in particular as rules are *not* logical constants but belong to the general structural framework on top of which logical constants are defined. Given that P has the dialogical role of claiming something to hold, and O the role of providing the assumptions under which something is supposed to hold, the rule $A \Rightarrow B$ means for P that B must be defended on the background A, whereas O only grants with $A \Rightarrow B$ the right to *use* it as a rule, without any propositional claim. This is exactly what is expressed in the dialogue rules for implications-as-rules presented in this paper.

A crucial aspect here is the significance which is given to modus ponens. For the implications-as-rules view, modus ponens is essential for the meaning of implication as it expresses the idea of *application*, which is the characteristic feature of a rule. In a natural-deduction setting with rules made explicit, the application of a rule $A_1, \ldots, A_n \Rightarrow B$

is framed as a generalized modus ponens, which, when applied to premisses $A_1, \ldots, A_n$, yields the conclusion B (Schroeder-Heister, 2012). The system LI° can be viewed as a calculus representing the idea of modus ponens at the sequent-calculus level. The standard interpretation of implication in the dialogical setting corresponds instead to the symmetric sequent calculus LI which is based on the 'implications-as-links' view. According to this view, an implication $A \to B$ which is introduced on the left side of the sequent sign by means of Gentzen's implication-left schema

$$(\to\vdash)\ \frac{\Gamma \vdash A \qquad B, \Delta \vdash C}{\Gamma, A \to B, \Delta \vdash C}$$

links an occurrence of A on the right side of the left premiss with an occurrence of B on the left side of the right premiss of this rule.

The standard dialogical approach favours sequent-style reasoning in the sense of $(\to\vdash)$. We have shown that natural-deduction style reasoning, into which the idea of implications-as-rules fits very neatly, and which can be given a sequent-style rendering via LI°, can be fully represented in the dialogical setting. This representation has the price that implications-as-rules receive an asymmetric treatment, which ultimately reflects differences between natural deduction and the symmetric sequent calculus LI rather than deficiencies of the dialogical setting or of the system LI° being modelled.

This situation is slightly complicated by the presence of cut. In order to achieve full deductive power, the presence of implications-as-rules required the use of (restricted) cut as a primitive rule. In the natural-deduction setting this is easily accommodated, as conclusions of applications of assumption rules can be premisses of elimination rules without creating a maximum formula. In the dialogical setting the handling of cut is difficult and by far not as plausible as in proof systems, since one has to model the claim of the cut formula by P and O according to the pattern of attack and defense. It should be remarked, however, that in a general natural-deduction setting with rules of arbitrary levels and general principles of definitional reflection, it might be reasonable to use a weaker notion of rule without the presupposition of cut, so that this problem disappears at the general level[6].

[6] This is investigated in forthcoming work by Lars Hallnäs and the second author.

Overall this paper demonstrates again that the dialogical framework is versatile enough to deal with approaches originally developed in the realm of proof-theoretic semantics. In the end, more general arguments are needed if one wants to give preference either to proofs or to dialogues as the appropriate foundational approach.

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Thomas Piecha and Peter Schroeder-Heister
Wilhelm-Schickard-Institut für Informatik, Universität Tübingen
Sand 13, 72076 Tübingen, Germany
e-mail: `piecha@informatik.uni-tuebingen.de`,
`psh@uni-tuebingen.de`
URL: `http://ls.inf.uni-tuebingen.de`