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Popper's Notion of Duality and His Theory of Negations

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Karl Popper developed a theory of deductive logic in the late 1940s. In his approach, logic is a metalinguistic theory of deducibility relations that are based on certain purely structural rules. Logical constants are then characterized in terms of deducibility relations. Characterizations of this kind are also called *inferential definitions* by Popper. In this paper, we expound his theory and elaborate some of his ideas and results that in some cases were only sketched by him. Our focus is on Popper's notion of duality, his theory of modalities, and his treatment of different kinds of negation. This allows us to show how his works on logic anticipate some later developments and discussions in philosophical logic, pertaining to trivializing (*tonk*-like) connectives, the duality of logical constants, dual-intuitionistic logic, the (non-)conservativeness of language extensions, the existence of a bi-intuitionistic logic, the non-logicality of minimal negation, and to the problem of logicality in general.

1. Introduction

We investigate Popper's theory of deductive logic that he developed in the late 1940s,[1] focusing on his treatment of different kinds of negation. An explanation of Popper's treatment of negation needs to take his conception of duality into account. While Popper did not declare himself explicitly on the subject of duality, it is still possible to give a rigorous formal definition of duality that completely agrees with Popper's use of the notion in propositional logic, extending into the treatment of several kinds of negation, as well as into the domain of modal logic. This wide applicability is only possible because his notion of duality does not depend on truth functions but is based on deducibility, and it illustrates the importance of Popper's notion of duality as a structuring principle in various areas of logic.

Moreover, we will show that some important results obtained by Popper foreshadow later developments in logic:

- Anticipating the discussion of connectives like *tonk*[2], Popper considered a *tonk*-like connective called the *opponent* of a statement, which leads, if it is present in a logical system, to the triviality of that system.
- He suggested to develop a system of dual-intuitionistic logic, which was then first formulated and investigated by *Cohen* (*1953*).
- He already discussed (non-)conservative language extensions. He recognized, for example, that the addition of classical negation to a system containing implication can change the set of deducible statements containing only implications, and he gave a definition of implication with the help of Peirce's rule that together with intuitionistic negation yields classical logic.

[1] See *Popper 1947a,b,c, 1948a,b, 1949*. A collected edition of Popper's works on logic is currently prepared by Peter Schroeder-Heister and the authors; it will appear in 2017.
[2] Cf. *Prior 1960*.

- He also considered the addition of classical negation to a language containing intuitionistic as well as dual-intuitionistic negation, whereby all three negations become synonymous. This is an example of a non-conservative extension where classical laws also hold for the weaker negations.
- Popper was probably the first to present a system that contains an intuitionistic negation as well as a dual-intuitionistic negation. By proving that in the system so obtained these two kinds of negation do not collapse (i.e., do not become synonymous), he gave the first formal account of a bi-intuitionistic logic.
- He provided an analysis of logicality, in which for example minimal negation[3] and subminimal negation (as it is now called[4]) turn out not to be logical constants.

The philosophical interpretation of Popper's theory that we will follow is closest to the one given by *Schroeder-Heister* (*2006*), who gave a reconstruction of Popper's theory in terms of the system developed by *Koslow* (*1992*). This reconstruction yielded important insights into Popper's approach,[5] and helped to better locate Popper's theory in the history and philosophy of logic.

In our investigation, however, we will not attempt to reconstruct Popper's theory in terms of a more modern system. Instead, we will stay as close as possible to Popper's terminological setting. Our aim is to present Popper's theory in its own terms in a hopefully concise and clear manner, and to elaborate some of Popper's ideas and results on negation that in some cases were only sketched by him.

Popper's formal treatment of negation might help to elucidate questions that arise in Popper's philosophy of science, or to answer questions on the relation between his treatment of negation in the context of deductive logic and the role of falsifications in the enterprise of empirical science. These questions are, however, outside the scope of this paper. Another limitation in scope concerns the logical constants: We restrict ourselves to propositional and modal logic, and do not consider first-order quantifiers.

1.1. *Short survey of the literature*

Reviews of Popper's articles on logic were written by *Ackermann* (*1948, 1949a,b*), *Beth* (*1948*), *Curry* (*1948a,b,c,d, 1949*), *Hasenjaeger* (*1949*), *Kleene* (*1948, 1949*) and *McKinsey* (*1948*). While several of these reviews give only a summary, the reviews by *Curry* (*1948a*), *Hasenjaeger* (*1949*), *Kleene* (*1948*) and *McKinsey* (*1948*) contain some serious criticisms of certain aspects of Popper's approach. An in-depth discussion of these criticisms can be found in *Schroeder-Heister 1984* (§3), which also contains further references to discussions of Popper's works on logic. Some of these criticisms will be discussed here, too. Brouwer, on the other hand, responded quite positively.[6] Detailed investigations of Popper's theory are given in the articles of *Lejewski* (*1974*) and *Schroeder-Heister* (*1984, 2006*), and in an unpublished thesis by *Cohen* (*1953*).[7]

Lejewski (*1974*) pursues an indication of Bernays, who stated that Popper's theory bears a close relationship with Tarski's theory of consequence developed in the 1930s.[8] Lejewski then reconstructs both Popper's and Tarski's theories in the logical system of Leśniewski. As a

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[3] See *Johansson 1937*.

[4] See e.g. *Dunn 1999* and the references therein.

[5] See also *Lejewski's* (*1974*) reconstruction in the system of Leśniewski.

[6] See *Brouwer's* (*December 10th, 1947*) letter to Popper: 'Your duality construction and your new definition of intuitionistic negation have delighted me, and I have presented your last paper on November 29th.' Brouwer presented *Popper 1947c* on October 25th, 1947 and *Popper 1948a,b* on November 29th, 1947 to the Koninklijke Nederlandse Akademie van Wetenschappen.

[7] See also the two unpublished theses by *Brooke-Wavell* (*1958*) and by *Dunn* (*1963*); cf. *Schroeder-Heister 1984*. A more recent contribution is *Bar-Am 2009*. We thank Mrs. Cohen for making *Cohen 1953* available.

[8] The remark of Bernays can be found in *Popper 1947b* (fn on p. 204), where Bernays refers to *Tarski 1930a*. Lejewski also considers *Tarski 1930b, 1935, 1936b*.

result, certain assumptions that were used only tacitly by Popper could be made explicit. *Popper* (*1974*) acknowledged the technical merit of Lejewski's work but added that it did not capture the philosophical and foundational spirit of his articles.[9]

Schroeder-Heister (*1984*) considers Popper's works as a contribution to the problem of distinguishing between logical and non-logical signs that arises in *Tarski's* (*1936a*) analysis of the concept of logical consequence. For Popper, he argues, a logical constant is an operation that can be characterized by an inferential definition. He therefore proposes to read Popper's inferential definitions not as definitions, but as adequacy conditions for logical constants. *Schroeder-Heister* (*1984*) also discusses objections against Popper's foundationalist claims that were put forth by the reviewers and by Lejewski. He ultimately agrees with these objections and concludes that Popper's theory cannot be considered to be a new foundation for logic.

Schroeder-Heister (*2006*) revises his former interpretation in certain aspects. He now sees Popper's approach not only as a contribution to the problem of the logicality of constants but, more importantly, as 'a structuralist approach according to which logic is a metalinguistic theory of deducibility relations, in terms of which logical operations are characterized' (ibid., p. 17). He discusses the characterization of the deducibility relation by the rules of a basis, and compares these rules to Gentzen's structural rules. Moreover, he outlines Popper's characterization of classical negation by using deducibility with multiple succedents (i.e. Popper's notion of relative demonstrability). We will elaborate on these topics in Sections 2.4 (see also Appendix A) and 6.

Cohen's (*1953*) unpublished thesis is not considered in any of the works mentioned so far, despite being of great importance to the matters here discussed. Its second part concerns the development of a system of dual-intuitionistic logic.[10] Cohen starts from Gentzen's observation that intuitionistic logic can be obtained from the sequent calculus for classical logic by restricting the number of formulas occurring in the succedent of sequents to at most one. He then formulates a sequent calculus where the number of formulas occurring in the antecedent of sequents is restricted to at most one (without restricting succedents), which he calls the *dualintuitionistic restricted predicate calculus GL2*. The idea of developing this system was suggested to him by Popper.[11] The inception of dual-intuitionistic logic can thus be attributed to Popper, and Cohen was the first to develop and investigate a system of dual-intuitionistic logic. Cohen's sequent calculus GL2 is, however, not exactly dual to intuitionistic logic, since it contains rules for both the anti-conditional and the conditional.[12]

1.2. *Overview*

In Section 2 we describe the general framework of Popper's definitions of logical constants. We comment on the kind of object languages Popper has in mind, the status of his fundamental

[9] *Popper* (*1974*, p. 1095) writes: '[...] he has not brought out, or even briefly hinted at, what my intentions were in writing these bad and ill-fated papers. By putting them in Leśniewski's language he has, in fact, done something extremely interesting, but at the same time something that goes contrary to my intentions.'

[10] *Popper* (*2004*, fn 8, p. 431f.) cites this system as an answer to his question 'whether we can construct a system of logic in which contradictory statements do not entail every statement.' However, '[i]n [Popper's] opinion, such a system [which lacks e.g. modus ponens] is of no use for drawing inferences.'

[11] As remarked by *Cohen* (*1953*, p. 188). *Popper* (*2004*, fn 8, p. 431f.) first refers to his *1948a, 1948b*, and then mentions that '[...] Cohen has developed the system [of dual-intuitionistic logic] in some detail.' In fact, Cohen also gave a full analysis of this system, including a proof of cut elimination. Popper (ibid.) has 'a simple interpretation of this calculus. All the statements may be taken to be modal statements asserting possibility.' This interpretation can already be found in *Popper's* (*July 28th, 1953*) letter to Cohen, where he interprets every statement a as 'a is possible' or, equivalently, as 'a is satisfiable'. In this letter Popper says that he 'consider[s] publishing these results', without, however, doing so. *Cohen* (*1953*, p. 208f.) names *Popper's* (*1948a*, p. 181) interpretation of sequents in terms of relative demonstrability (cf. Section 3.4) as the most suitable.

[12] *Cohen* (*1953*, part II, §3) first formulates the system GL1, which includes the rules FS and FA for the conditional. This system is then extended to his dualintuitionistic restricted predicate calculus GL2 by adding the rules GS and GA for the anti-conditional; see *Cohen 1953* (part II, §4). Cf. *Kapsner et al. 2014* and *Kapsner 2014* (p. 128, fn 6).

concept of deducibility, the means employed in the metalanguage and the characterization of the deducibility relation by a basis. Popper's Basis I will be our main interest; other bases are discussed in Appendix A. In Section 3 we present central properties and relations of object languages that are based on Popper's notion of deducibility. They belong to what Popper calls the *general theory of derivation*.[13] In Section 4 we consider Popper's *special theory of derivation*, which deals with definitions of logical constants. In Section 5 we propose a certain interpretation of Popper's notion of duality, which we make precise by use of a duality function, and we give an account of Popper's theory of modalities. Based on that, we give a detailed account of Popper's theory of negations in Section 6.

2. Popper's framework

Popper considers pairs of an object language $\mathcal{L}$ and a deducibility relation defined on $\mathcal{L}$.[14] The set of axioms characterizing a deducibility relation is called *basis*. Popper uses mainly two alternative bases, called Basis I and Basis II, which also occur in different versions. We use a version of Basis I as the foundation for the rest of this paper.[15] The axioms of the basis as well as subsequent inferential definitions are formulated in a restricted metalanguage.

2.1. *Object languages*

In distinction to modern approaches that proceed by giving an alphabet of the object language under consideration and either a grammar or inductive definition of those expressions that are to be counted as terms and formulas of the object language, Popper's approach does not presuppose any knowledge about the form or syntactic structure of the object language under consideration.[16] It is indeed supposed to be an approach that is not only applicable to formally defined languages but also to natural language. For example, the conjunction of two statements a and b of the object language need not have any particular syntactic form like '$a \wedge b$' or 'a and b'.[17]

We will write $\mathcal{L}$ for an object language and use small Latin letters $a, b, c, \ldots$ (also with indices) as variables ranging over $\mathcal{L}$. The members of an object language are assumed to be statements so that it makes sense to say that some of them are deducible from others. Popper (1947b, p. 204) calls them 'expressions of which we might reasonably say that they are true or that they are false'.[18] We furthermore assume that any object language considered is nonempty.

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[13] See Popper 1947b (§3.2).

[14] Schroeder-Heister (2006) uses the term 'deducibility structure' for such pairs.

[15] This version of Basis I can be found in Popper 1947b.

[16] Popper is aware of this difference; he gives a sketch of the classical approach at the beginning of Popper 1949. The idea that logical analysis is not only applicable to formally specified languages is still present later, in Popper 2004 (addendum 5 from 1963), where he stresses this fact in the context of discussing Tarski's theory of truth: 'It has often been said that Tarski's theory of truth is applicable only to formalized language systems. We do not believe that this is correct. Admittedly it needs a language—an object language—with a certain degree of artificiality; and it needs a distinction between an object-language and a meta-language [...] [N]ot every language which is subject to some stated rules, or based on more or less clearly formulated rules [...] need be a fully formalized language. The recognition of the existence of a whole range of more or less artificial though not formalized languages seems to me a point of considerable importance, and specially important for the philosophical evaluation of the theory of truth.'

[17] See Popper 1947b (p. 205): '[Forming the conjunction of a and b] is done, in English, by linking them together with the help of the word "and". But we need not suppose that any such word exists: the link may be effected in very different ways; moreover, the new statement need not even contain the old ones as recognizable separate parts (or "components").'
To make this point clearer, we consider as an example the propositional language containing only signs for negation ($\neg$) and disjunction ($\vee$) together with a calculus in which all classically valid formulas of this restricted language are derivable. This language nevertheless contains for any two formulas ϕ and ψ a conjunction of ϕ and ψ. One such conjunction may be $\neg(\neg\phi \vee \neg\psi)$; but other variants are possible, and there is in general no way to specify *the* canonical form of conjunction in that language.

[18] He states explicitly (ibid.): 'Nothing is presupposed of our $a, b, c, \ldots$ except that they are statements, and our theory shows, thereby, that there exists a rudimentary theory of inference for any language that contains statements, whatever their logical structure or lack of structure may be.'

2.2. *The concept of deducibility*

Popper's approach is based on the concept of deducibility (resp. derivability). It is the only undefined notion as far as propositional and modal logic are concerned.[19] Deducibility is a relation, written with the solidus $/$, that ranges over the object language and holds between finitely many premises (including the case of no premises) $a_1, \ldots, a_n$ and exactly one conclusion b. In $/$-notation, Popper writes

$$a_1, \ldots, a_n / b$$

to express that the statement b can be deduced from the statements $a_1, \ldots, a_n$. The $/$-notation is introduced as a horizontal variant of the notation

$$\begin{array}{c} a_1 \\ \vdots \\ a_n \\ \hline b \end{array}$$

that is often used to state rules of inference like, for example,

$$\begin{array}{c} \text{If } A\text{, then } B \\ A \\ \hline B \end{array}$$

which says that from the premise 'If A, then B' together with the premise A the conclusion B can be deduced.[20] The case $n = 0$, in which no premises occur, was not yet considered in *Popper 1947b*. It was added later in *Popper 1948a*, where the so-called *D-notation* is introduced. In this notation, $D(a_1, a_2, \ldots, a_n)$ stands for $a_2, \ldots, a_n / a_1$, and the special case $D(a_1)$ corresponds to $/a_1$, meaning that a_1 is deducible without premises.

In the context of deducibility neither the order of premises nor the multiplicity of identical premises is relevant, since deducibility enjoys the following structural properties:

Lemma 2.1:

(1) Exchange of premises: If $a_1, \ldots, a_i, a_{i+1}, \ldots, a_n / b$*, then* $a_1, \ldots, a_{i+1}, a_i, \ldots, a_n / b$.
(2) Contraction of premises: If $a_1, \ldots, a_i, a_i, \ldots, a_n / b$*, then* $a_1, \ldots, a_i, \ldots, a_n / b$.

These properties are consequences of Popper's characterization of deducibility by his Basis I.[21] Premises $a_1, \ldots, a_n$ can thus be understood as a set $\{a_1, \ldots, a_n\}$.

2.3. *The metalanguage*

In order to formulate rules that characterize the concept of deducibility and to give definitions of logical constants, Popper has to make use of a metalanguage. To improve readability, we use the following symbolic notation for it, which is similar to Popper's:

Symbol	$\rightarrow$	$\leftrightarrow$	&	$\bigvee$	(a)	$(\exists a)$
Meaning	if-then	if and only if	and	or	for all a	there is an a

The quantifiers (a) and $(\exists a)$ range over statements a of the object language. Popper notes that as long as one does not want to introduce modal connectives, one can do without disjunction.[22]

[19] An operation of substitution is added for the treatment of first-order logic, which, however, is not considered in this paper.
[20] See *Popper 1947b* (p. 194).
[21] Cf. also *Popper 1947b*.

He also notes that the introduction of modal connectives requires metalinguistic negation.[23] Interestingly though, negation is not used in defining them, whereas disjunction is indeed essential. We have therefore omitted metalinguistic negation in our list of symbols.

Due to the fact that the metalanguage is quite restricted and does not contain negation, the following lemma holds:

Lemma 2.2: *In any nonempty object language with a trivial deducibility relation (i.e.* $a_1,\ldots,a_n/b$ *holds for any* $a_1,\ldots,a_n,b$*) every formula of the metalanguage is satisfied.*

Proof By a simple induction on formulas of the metalanguage. □

Hence, even if a definition stated in the metalanguage trivializes the deducibility relation / on the object language, this definition cannot make the metalanguage inconsistent.[24]

For certain formulas of the metalanguage Popper uses a special vocabulary. Atomic formulas of the form

$$a_1,\ldots,a_n/b$$

are also called *absolute rules of derivation*, and formulas of the form

$$a_1,\ldots,a_n/b \rightarrow c_1,\ldots,c_m/d$$

and iterated versions thereof, are also called *conditional rules of derivation* or just *rules of derivation*. Note that rules of derivation are not rules in a calculus, understood as a proof system. Popper does not develop a calculus in this sense, and what he calls rules of derivation are metalinguistically formulated statements about the deducibility relation.[25] We follow Popper in using the term *rule* to speak about metalinguistic formulas.

2.4. *The characterization of deducibility by a basis*

So far, the deducibility relation / has only been defined by saying that it ranges over an object language $\mathcal{L}$. The next step consists in providing what Popper calls a *basis* for this relation.[26]

A basis is a complete and independent set of rules, formulated in the metalanguage, that axiomatizes the deducibility relation /. Completeness is here defined with respect to Popper's notion of *absolute validity*, which is similar to the notion of validity obtained by allowing only structural rules of inference.[27] Popper's idea seems to be that even if we abstract away from

[22] See *Popper 1947c* (p. 1216): '[...] it may be mentioned that the only logical rules needed in the metalanguage (except where we treat modalities) are those of the positive part of the HILBERT–BERNAYS calculus of propositions as far as they pertain to "if-then", "if, and only if", and to "and" [...], and the rules for identity. The rules for negation need not be assumed [...]; but we need rules for universal quantification, especially the rule of specification [...].'

[23] See *Popper 1947c* (p. 1223): 'For the development of the theory of modality, metalinguistic negation is also needed, together with at least its intuitionistically valid rules.'

[24] This will become relevant in the discussion of the logical constant *opp* later on. Another simple example is the metalinguistic formula $(\exists a)(\vdash a \,\&\, \urcorner a)$, where $\vdash a$ stands for the demonstrability of a and $\urcorner a$ for the refutability of a (see Sections 3.2 and 3.3). This formula is also only true for object languages with a trivial deducibility relation, although it is a satisfiable formula of the metalanguage.

[25] *Popper* (*1948b*, p. 327) once uses the term 'metalinguistic calculus'.

[26] *Popper 1947c* is an exception to this approach: The requirement to specify a basis for the deducibility relation is dropped, and the burden of making sure that the logic has certain structural properties is shifted to the definitions of the logical constants. For this reason he uses an extended definition of conjunction, called basic definition (DB2), to ensure reflexivity and transitivity of / as well as exchangeability of premises. Unfortunately, (DB2) corresponds to the defective Basis II of *Popper 1947b*, and is therefore just as problematic (cf. Appendix A).

[27] Popper gives his analysis of absolute validity in terms of so-called *statement preserving interpretations* in *Popper 1947a*, where he writes about absolutely valid inferences (ibid, p. 274): 'There are inferences which are valid according to all our definitions, in spite of the fact that the logical form of the statements involved is irrelevant.'

any concrete logical system (containing a specific set of logical constants) under consideration, we still have a rudimentary residuum of deduction consisting of structural[28] inferences, like the inference from a statement a to the statement a.

Popper's Basis I is given by a *generalized reflexivity principle*, called (Rg), together with a *generalized transitivity principle*, called (Tg):[29]

$$a_1, \ldots, a_n/a_i \qquad (1 \leq i \leq n) \tag{Rg}$$

$$\left\{\begin{array}{l} \quad a_1, \ldots, a_n/b_1 \\ \& \ a_1, \ldots, a_n/b_2 \\ \vdots \qquad \quad \vdots \\ \& \ a_1, \ldots, a_n/b_m \end{array}\right\} \to (b_1, \ldots, b_m/c \to a_1, \ldots, a_n/c) \tag{Tg}$$

The principle (Tg) is not a rule in the strict sense, but rather a schematic rule. This means that (Tg) has, depending on the value of m, many other rules as instances. There is therefore no way to directly express the content of (Tg) by means of a single formula of the metalanguage. This fact was very unsatisfactory for Popper. He searched for a replacement of (Tg) that could be expressed directly in his metalanguage. We discuss his proposals and their problems in Appendix A.

3. The general theory of derivation

Popper's general theory of derivation does not refer to any logical signs of the object language. It studies properties of statements and relations on statements that can be defined using only the deducibility relation. Examples of such properties are being a theorem in the deducibility structure, i.e. being a demonstrable statement, or being a statement that is refutable.

We discuss the following properties and relations: mutual deducibility, complementarity, demonstrability, contradictoriness, refutability and relative demonstrability. Our focus is on mutual deducibility and relative demonstrability. Relative demonstrability in particular contains the notions of complementarity, demonstrability, contradictoriness and refutability as special cases.

3.1. *Mutual deducibility*

The relation of mutual deducibility // is explicitly defined as follows:

$$a//b \leftrightarrow (a/b \,\&\, b/a) \tag{D//}$$

It is an equivalence relation, as one can see by checking the rules of Basis I. Two mutually deducible statements a and b are said to have the same *logical force*.[30] The equivalence classes induced by // are thus logical forces.

Popper (*1947b*, p. 203) calls two mutually deducible statements also *logically equivalent*, and then calls (D//) the *substitutivity principle for logical equivalence*. The following substitution lemma holds:

Lemma 3.1: *If a and b are mutually deducible, then we may substitute b for a in every deducibility relation, i.e. the following two statements are true:*

[28] Although Popper does not use the term 'structural'.

[29] See *Popper 1947b*.

[30] See *Popper 1947a* (p. 282).

(1) $a//b \to (a_1,\dots,a_n/a \to a_1,\dots,a_n/b)$.
(2) $a//b \to (a_1,\dots,a_n,a,a_{n+1},\dots,a_m/c \to a_1,\dots,a_n,b,a_{n+1},\dots,a_m/c)$.

Proof (1) follows directly from (Tg), and (2) follows from (Tg) and Lemma 2.1. □

Popper also considers the following definition of //:[31]

$$a//b \leftrightarrow (c)(a/c \leftrightarrow b/c) \tag{D//$'$}$$

For this alternative definition we can prove the following lemma:

Lemma 3.2: *In the presence of* (Tg) *and* (Rg), (D//) *is equivalent to* (D//′).

Proof We have to show that $(a/b \,\&\, b/a) \leftrightarrow (c)(a/c \leftrightarrow b/c)$ is true. The proof from left to right uses (Tg), and the proof from right to left uses (Rg). □

3.2. *Complementarity and demonstrability*

The notion of complementarity for statements $a_1,\dots,a_n$, written $\vdash a_1,\dots,a_n$, is given as follows:

$$\vdash a_1,\dots,a_n \leftrightarrow (b)(c)((a_1/c \,\&\dots\&\, a_n/c) \to b/c) \tag{D$\vdash$1}$$

If we let $n = 1$, we obtain the definition of a self-complementary or demonstrable statement, written $\vdash a$:

$$\vdash a \leftrightarrow (b)(c)(a/c \to b/c) \tag{D$\vdash$1$'$}$$

Intuitively, what is expressed by the complementarity of the statements $a_1,\dots,a_n$ is that at least one of them has to be true, i.e. that taken together they exhaust all possible states of affairs. This idea is captured by saying that if the statements $a_1,\dots,a_n$ are complementary, then any statement c that follows from each of the statements $a_1,\dots,a_n$ individually follows from any statement b. From (D⊢1′) one can thus obtain

$$\vdash a \leftrightarrow (b)(b/a)$$

by instantiating c by a and by using basic rules. A demonstrable statement is thus a statement that follows from any statement whatsoever.[32]

3.3. *Contradictoriness and refutability*

The notion of contradictoriness for statements $a_1,\dots,a_n$, written $\nearrow a_1,\dots,a_n$, is given as follows:

$$\nearrow a_1,\dots,a_n \leftrightarrow (b)(c)((b/a_1 \,\&\dots\&\, b/a_n) \to b/c) \tag{D$\nearrow$}$$

For $n = 1$ we get the notion of refutability of a statement a, written $\nearrow a$:

$$\nearrow a \leftrightarrow (b)(c)(b/a \to b/c) \tag{D$\nearrow'$}$$

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[31] See *Popper 1947c* (def. (DB1)) or *Popper 1947a* (def. (7.1)).

[32] For the definition of demonstrability see *Popper 1947b* (def. (D8.2+)) or *Popper 1947c* (def. (D11)). For the definition of complementarity see *Popper 1948a* (def. (D3.1)).

The intuition lying behind these notions is the following: A statement is refutable if it is false no matter the state of affairs, and a sequence of statements $a_1,\ldots,a_n$ is contradictory if its members cannot be true together. Contradictoriness of the statements $a_1,\ldots,a_n$ is therefore defined by saying that any statement b that implies each of the statements $a_1,\ldots,a_n$ individually implies any statement c.[33] From (D$\urcorner'$) one can thus obtain

$$\urcorner a \leftrightarrow (c)(a/c)$$

by substituting a for b and by using basic rules. This is the definition of a self-contradictory statement, i.e. of a refutable statement. From such a statement any other statement follows.

3.4. *Relative demonstrability*

Complementarity and contradictoriness can be combined in one relation that holds between statements $a_1,\ldots,a_n$ and $b_1,\ldots,b_m$. This relation is called *relative demonstrability* (or *relative refutability*), written $a_1,\ldots,a_n \vdash b_1,\ldots,b_m$. One of the definitions given by Popper is:[34]

$$a_1,\ldots,a_n \vdash b_1,\ldots,b_m \leftrightarrow (c)((b_1/c \,\&\, \ldots \,\&\, b_m/c) \rightarrow a_1,\ldots,a_n/c) \qquad (\mathrm{D}{\vdash}2)$$

For technical reasons, which are explained in Appendix B, we will use the following slightly modified definition[35] of relative demonstrability in the remainder of this paper:

Definition 3.3: *Relative demonstrability* $a_1,\ldots,a_n \vdash b_1,\ldots,b_m$ is defined by

$$\begin{aligned} &a_1,\ldots,a_n \vdash b_1,\ldots,b_m \leftrightarrow \\ &\quad (c)(d_1)\ldots(d_k)((b_1,d_1,\ldots,d_k/c \,\&\, \ldots \,\&\, b_m,d_1,\ldots,d_k/c) \rightarrow a_1,\ldots,a_n,d_1,\ldots,d_k/c) \end{aligned} \qquad (\mathrm{D}{\vdash}3)$$

Note that in distinction to Popper's (D⊢2) the definiens in (D⊢3) is now formulated with additional context statements $d_1,\ldots,d_k$ for $0 \le l \le k$, which occur as additional premises in each of the deducibility relations. Definition (D⊢3) is thus more general than (D⊢2).

Lemma 3.4: *The concept of relative demonstrability contains, as special cases, the concepts of complementarity, demonstrability, contradictoriness and refutability.*

Proof Let $k = 0$. For complementarity, let $n = 0$. For demonstrability, let $n = 0$ and $m = 1$. For contradictoriness, let $m = 0$. For refutability, let $n = 1$ and $m = 0$. ☐

Lemma 3.5: *For all* $a_1,\ldots,a_n,b$: $a_1,\ldots,a_n/b \leftrightarrow a_1,\ldots,a_n \vdash b$.

Proof For $m = 1$ in (D⊢3) we have to show that $(c)(d_1)\ldots(d_k)(b_1,d_1,\ldots,d_k/c \rightarrow a_1,\ldots,a_n,d_1,\ldots,d_k/c)$ is equivalent to $a_1,\ldots,a_n/b_1$. Let $k = 0$. Instantiating c by b_1 yields $b_1/b_1 \rightarrow a_1,\ldots,a_n/b_1$, and by (Rg) we get $a_1,\ldots,a_n/b_1$. Likewise for the other direction. ☐

This lemma allows us to replace / by ⊢ in all formulas of the metalanguage. This will be necessary further on to bring some of Popper's definitions into a form that makes them dualizable.

[33] For the definition of refutability, see *Popper 1947b* (def. (D8.3)) or *Popper 1947c* (def. (D12)). For the definition of contradictoriness, see *Popper 1948a* (def. (D3.2) and (D3.2′)).

[34] See *Popper 1948a* (def. (D3.3′)).

[35] Cf. *Schroeder-Heister 2006* (app. 2).

Note that, conversely, $\vdash$ can only be replaced by $/$ if $\vdash$ has exactly one succedent b. For mutual deducibility we have $a//b \leftrightarrow (a \vdash b \;\&\; b \vdash a)$.

Lemma 3.6: *The following structural rules hold for* $a_1,\ldots,a_n \vdash b_1,\ldots,b_m$*:*

(1) Weakening on the left (LW) *and weakening on the right* (RW)*:*

$$a_1,\ldots,a_n \vdash b_1,\ldots,b_m \rightarrow a_1,\ldots,a_n,a_{n+1} \vdash b_1,\ldots,b_m \tag{LW}$$

$$a_1,\ldots,a_n \vdash b_1,\ldots,b_m \rightarrow a_1,\ldots,a_n \vdash b_1,\ldots,b_m,b_{m+1} \tag{RW}$$

(2) Exchange on the left (LE) *and exchange on the right* (RE)*:*

$$a_1,\ldots,a_i,a_{i+1},\ldots,a_n \vdash b_1,\ldots,b_m \rightarrow a_1,\ldots,a_{i+1},a_i,\ldots,a_n \vdash b_1,\ldots,b_m \tag{LE}$$

$$a_1,\ldots,a_n \vdash b_1,\ldots,b_j,b_{j+1},\ldots,b_m \rightarrow a_1,\ldots,a_n \vdash b_1,\ldots,b_{j+1},b_j,\ldots,b_m \tag{RE}$$

(3) Contraction on the left (LC) *and contraction on the right* (RC)*:*

$$a_1,\ldots,a_i,a_i,\ldots,a_n \vdash b_1,\ldots,b_m \rightarrow a_1,\ldots,a_i,\ldots,a_n \vdash b_1,\ldots,b_m \tag{LC}$$

$$a_1,\ldots,a_n \vdash b_1,\ldots,b_j,b_j,\ldots,b_m \rightarrow a_1,\ldots,a_n \vdash b_1,\ldots,b_j,\ldots,b_m \tag{RC}$$

(4) If there are i,j *(for* $1 \le i \le n$ *and* $1 \le j \le m$*) such that* $a_i = b_j$*, then* $a_1,\ldots,a_n \vdash b_1,\ldots,b_m$*.*

(5) The cut rule:

$$a_1,\ldots,a_n \vdash b_1,\ldots,b_m,c \rightarrow (c,a_1,\ldots,a_n \vdash b_1,\ldots,b_m \rightarrow a_1,\ldots,a_n \vdash b_1,\ldots,b_m) \tag{Cut}$$

Proof Consider the definiens of $a_1,\ldots,a_n \vdash b_1,\ldots,b_m$:

$$(c)(d_1)\ldots(d_k)((b_1,d_1,\ldots,d_k/c \;\&\; \ldots \;\&\; b_m,d_1,\ldots,d_k/c) \rightarrow a_1,\ldots,a_n,d_1,\ldots,d_k/c).$$

(1) We can always strengthen the antecedent of the implication or weaken its succedent. Left weakening of $/$ follows from Basis I. We thus get (LW) by weakening the succedent $a_1,\ldots,a_n,d_1,\ldots,d_k/c$ to $a_1,\ldots,a_n,a_{n+1},d_1,\ldots,d_k/c$. The antecedent can be strengthened by adding the conjunct $b_{m+1},d_1,\ldots,d_k/c$, which gives us (RW).

(2) Exchange on the left (LE) is due to Lemma 2.1 (1). Exchange on the right (RE) follows from the commutativity of metalinguistic conjunction.

(3) Contraction on the left (LC) is due to Lemma 2.1 (2). Consider $a,a/b$; by (Rg) we have a/a, and (Tg) yields a/b. By replacing $/$ by $\vdash$ and subsequent applications of weakening and exchange we obtain (LC). Contraction on the right (RC) follows from the idempotence of metalinguistic conjunction.

(4) For $a_i = b_j$ we have a_i/b_j by (Rg). Lemma 3.5 gives $a_i \vdash b_j$. By applications of (LW), (RW), (LE) and (RE) we get $a_1,\ldots,a_n \vdash b_1,\ldots,b_m$.

(5) See Appendix B, Theorem B.3. □

Relative demonstrability $a_1,\ldots,a_n \vdash b_1,\ldots,b_m$ can be interpreted as derivability of the disjunction of $b_1,\ldots,b_m$ from the conjunction of $a_1,\ldots,a_n$. This interpretation is justified by the fact that for object languages containing conjunction $\wedge$ and disjunction $\vee$ one can show the following:

$$a_1,\ldots,a_n \vdash b_1,\ldots,b_m \leftrightarrow a_1 \wedge \ldots \wedge a_n \vdash b_1 \vee \ldots \vee b_m$$

This is a consequence of Lemma 5.3, proved below. The concept of relative demonstrability gives thus an interpretation of *Gentzen's* (1935) sequents.[36]

4. The special theory of derivation

The subject of Popper's special theory of derivation are definitions of logical constants. In distinction to the general theory, which studies relations on statements defined by deducibility, the special theory deals with relations between logically complex statements and their components.

4.1. *Definitions of logical constants*

Since object languages $\mathcal{L}$ are not specified syntactically, it is not possible to introduce logical connectives by stipulating that for any two statements a and b in $\mathcal{L}$ there exists a statement having some specific syntactic form, say $a \wedge b$. Instead, logical constants have to be characterized by definitions in terms of the role they play with respect to the deducibility relation $/$. Popper calls such definitions *inferential definitions*, and a sign of an object language $\mathcal{L}$ is a *logical constant* if, and only if, it can be defined by an inferential definition.[37]

In Popper's special theory, definitions of logical constants have the following form (we use $\circ$ as a placeholder for an arbitrary binary connective):

$$a//a_1 \circ a_2 \leftrightarrow \mathcal{R}(a, a_1, a_2) \tag{D$\circ$}$$

where the right part $\mathcal{R}(a, a_1, a_2)$ is a formula of the metalanguage containing (among others) the statements a, a_1, a_2 and as relations the deducibility relation $/$ as well as the defined relations $\vdash$ and $\not\,$. In the course of a logical argument, definitions of logical constants containing $\vdash$ and $\not\,$ can always be replaced by definitions that contain only $/$.[38] Popper calls (D$\circ$) an *explicit definition* of the connective $\circ$. But these explicit definitions are not always very handy to work with, so we will often consider only the right part of them, replacing a by $a_1 \circ a_2$ in $\mathcal{R}$:

$$\mathcal{R}(a_1 \circ a_2, a_1, a_2) \tag{C$\circ$}$$

This is called the *characterizing rule* (C$\circ$), which corresponds to the definition (D$\circ$).

4.2. *A remark on the use of the notation $a \circ b$*

Throughout his articles Popper uses notation like $a \wedge b$, $a \vee b$ and $a > b$ to speak about conjunctions, disjunctions and implications, respectively, of some object language $\mathcal{L}$. The use of this notation was criticized, for example, by *Kleene* (1948). *Lejewski* (1974, §§5–6) and *Schroeder-Heister* (1984, §3.1; 2006, §2.2) discuss the problem of the interpretation of $a \wedge b$ at some length. We add some further remarks, using $\wedge$ as a representative example.

The symbol $\wedge$ cannot be thought of as a symbol of the alphabet of $\mathcal{L}$, since Popper's approach does not specify anything about the object language $\mathcal{L}$ under consideration apart from the fact that it consists of statements. Neither can it be a kind of abstract replacement for a symbol of conjunction of $\mathcal{L}$, because he does not require that a language that contains a conjunction for any

[36] See *Popper 1948a* (p. 181); cf. *Cohen 1953* (p. 69f. and p. 208f.).

[37] See e.g. *Popper 1947a* (p. 286). In his terminology, logical constants are called *formative signs*, in distinction to what he calls *descriptive signs*, such as 'mountain' or 'elderly disgruntled newspaper reader'; see *Popper 1947a* (p. 257). According to *Popper* (*1947a*, p. 286), '*inferential definitions* […] are characterized by the fact that they define a formative sign by its logical force which is defined, in turn, by a *definition in terms of inference* (i.e. of "/").'

[38] Whenever $\vdash$ is replaced by its definiens, an appropriate choice for the number k of context statements has to be made.

two statements also contains a sign of conjunction. Indeed, he allows the use of expressions like $a \wedge b$ for languages that do not contain a sign for conjunction at all.

Reading $a \wedge b$ as a kind of definite description of the statement of $\mathcal{L}$ that is *the* conjunction of a and b does not work either, since object languages may not only have one conjunction for any two given statements a and b but several. In these cases the uniqueness condition of definite descriptions cannot be fulfilled.

Schroeder-Heister (*1984*, p. 85) proposes to read $a \wedge b$ as an abstract term denoting the equivalence class of conjunctions of a and b. But as this interpretation would not justify the use of $a \wedge b$ in a deducibility statement (e.g. $a \wedge b, c/d$), he goes on to say that in the context of such a deducibility statement the term $a \wedge b$ denotes an arbitrary representative of the equivalence class of conjunctions.

Elaborating on this idea we propose to read $a \wedge b$ as an ε-term as used in the epsilon calculus of Hilbert, i.e., $a \wedge b$ is defined as $\varepsilon c \mathcal{R}_{\wedge}(c, a, b)$. This means that $a \wedge b$ just picks one element, if it exists, of the equivalence class defined by the inferential definition of $\wedge$. To avoid non-denoting terms, we therefore demand that the use of $a \wedge b$ requires a theorem or postulate about the non-emptiness of the equivalence class of conjunctions of a and b. But once the availability of a conjunction for any two statements a and b is guaranteed, the use of $a \wedge b$ and the instantiation of universally quantified formulas of the metalanguage with terms of the form $a \wedge b$ is no longer problematic. Reading $a \wedge b$ as an ε-term also makes the existential presupposition explicit in cases where $a \wedge b$ is the result of an instantiation of a universal quantifier of the metalanguage.

4.3. *Popper's definitional criterion of logicality*

The question we are facing now is what form characterizing rules like $\mathcal{R}(a_1 \circ a_2, a_1, a_2)$ might be allowed to take. Should certain rules be disallowed because their use in a definition of form (D$\circ$) does not in fact define a logical constant, or does any rule $\mathcal{R}$ give rise to a definition of a logical constant?

One can find seemingly conflicting utterances on this question in Popper's articles. On the one hand, he writes that the definitions have to meet no restrictions whatsoever. He thus allows, for example, the following definition for 'opponent' (*opp*), with its characterizing rule:

$$a//opp(b) \leftrightarrow (c)(b/a \,\&\, a/c) \tag{Dopp}$$

$$(c)(b/opp(b) \,\&\, opp(b)/c) \tag{Copp}$$

This obviously trivializes any system, since it implies $(c)(b/c)$. But this does not lead Popper to reject (D*opp*) as a definition.[39] Historically, it is interesting to note that the connective *opp* is quite similar to the connective *tonk*, which was later introduced into the philosophical discussion by Prior.[40]

Lejewski (*1974*, p. 644) considers an even stronger version of *opp*, which is also discussed by *Schroeder-Heister* (*1984*, p. 89), where it is called opp^*. This supposed logical constant not only turns the object language inconsistent but also the metalanguage. Its definition and characterizing

[39] See *Popper 1947a* (p. 284): 'We need not make sure, in any other way, that our system of definitions is consistent. For example, we may define (introducing an arbitrarily chosen name "opponent"): [see (D*opp*)]. This definition has the consequence that every language which has a sign for "opponent of *b*" [...] will be inconsistent [...]. But this need not lead us to abandon [(D*opp*)]; it only means that no consistent language will have a sign for "opponent of *b*".'

[40] *Prior* (*1960*) cites *Popper 1947a*, but only for giving an example of what he understands by an analytically valid inference. He does not mention the fact that *Popper* (*1947a*) discusses the *tonk*-like connective *opp*. Neither did *Belnap* (*1962*) and *Stevenson* (*1961*) notice this in their responses to Prior's article, even though they both also explicitly mention Popper's article. It is noted, however, by *Merrill* (*1962*).

rule is supposed to be:

$$a//opp^*(b) \leftrightarrow (c)(b/a \,\&\, \neg(b/c)) \tag{Dopp^*}$$

$$(c)(b/opp^*(b) \,\&\, \neg(b/c)) \tag{Copp^*}$$

The definition (Dopp^*) is formulated with metalinguistic negation ($\neg$). This contrasts with all definitions considered by Popper, who never uses negation in his metalinguistic definitions of logical constants. So while it is true that the definition of opp^* would turn any metalanguage inconsistent in which it is stated, this criticism cannot be applied to the system of Popper. Furthermore, in view of Lemma 2.2, there can be no such trivializing definition in the restricted metalanguage specified in Section 2.3.

On the other hand, there is one condition that Popper seems to consider to be essential for any definition of a logical constant, namely uniqueness.[41] He uses the expression 'fully characterizing rule' to single out those rules that allow us to establish that any two statements satisfying the rule are interdeducible.[42] In other words, fully characterizing rules are exactly those rules that satisfy uniqueness. They can be defined as follows:

Definition 4.1: A rule $\mathcal{R}(c, a_1, \ldots, a_n)$ is *fully characterizing* if, and only if

$$(\mathcal{R}(a, a_1, \ldots, a_n) \,\&\, \mathcal{R}(b, a_1, \ldots, a_n)) \rightarrow a//b.$$

In other words, if, and only if, a rule $\mathcal{R}$ characterizes a statement c up to mutual deducibility (i.e. if c is unique), then $\mathcal{R}$ is fully characterizing c.

It is the existence of fully characterizing rules that distinguishes logical constants from non-logical constants, and it is this criterion of logicality that leads Popper to reject e.g. minimal negation as a logical constant (see Section 6.5). For an extensive discussion of Popper's criterion of logicality we refer to *Schroeder-Heister 1984, 2006*.

Another question concerning the form of definitions of logical constants is the following: For two given alternative rules, say $\mathcal{R}$ and $\mathcal{R}'$, that are equivalent, is one preferable over the other in defining a logical constant?

Popper often considers more than just one possible definition of a logical constant, showing (or in some cases only indicating) that these alternative definitions are equivalent. There seems to be no logical or philosophical criterion that makes Popper prefer one definition rather than another, equivalent definition. Often, it seems, he just prefers the brevity of some definition, or the ease with which he is able to explain it.[43]

This contrasts with more modern approaches concerning definitions of logical constants. Especially in the tradition of Gentzen, Prawitz and Dummett there is a strong sense of preferring one special kind of rules (either introduction rules or elimination rules) as being constitutive of the meaning of logical constants. Such considerations play no part in Popper's definitions of logical constants. For example, *Popper's* (*1947b*, p. 228) definition of the universal quantifier resembles its elimination rule, while his definition of the existential quantifier resembles its introduction rule.

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[41] Cf. the discussion in *Schroeder-Heister 2006* (p. 20) on the uniqueness condition (1.4) and the existence condition (1.5). According to *Schroeder-Heister* (*2006*, p. 31), 'Popper's theory is a *radical* structuralist theory in that just the inferential role of logical compounds is uniquely described without any further constraints.'

[42] See *Popper 1948b* (p. 324).

[43] This is in contrast to the theory of Koslow as presented by *Schroeder-Heister* (*2006*, p. 21): 'In Koslow's theory inferential characterizations [. . .] have a specific syntactic form. They provide inferential conditions corresponding to elimination rules in natural deduction and then require [the characterized statement] to be the weakest object satisfying these conditions.'

But the fact that Popper does not have a philosophical criterion for preferring one form of definition over another does not preclude the possibility that he has a logical one. In the more mature version of his theory presented in *Popper 1948a,b*, he sometimes states definitions of logical constants in a form that highlights the duality of certain of his definitions. We will follow this lead and give the definitions in a way that allows for the formulation of a duality function that transforms any definition of a logical constant into a definition of a dual logical constant.

5. Popper's notion of duality

Although Popper makes frequent use of duality,[44] he nowhere gives an explanation of his notion of duality. We think such an explanation is needed, since Popper does not only discuss duality in the context of classical logic, where the well-known duality based on truth functions can be applied, but also in the context of non-classical logics such as intuitionistic logic, where a different notion of duality is called for.

We propose to understand his notion of duality as being based on his concept of relative demonstrability ($\vdash$). More precisely, an inferential definition is said to be dual to another inferential definition, if it results from exchanging all statements on the left side of $\vdash$ with the statements on its right side, i.e. by transforming $a_1, \ldots, a_n \vdash b_1, \ldots, b_m$ into $b_1, \ldots, b_m \vdash a_1, \ldots, a_n$.[45] In the case of binary connectives we have to swap the arguments to produce its dual.[46] We make this understanding of the notion of duality precise by using a duality function, defined as follows:

Definition 5.1: Let $\star$ be a unary connective and $\circ$ a binary connective. The *duality function* δ is defined by the following clauses (where Γ and Π are lists of statements; Γ^δ means that δ is applied to each member of Γ):

$$a^\delta =_{\text{df}} a, \quad (\star a)^\delta =_{\text{df}} \star^\delta a^\delta, \quad (a \circ b)^\delta =_{\text{df}} b^\delta \circ^\delta a^\delta \quad \text{and} \quad (\Gamma \vdash \Pi)^\delta =_{\text{df}} \Pi^\delta \vdash \Gamma^\delta.$$

There are no clauses for $/$ and $/\!/$. This does not restrict the range of applicability of δ, since $/$ can always be replaced by $\vdash$ (cf. Lemma 3.5). In the following we show that the function δ maps definitions of logical constants to definitions of what Popper considers to be their duals. We do this for conjunction and disjunction, conditional and anti-conditional, and for modal connectives.

5.1. *Conjunction and disjunction*

As was already mentioned, Popper gives several characterizing rules for conjunction ($\wedge$). We choose the following definition:

$$a /\!/ b \wedge c \leftrightarrow (d)(a \vdash d \leftrightarrow b, c \vdash d) \tag{D$\wedge$}$$

[44] He uses expressions like '… is the dual of …' and '… is a kind of dual to …', or speaks of a 'dual notion / rule / definition'.

[45] Cf. *Kapsner 2014* (p. 76), who also mentions this possibility. There is, however, a difference between the semantical concept $\models$ that Kapsner considers and Popper's defined concept $\vdash$.
Note that it is in general not possible to invert the direction of the sign of derivability ($/$), for it allows multiple statements on the left but only one statement on the right (for alternative versions see Appendix A). As already mentioned in Section 3.1, the replacement of $/$ by $\vdash$ is allowed. One can therefore formulate definitions of logical constants in terms of relative demonstrability instead of deducibility. This allows us to make the duality of logical constants obvious. To our knowledge, Popper mentions this duality based on relative demonstrability only once, in *Popper 1948a* (p. 181): 'For certain purposes — especially if we wish to emphasize the duality or symmetry between "$\vdash$" and "7" — the use of "$(\ldots)\vdash$" turns out to be preferable to that of "$7(\ldots)$".'

[46] Our definition of duality resembles *Cohen's* (*1953*, p. 82): 'We define the dual of a postulate of any Gentzen-type system as the result of writing each sequent $\mathrm{p}_1, \ldots, \mathrm{p}_{\underline{\mathrm{m}}} \Vdash \mathrm{q}_1, \ldots, \mathrm{q}_{\underline{\mathrm{n}}}$ as $\mathrm{q}_{\underline{\mathrm{n}}}, \mathrm{q}_{\underline{\mathrm{n}}-1}, \ldots, \mathrm{q}_1 \Vdash \mathrm{p}_{\underline{\mathrm{m}}}, \mathrm{p}_{\underline{\mathrm{m}}-1}, \ldots, \mathrm{p}_1$, of interchanging $\underline{\mathrm{K}}\,\mathrm{p}\,\mathrm{q}$ with $\underline{\mathrm{A}}\,\mathrm{q}\,\mathrm{p}$ (hence also interchanging $\underline{\mathrm{A}}\,\mathrm{p}\,\mathrm{q}$ with $\underline{\mathrm{K}}\,\mathrm{q}\,\mathrm{p}$), of interchanging $\underline{\mathrm{F}}\,\mathrm{p}\,\mathrm{q}$ with $\underline{\mathrm{G}}\,\mathrm{q}\,\mathrm{p}$ (hence also interchanging $\underline{\mathrm{G}}\,\mathrm{p}\,\mathrm{q}$ with $\underline{\mathrm{F}}\,\mathrm{q}\,\mathrm{p}$) […], and of reversing the order of premises whenever the postulate is a rule of inference having two premises.' Here $\underline{\mathrm{K}}$ stands for conjunction, $\underline{\mathrm{A}}$ for disjunction ('alternative'), $\underline{\mathrm{F}}$ for the conditional, and $\underline{\mathrm{G}}$ for the anti-conditional.

$$b \wedge c \vdash d \leftrightarrow b, c \vdash d \tag{C$\wedge$}$$

If we apply our duality function δ to the characterizing rule (C$\wedge$), we obtain:

$$d \vdash c \wedge^{\delta} b \leftrightarrow d \vdash b, c \tag{(C$\wedge$)$^\delta$}$$

which is equivalent to Popper's definition of disjunction ($\vee$):[47]

$$a /\!/ b \vee c \leftrightarrow (d)(d \vdash a \leftrightarrow d \vdash b, c) \tag{D$\vee$}$$

$$d \vdash b \vee c \leftrightarrow d \vdash b, c \tag{C$\vee$}$$

We immediately obtain the following introduction and elimination rules for conjunction and disjunction:

Lemma 5.2: *The following rules for conjunction and disjunction hold:*

(1) $a \wedge b / a$	*(4)* $a / a \vee b$
(2) $a \wedge b / b$	*(5)* $b / a \vee b$
(3) $a, b / a \wedge b$	*(6)* $(c)((a/c \;\&\; b/c) \rightarrow a \vee b / c)$

Proof For (1) consider the substitution instance $a \wedge b \vdash a \leftrightarrow a, b \vdash a$ of (C$\wedge$); $a, b \vdash a$ holds by (Rg). Thus $a \wedge b \vdash a$. Rules (2)–(5) are shown analogously. For (6) consider the substitution instance $a \vee b \vdash a \vee b \leftrightarrow a \vee b \vdash a, b$ of (C$\vee$); $a \vee b \vdash a \vee b$ holds by (Rg). Thus $a \vee b \vdash a, b$. From definition (D$\vdash$3) we get $(c)((a/c \;\&\; b/c) \rightarrow a \vee b/c)$. □

Lemma 5.3: *Conjunction and disjunction can be introduced and eliminated within contexts, i.e. the following equivalences hold:*

$$a_1, \dots, a_n, b, c \vdash d \leftrightarrow a_1, \dots, a_n, b \wedge c \vdash d$$

$$a_1, \dots, a_n \vdash b_1, \dots, b_m, c, d \leftrightarrow a_1, \dots, a_n \vdash b_1, \dots, b_m, c \vee d$$

Proof For the first equivalence consider the following two instances of (Tg):

$$\left\{\begin{array}{l} \quad a_1, \dots, a_n, b \wedge c \vdash a_1 \\ \& \; a_1, \dots, a_n, b \wedge c \vdash a_2 \\ \vdots \qquad\qquad \vdots \\ \& \; a_1, \dots, a_n, b \wedge c \vdash a_n \\ \& \; a_1, \dots, a_n, b \wedge c \vdash b \\ \& \; a_1, \dots, a_n, b \wedge c \vdash c \end{array}\right\} \rightarrow (a_1, \dots, a_n, b, c \vdash d \rightarrow a_1, \dots, a_n, b \wedge c \vdash d)$$

$$\left\{\begin{array}{l} \quad a_1, \dots, a_n, b, c \vdash a_1 \\ \& \; a_1, \dots, a_n, b, c \vdash a_2 \\ \vdots \qquad\qquad \vdots \\ \& \; a_1, \dots, a_n, b, c \vdash a_n \\ \& \; a_1, \dots, a_n, b, c \vdash b \wedge c \end{array}\right\} \rightarrow (a_1, \dots, a_n, b \wedge c \vdash d \rightarrow a_1, \dots, a_n, b, c \vdash d)$$

For the second equivalence we first show the direction from left to right:

[47] Our definitions (D$\wedge$) and (D$\vee$) correspond to *Popper's* (*1948a*) rules (3.71) and (3.72), respectively.

(1) From $a_1, \ldots, a_n \vdash b_1, \ldots, b_m, c, d$ and two applications of (Cut) with $c/c \vee d$ and $d/c \vee d$ we obtain $a_1, \ldots, a_n \vdash b_1, \ldots, b_m, c \vee d, c \vee d$.
(2) By (RC) we obtain $a_1, \ldots, a_n \vdash b_1, \ldots, b_m, c \vee d$.

For the direction from right to left:

(1) Assume $a_1, \ldots, a_n \vdash b_1, \ldots, b_m, c \vee d$.
(2) This is equivalent to $(e)((b_1/e \;\&\; \ldots \;\&\; b_m/e \;\&\; c \vee d/e) \rightarrow a_1, \ldots, a_n/e)$.
(3) Assume b_1/e to b_m/e, c/e and d/e.
(4) From c/e and d/e follows $c \vee d/e$.
(5) From b_1/e to b_m/e and $c \vee d/e$ we get $a_1, \ldots, a_n/e$ from (2).

Thus $a_1, \ldots, a_n \vdash b_1, \ldots, b_m, c, d \leftrightarrow a_1, \ldots, a_n \vdash b_1, \ldots, b_m, c \vee d$. □

Lemma 5.4: *Conjunction and disjunction are logical constants, i.e. their rules are fully characterizing.*

Proof For conjunction we have to show that

$$((d)(a_1 \vdash d \leftrightarrow b, c \vdash d) \;\&\; (d)(a_2 \vdash d \leftrightarrow b, c \vdash d)) \rightarrow a_1 /\!/ a_2$$

is true. Assuming the antecedent, we substitute a_2 for d in both its conjuncts to obtain $a_1 \vdash a_2 \leftrightarrow b, c \vdash a_2$ and $a_2 \vdash a_2 \leftrightarrow b, c \vdash a_2$. By (Rg) we obtain $b, c \vdash a_2$ from the second conjunct, and with $b, c \vdash a_2$ we obtain $a_1 \vdash a_2$ from the first conjunct. Likewise for $a_2 \vdash a_1$. The proof for disjunction is similar. □

5.2. *Conditional and anti-conditional*

The definitions of the conditional and the anti-conditional. Again, we choose those formulations of the characterizing rules that highlight the duality of the definitions. For the conditional (>) this definition is:

$$a /\!/ b > c \leftrightarrow (d)(d \vdash a \leftrightarrow d, b \vdash c) \tag{D>}$$

$$d \vdash b > c \leftrightarrow d, b \vdash c \tag{C>}$$

If we dualize (C>), we obtain:

$$c >^{\delta} b \vdash d \leftrightarrow c \vdash d, b \tag*{(C>)$^\delta$}$$

which is equivalent to Popper's definition of the anti-conditional ($\ngtr$):[48]

$$a /\!/ b \ngtr c \leftrightarrow (d)(a \vdash d \leftrightarrow c \vdash d, b) \tag*{(D$\ngtr$)}$$

$$c \ngtr b \vdash d \leftrightarrow c \vdash d, b \tag*{(C$\ngtr$)}$$

As an informal observation we may state that the conditional is characterized by the deduction theorem and the anti-conditional by the dual of the deduction theorem.

On the basis of these definitions we can show that modus ponens holds for the conditional, and that a dual to modus ponens holds for the anti-conditional.

Lemma 5.5: *The following rules hold:*

[48] Our definitions (D>) and (D$\ngtr$) correspond to *Popper's* (*1948a*) rules (3.81) and (3.82), respectively.

(1) $b, b > c \vdash c$ *(modus ponens),*
(2) $c \vdash c \not> b, b$ *(a dual rule to modus ponens).*

Proof (1) In (C>) let d be $b > c$ to obtain $b > c \vdash b > c \leftrightarrow b > c, b \vdash c$. By (Rg), $b > c, b \vdash c$ follows.

(2) In (C$\not>$) let d be $c \not> b$ to obtain $c \not> b \vdash c \not> b \leftrightarrow c \vdash c \not> b, b$. By (Rg), $c \vdash c \not> b, b$ follows. □

Lemma 5.6: *Conditional and anti-conditional are logical constants, i.e. their rules are fully characterizing.*

Proof For the conditional we have to show that

$$((d)(d \vdash a_1 \leftrightarrow d, b \vdash c) \;\&\; (d)(d \vdash a_2 \leftrightarrow d, b \vdash c)) \rightarrow a_1 /\!/ a_2$$

is true. Assuming the antecedent, we instantiate d by a_1 in both its left and right conjunct. From the left one gets $a_1 \vdash a_1 \leftrightarrow a_1, b \vdash c$, and thus $a_1, b \vdash c$ by (Rg). From the right one gets $a_1 \vdash a_2 \leftrightarrow a_1, b \vdash c$, and thus $a_1 \vdash a_2$. Likewise for $a_2 \vdash a_1$. The proof for the anti-conditional is similar. □

Following Popper, we will use the terms ‘implication’ and ‘conditional’ interchangeably to refer either to the connective $>$ or to statements of the form $a > b$.

Classical implication in the absence of classical negation. As is well known, if implication is defined solely by the usual implication introduction and elimination rules of natural deduction, then the addition of classical negation by its natural deduction rules is a non-conservative extension of the logic. For example, Peirce’s law $((a > b) > a) > a$ is not derivable using only the introduction and elimination rules for implication, but becomes derivable if the rules of classical negation are added.[49]

Popper (1947b) already made this observation in the context of his characterizing rules.[50] In order to make statements such as Peirce’s law deducible without invoking negation, Popper considers the use of the following additional rule for the conditional:

$$a, b > c / b \leftrightarrow a / b \tag{4.2e}$$

Reading (4.2e) from left to right amounts to a characterization of implication by Peirce’s rule, i.e. a rule version of Peirce’s law.[51] This becomes more obvious if we replace $/$ by $\vdash$ and drop the context statement a[52]:

$$b > c \vdash b \rightarrow \; \vdash b \tag{Peirce's rule}$$

[49] See Dummett 1991 (p. 291f.); cf. de Campos Sanz, Piecha, and Schroeder-Heister 2014.

[50] He calls *positive logic* that part of propositional logic obtained by the definitions of the logical constants excluding negation, and remarks in Popper 1947b (p. 215): ‘Positive logic as defined by these rules does not yet contain all valid rules of inferences in which no use is made of negation: there is a further region which we may call the “extended positive logic” […].’ *Kleene’s* (1948) review of Popper 1947c cites Popper’s observation and turns it into a critique of Popper’s claim to give explicit definitions of the logical constants: ‘If his definitions were “explicit,” as he claims, it should make no difference whatsoever, in the case of a formula containing only “>”, whether or not the definition of classical negation has been stated.’ But it is clearly not enough to just ‘state’ the definition. It is the existence postulate which tells us that for every statement there exists a classical negation of it in the object language which permits the proof of the demonstrability of, for example, Peirce’s law to go through.

[51] Whether Popper was the first to consider Peirce’s rule is not completely clear. Seldin (2008, §3) ‘think[s …] that Popper and Curry thought of this rule independently.’ Note that Popper did not call his rule ‘Peirce’s rule’.

[52] Which is only present, we think, because in Popper 1947b the left side of $/$ must not be empty.

Popper also mentions that in the presence of (4.2e), the rules for classical negation can be obtained from the definitions of intuitionistic negation.[53]

We can therefore ascribe to Popper the following insights, which were far from trivial at the time: (1) The addition of classical negation to a logic containing implication defined by its usual introduction and elimination rules forms a non-conservative extension. (2) Alternatively, implication can be characterized by a stronger set of rules containing a version of Peirce's rule. (3) If this stronger set of rules for implication is used, then intuitionistic and classical negation coincide.

5.3. *The modal connectives*

Modal logic is introduced in *Popper 1947c* (§IX). Although modal logic is not our main focus, it is nevertheless necessary to introduce it here, because Popper uses modal connectives in his proof of the compatibility of intuitionistic and dual-intuitionistic logic, which will be dealt with in Section 6.

Popper considers the following six modal connectives: *necessary*, *impossible*, *logical*, *contingent*, *possible* and *uncertain*. They are taken from *Carnap 1947* (p. 175), and the definitions given for them by Popper are strongly influenced by Carnap's treatment of modality. Popper's definitions and characterizing rules for the modal connectives all have the following form:

$$a /\!/ \mathbf{M}b \leftrightarrow (\vdash a \bigvee \neg a) \;\&\; \mathcal{R}(a,b) \tag{DM}$$

$$(\vdash \mathbf{M}b \bigvee \neg \mathbf{M}b) \;\&\; \mathcal{R}(\mathbf{M}b,b) \tag{CM}$$

where $\mathbf{M}$ stands for any of the six modal connectives, and $\mathcal{R}(a,b)$ varies depending on the modal connective to be defined.

Table 1 combines the table given by *Carnap* (*1947*, p. 175)[54] and the list of definitions given by *Popper* (*1947c*, p. 1223). Carnap's half of the table contains his names for the modal connectives, their definition in terms of necessity $\Box$ and possibility $\Diamond$ and the semantic property to which they correspond. In Carnap's system every statement (or sentence, to use Carnap's terminology) falls into one of three disjoint categories. It can either be L-true, L-false or factual. A statement is called *L-determinate*, if it is either L-true or L-false. If a statement is not L-true, then it is either L-false or factual and so on. Popper's half of the table contains his names for the modal connectives, the corresponding symbol and the part $\mathcal{R}(\mathbf{M}b,b)$ of the definition that varies with the modal connectives.

If L-truth is matched with demonstrability and L-falsity with refutability, then the close correspondence between the semantic properties given by Carnap's and Popper's definitions becomes obvious. For example, in the case of a noncontingent (Carnap's terminology) or logical (Popper's terminology) statement, the semantic property of being L-determinate corresponds to the property of being either demonstrable or refutable according to Popper's definition.

We observe that the part $\vdash \mathbf{M}b \bigvee \neg \mathbf{M}b$ in Popper's definitions (**CM**) of the modal connectives $\mathbf{M}$ corresponds to the fact that in Carnap's system all modal statements (i.e. statements whose outermost logical constant is a modal constant) are L-determinate.[55] Carnap's modal logic is S5, which can be axiomatized by the axioms K, T and 5. The same is the case for Popper's modal logic, as the following theorem shows.

[53] See *Popper 1947b* (p. 216): 'Also we do not need, in the presence of rule 4.2e, the whole force of our rule of negation 4.6 [i.e. classical negation], but can obtain this rule as a secondary rule from some weaker rules [...] (from the rules of the so-called intuitionist logic).' *Popper* (*1947b*, p. 200) calls a rule *secondary* if it is derivable from a set of given so-called primary rules, i.e. if 'every inference which is asserted as valid by the secondary rule could be drawn merely by force of the primary rules alone.'

[54] We have replaced N by $\Box$ here.

[55] *Carnap* (*1947*, p. 174) mentions this explicitly for necessity.

Table 1. The modal connectives from Carnap and Popper.

Carnap			*Popper*		
Name	$\Box,\Diamond$*-Definition*	*Semantic property*	*Sign*	*Name*	$\mathcal{R}(\mathbf{M}b,b)$
Necessary	$\Box p$ $\neg\Diamond\neg p$	L-true	**N**	Necessary	$\vdash \mathbf{N}b \leftrightarrow \vdash b$
Impossible	$\Box\neg p$ $\neg\Diamond p$	L-false	**I**	Impossible	$\vdash \mathbf{I}b \leftrightarrow \, \urcorner b$
Contingent	$\neg\Box p \wedge \neg\Box\neg p$ $\Diamond\neg p \wedge \Diamond p$	factual	**C**	Contingent	$\urcorner\, \mathbf{C}b \leftrightarrow (\vdash b \bigvee \urcorner\, b)$
Non-necessary	$\neg\Box p$ $\Diamond\neg p$	not L-true	**U**	Uncertain	$\urcorner\, \mathbf{U}b \leftrightarrow \vdash b$
Possible	$\neg\Box\neg p$ $\Diamond p$	not L-false	**P**	Possible	$\urcorner\, \mathbf{P}b \leftrightarrow \, \urcorner\, b$
Noncontingent	$\Box p \vee \Box\neg p$ $\neg\Diamond\neg p \vee \neg\Diamond p$	L-determinate	**L**	Logical	$\vdash \mathbf{L}b \leftrightarrow (\vdash b \bigvee \urcorner\, b)$

Theorem 5.7 : *From the definitions* (D**N**) *and* (D>) *we can prove the S5 axioms K, T and 5.*

Proof We consider axiom 5, i.e. $\vdash \mathbf{N}b > \mathbf{N}\mathbf{N}b$, which is shown as follows (K and T can be shown similarly):

(1) $\vdash \mathbf{N}\mathbf{N}b \leftrightarrow \vdash \mathbf{N}b$, from (D**N**).
(2) $\vdash \mathbf{N}b \bigvee \urcorner\, \mathbf{N}b$, from (D**N**).
(3) If $\vdash \mathbf{N}b$, then $\vdash \mathbf{N}\mathbf{N}b$, from (1).
(4) If $\vdash \mathbf{N}\mathbf{N}b$, then $\mathbf{N}b \vdash \mathbf{N}\mathbf{N}b$, by (LW).
(5) If $\urcorner\, \mathbf{N}b$, then $(c)\,\mathbf{N}b \vdash c$, hence $\mathbf{N}b \vdash \mathbf{N}\mathbf{N}b$, by instantiating c by $\mathbf{N}\mathbf{N}b$.
(6) Therefore $\mathbf{N}b \vdash \mathbf{N}\mathbf{N}b$, from (2), (3) and (5).
(7) Therefore $\vdash \mathbf{N}b > \mathbf{N}\mathbf{N}b$, from (6) and (D>).

The proof can be generalized for all modal connectives **M** defined by Popper, i.e. $\vdash \mathbf{M}b > \mathbf{N}\mathbf{M}b$ holds for any **M**. □

The following result will be important in Section 6.4, where it is used to show that the interpretation of intuitionistic negation by **I** and of dual-intuitionistic negation by **U** works.

Theorem 5.8 : *Uncertainty* **U** *and impossibility* **I** *are dual modal notions.*

Proof By applications of the duality function δ (Definition 5.1). □

Each of the considered modal connectives is a logical constant in Popper's sense. We show this for **N** as an example:

Lemma 5.9: **N** *is a logical constant, i.e. its rule is fully characterizing.*

Proof We have to show that

$$((\vdash a_1 \bigvee \urcorner\, a_1)\ \&\ (\vdash a_1 \leftrightarrow \vdash b)\ \&\ (\vdash a_2 \bigvee \urcorner\, a_2)\ \&\ (\vdash a_2 \leftrightarrow \vdash b)) \to a_1 /\!/ a_2$$

is true. Assume $\vdash a_1$. From $\vdash a_1 \leftrightarrow \vdash b$ we then get $\vdash b$. From $\vdash b$ and $\vdash a_2 \leftrightarrow \vdash b$ we get $\vdash a_2$. From $\vdash a_2$ we get $a_1 \vdash a_2$. Assume $\urcorner\, a_1$. Hence $(c)(a_1/c)$, and by instantiating c by a_2 we get a_1/a_2. Therefore $a_1 \vdash a_2$. The proof of a_2/a_1 is similar. □

6. Popper's theory of negations

Popper considered several different kinds of negation. We discuss their definitions and investigate how they relate to each other. The definitions are taken from *Popper 1948b* unless indicated otherwise.[56] Some of the following results were only stated without proof by Popper, and for some he gave only proof sketches. We will provide more details and fill in the missing proofs.

6.1. *Classical negation*

For classical negation ($\neg_k$) Popper considers several definitions, among them the following two:[57]

$$a /\!/ \neg_k b \leftrightarrow (a,b \vdash \ \& \ \vdash a,b) \tag{D$\neg_k$1}$$

$$a /\!/ \neg_k b \leftrightarrow (c)(d)(d,a \vdash c \leftrightarrow d \vdash b,c) \tag{D$\neg_k$2}$$

with the following two characterizing rules:

$$\neg_k b, b \vdash \ \& \ \vdash \neg_k b, b \tag{C$\neg_k$1}$$

$$(c)(d)(d, \neg_k b \vdash c \leftrightarrow d \vdash b,c) \tag{C$\neg_k$2}$$

These two definitions reflect two different ways of characterizing classical negation. Definition (D$\neg_k$1) is based on the idea that the classical negation of a statement b is a statement which is at the same time complementary and contradictory to b, whereas (D$\neg_k$2) is very similar to the rules for negation used in classical sequent calculus.

Lemma 6.1: *The definitions* (D$\neg_k$1) *and* (D$\neg_k$2) *are equivalent.*

Proof We consider the characterizing rules, and show first that (C$\neg_k$1) follows from (C$\neg_k$2):

In $(c)(d)(d,\neg_k b \vdash c \leftrightarrow d \vdash b,c)$ let d be b to obtain $(c)(b,\neg_k b \vdash c \leftrightarrow b \vdash b,c)$. By (Rg) one obtains $(c)(b,\neg_k b \vdash c)$, i.e. $b,\neg_k b \vdash$; hence $\neg_k b, b \vdash$ by (LE). In $(c)(d)(d,\neg_k b \vdash c \leftrightarrow d \vdash b,c)$ let c be $\neg_k b$ to obtain $(d)(d,\neg_k b \vdash \neg_k b \leftrightarrow d \vdash b,\neg_k b)$. By (Rg) one obtains $(d)(d \vdash b,\neg_k b)$, i.e. $\vdash b,\neg_k b$; hence $\vdash \neg_k b, b$ by (RE).

To show that (C$\neg_k$2) follows from (C$\neg_k$1) it is sufficient to show that the two implications $\vdash \neg_k b, b \rightarrow (d,\neg_k b \vdash c \rightarrow d \vdash b,c)$ and $\neg_k b, b \vdash \rightarrow (d \vdash b,c \rightarrow d,\neg_k b \vdash c)$ hold. This can be done by using (Cut). To show the first implication we assume $\vdash \neg_k b, b$, from which we obtain $d \vdash \neg_k b, b$ by (LW). Assuming $d,\neg_k b \vdash c$ we get $d,d,b \vdash c$ by (Cut), from which we get $d,b \vdash c$ by (LC). The second implication is a direct instance of (Cut), where the cut formula is the statement b. □

The use of (Cut) can be justified by using our definition (D$\vdash$3) for relative demonstrability ($\vdash$) instead of Popper's definition (D$\vdash$2), or by assuming that the object language contains conjunction ($\wedge$), disjunction ($\vee$) and the conditional ($>$); see Appendix B.[58] We take the first option here, and presuppose our definition (D$\vdash$3). The rule (Cut) is discussed in Appendix B.

Lemma 6.2: *Classical negation* $\neg_k$ *is self-dual.*

Proof By applying the duality function δ to (D$\neg_k$1). □

[56] Popper's notation for negations varies; we write $\neg_k a$, for example, where Popper would e.g. write a^k.

[57] In *Popper 1948b*, (D$\neg_k$1) is (D4.3), and (D$\neg_k$2) is (4.31).

[58] Popper states that (D$\neg_k$1) and (D$\neg_k$2) are equivalent if the existence of either conjunction or disjunction is guaranteed for arbitrary statements.

6.2. *Intuitionistic negation and dual-intuitionistic negation*

The study of formalized intuitionistic logic started with Heyting's set of axioms.[59] Originally, Heyting's formalization was written in response to a prize question proposed by Mannoury in 1927, which also asked 'to investigate whether from the system to be constructed [for intuitionistic logic] a dual system may be obtained by (formally) interchanging the *principium tertii exclusi* and the *principium contradictionis*.'[60] Thus the idea of also somehow dualizing intuitionistic logic was present from the very beginning of its development.[61] The principium tertii exclusi is in Popper's theory mirrored by the property of a negation of b to be complementary to b, and the principium contradictionis is mirrored by the contradictoriness of b and the negation of b. Popper uses the term 'minimum definable negation' when he writes about what we call dual-intuitionistic negation ($\neg_m$), but still mentions the fact that it forms some kind of dual to intuitionistic negation.[62] For intuitionistic negation ($\neg_i$), Popper gives the following definition and characterizing rule:[63]

$$a /\!/ \neg_i b \leftrightarrow (c)(c \vdash a \leftrightarrow c, b \vdash) \tag{D$\neg_i$}$$

$$c \vdash \neg_i b \leftrightarrow c, b \vdash \tag{C$\neg_i$}$$

If we dualize (D$\neg_i$), we get

$$a /\!/ \neg_i{}^{\delta} b \leftrightarrow (c)(a \vdash c \leftrightarrow \; \vdash c, b) \tag{D$\neg_i$)$^\delta$ \text{(}$$

which is identical to Popper's definition for dual-intuitionistic negation ($\neg_m$):[64]

$$a /\!/ \neg_m b \leftrightarrow (c)(a \vdash c \leftrightarrow \; \vdash c, b) \tag{D$\neg_m$}$$

$$\neg_m b \vdash c \leftrightarrow \; \vdash c, b \tag{C$\neg_m$}$$

Lemma 6.3: *For intuitionistic negation $b, \neg_i b \vdash$ holds, and for dual-intuitionistic negation $\vdash b, \neg_m b$ holds.*

Proof In (C$\neg_i$) let c be $\neg_i b$; by (Rg) and (LE) we get $b, \neg_i b \vdash$. In (C$\neg_m$) let c be $\neg_m b$; by (Rg) and (RE) we get $\vdash b, \neg_m b$. □

Lemma 6.4: *Intuitionistic negation $\neg_i$ and dual-intuitionistic negation $\neg_m$ are logical constants, i.e. their rules are fully characterizing.*

Proof For intuitionistic negation we have to show that

$$((c)(c \vdash a_1 \leftrightarrow c, b \vdash) \;\&\; (c)(c \vdash a_2 \leftrightarrow c, b \vdash)) \rightarrow a_1 /\!/ a_2$$

is true. In both conjuncts let c be a_1 to obtain $a_1 \vdash a_1 \leftrightarrow a_1, b \vdash$ and $a_1 \vdash a_2 \leftrightarrow a_1, b \vdash$. From the first conjunct we get $a_1, b \vdash$ by (Rg), and from $a_1, b \vdash$ and the second conjunct we obtain $a_1 \vdash a_2$. The proof of $a_2 \vdash a_1$ is similar.

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[59] See *Heyting 1930*.

[60] Quoted from *Troelstra 1990* (p. 4).

[61] Cf. *Kapsner 2014* (p. 128).

[62] See *Popper 1948b* (p. 323). *Cohen* (*1953*, p. 188) remarks that '[t]he negation functor in the dualintuitionistic restricted predicate calculus GL2 [developed by Cohen] has the same properties as Popper's "minimum definable (non-modal) negation [$\neg_m$]."'

[63] See *Popper 1948b* (def. (D4.1)).

[64] See *Popper 1948b* (def. (D4.2)).

For dual-intuitionistic negation we have to show that

$$((c)(a_1 \vdash c \leftrightarrow \ \vdash c, b) \ \& \ (c)(a_2 \vdash c \leftrightarrow \ \vdash c, b)) \rightarrow a_1 /\!/ a_2$$

is true. In both conjuncts let c be a_1 to obtain $a_1 \vdash a_1 \leftrightarrow \ \vdash a_1, b$ and $a_2 \vdash a_1 \leftrightarrow \ \vdash a_1, b$. From the first conjunct we obtain $\vdash a_1, b$ by (Rg), and from $\vdash a_1, b$ and the second conjunct we get $a_2 \vdash a_1$. The proof of $a_1 \vdash a_2$ is similar. □

6.3. *Non-conservative language extensions*

Popper (1948b, §V) considers non-conservative language extensions.[65] An example is the addition of classical negation $\neg_k$ to a language containing both intuitionistic negation $\neg_i$ and dual-intuitionistic negation $\neg_m$. Due to the following theorem this addition is a non-conservative extension, since it makes classical laws hold for the two weaker negations $\neg_i$ and $\neg_m$.

Theorem 6.5 : *In the presence of $\neg_k$ we have $\neg_k a /\!/ \neg_i a$, $\neg_k a /\!/ \neg_m a$ and $\neg_i a /\!/ \neg_m a$. In other words, the three negations $\neg_k$, $\neg_i$ and $\neg_m$ collapse (i.e. they become synonymous).*

Proof Classical negation $\neg_k$ satisfies the rules for $\neg_i$ and $\neg_m$, i.e. we have $a \vdash \neg_k b \leftrightarrow a, b \vdash$ and $\neg_k a \vdash b \leftrightarrow \ \vdash a, b$, respectively. Both equivalences are direct consequences of (C$\neg_k$1) and (C$\neg_k$2), by using (Cut). Since the rules for $\neg_i$ and $\neg_m$ are fully characterizing, we have $\neg_k a /\!/ \neg_i a$ and $\neg_k a /\!/ \neg_m a$. Hence also $\neg_i a /\!/ \neg_m a$, for any object language containing $\neg_k$, $\neg_i$ and $\neg_m$. □

Popper (1948b, p. 324) also considers the more general situation where two logical functions S_1 and S_2 have been introduced by sets of primitive rules R_1 and R_2, respectively, such that $R_2 \subset R_1$. If both S_1 and S_2 are definable, and S_1 is given, then one can show that S_1 and S_2 are equivalent.

This can be generalized further, since R_2 need not be a subset of R_1; it is sufficient that R_1 implies R_2. Consider the following setting[66] with two fully characterizing rules $\mathcal{R}_1$ and $\mathcal{R}_2$ for two n-ary constants such that $\mathcal{R}_1$ implies $\mathcal{R}_2$:

$$\mathcal{R}_1(a, a_1, \ldots, a_n) \tag{1}$$

$$\mathcal{R}_2(a, a_1, \ldots, a_n) \tag{2}$$

$$\mathcal{R}_1(a, a_1, \ldots, a_n) \rightarrow \mathcal{R}_2(a, a_1, \ldots, a_n) \tag{3}$$

Now assume that $\mathcal{R}_2(b, a_1, \ldots, a_n)$ holds. From (1) and (3) we get $\mathcal{R}_2(a, a_1, \ldots, a_n)$, which implies $a /\!/ b$, since we have fully characterizing rules. Hence $\mathcal{R}_1(b, a_1, \ldots, a_n)$. The two characterized constants become thus synonymous; in other words, adding $\mathcal{R}_1$ yields a non-conservative extension of systems containing $\mathcal{R}_2$.

Popper's treatment of conservativeness also throws some light on his logical approach in general. Based on the fact that Popper does not use conservativeness as a criterion for accepting characterizing rules, Schroeder-Heister (2006, §3) argues that Popper is not aiming at a semantic justification of logical theories, since from a semantic theory we expect that the introduction of a new constant is always a conservative extension. Instead, Popper's theory is rather a means to metalinguistically describe logical theories.

[65] Without using these terms. See also Popper 1947a (p. 282, fn 20), where it is observed that 'if, in one language, a classical as well as an intuitionistic negation exists of every statement, then the latter becomes equivalent to the former, or in other words, classical negation then absorbs or assimilates its weaker kin.' Moreover, he observes that this does not happen if classical negation is put together with minimal negation (or with yet another negation, 'the impossibility of *b*', proposed by him ibid., p. 283). However, minimal negation is not a logical constant in Popper's sense (cf. Theorem 6.20).

[66] See Schroeder-Heister 2006 (§3).

6.4. *The compatibility of intuitionistic and dual-intuitionistic negation*

The logical constants of intuitionistic and dual-intuitionistic negation are compatible in the sense that there is a logic containing both, without them collapsing into classical negation. A proof of this result was sketched by Popper (1948b, §V). We give a full proof in what follows. It consists in the exposition of the logic $\mathcal{L}_1$, for which we show that it has the following, desired properties:

- It satisfies the rules of Basis I.
- It contains for any two statements a and b also $a \wedge b$, $a \vee b$, $a > b$, $a \ngtr b$, $\mathbf{I}\,a$ and $\mathbf{U}\,a$.
- It contains for every statement a its intuitionistic negation $\neg_i\, a$ and its dual-intuitionistic negation $\neg_m\, a$.
- It satisfies all the inferential definitions of the logical connectives it contains.
- Both $\neg_i\, a /\!/ \mathbf{I}\,a$ and $\neg_m\, a /\!/ \mathbf{U}\,a$ holds.
- The duals $\mathbf{I}\,a$ and $\mathbf{U}\,a$ (see Theorem 5.8) do not collapse.

Definition 6.6: The *logic* $\mathcal{L}_1$ is three-valued with truth-values d, c and r.[67] It contains a statement s with constant truth-value c. The truth-values of the compound statements of $\mathcal{L}_1$ and of s are given by the following truth-tables; for completeness, we add the truth-tables for $\mathbf{N}, \mathbf{P}, \mathbf{L}$ and $\mathbf{C}$, which are not included in Popper 1948b:

a	b	$a \wedge b$	$a \vee b$	$a > b$	$a \ngtr b$
d	d	d	d	d	r
d	c	c	d	c	d
d	r	r	d	r	d
c	d	c	d	d	r
c	c	c	c	d	r
c	r	r	c	r	c
r	d	r	d	d	r
r	c	r	c	d	r
r	r	r	r	d	r

s
c

a	$\mathbf{I}\,a$	$\mathbf{U}\,a$
d	r	r
c	r	d
r	d	d

a	$\mathbf{N}\,a$	$\mathbf{P}\,a$	$\mathbf{L}\,a$	$\mathbf{C}\,a$
d	d	d	d	r
c	r	d	r	d
r	r	r	d	r

A deducibility relation $a_1, \ldots, a_n / b$ is $\mathcal{L}_1$*-valid*, if the following conditions hold:

- If $a_1 \wedge \ldots \wedge a_n$ has truth-value d, then b must have truth-value d as well.
- If $a_1 \wedge \ldots \wedge a_n$ has truth-value c, then b must have either truth-value c or truth-value d.
- If $a_1 \wedge \ldots \wedge a_n$ has truth-value r, then b can have any truth-value.

The truth-values d, c and r of $\mathcal{L}_1$ reflect Carnap's tripartition into demonstrable, factual (contingent), and refutable statements, respectively. That is, if d is read as L-true, c as factual in the sense of Carnap, and r as L-false, then the given truth-tables for $\mathcal{L}_1$ are an adequate semantics for this reading.

Lemma 6.7: *$\mathcal{L}_1$ satisfies the rules of Basis I.*

Proof That $\mathcal{L}_1$ satisfies (Rg) and (Tg) is a direct consequence of the definition of $\mathcal{L}_1$-validity. □

Lemma 6.8: *The following propositions are true:*

(1) A statement of $\mathcal{L}_1$ is demonstrable if, and only if, it has the value d *for all valuations.*
(2) A demonstrable statement exists in $\mathcal{L}_1$.
(3) A statement is refutable if, and only if, it has the value r *for all valuations.*

[67] Popper (1948b) uses the truth-values 1, 2 and 3, which are here replaced by d, c and r, respectively.

(4) A refutable statement exists in $\mathcal{L}_1$.
(5) It is $a//b$ if, and only if, the value of a is identical with the value of b.

Proof (1), (3) and (5) follow directly from the definitions of demonstrability and refutability (see Sections 3.2 and 3.3), together with the interpretation of deducibility in $\mathcal{L}_1$. A witness for (2) is the demonstrable statement $a > a$. A witness for (4) is the refutable statement $a \not> a$. □

Lemma 6.9: *The logical constants in $\mathcal{L}_1$ satisfy their respective inferential definitions* (D∧)*,* (D∨)*,* (D>)*,* (D$\not>$)*,* (D**I**) *and* (D**U**)*.*

Proof This is only shown for **U**, the other cases are analogous. The characterizing rule for **U** is:

$$(\vdash \mathbf{U}b \bigvee \urcorner \mathbf{U}b) \; \& \; (\urcorner \mathbf{U}b \leftrightarrow \; \vdash b)$$

That the left conjunct is satisfied can be seen by looking at the truth-table for **U**, which contains only either d or r. If $\mathbf{U}a$ has the truth-value d, then it is demonstrable, and if it has the truth-value r, then it is refutable.

For the right conjunct, we consider first the part $\urcorner \mathbf{U}b \to \; \vdash b$, which is also satisfied: If $\mathbf{U}b$ is refutable, then it has the value r, and if it has the value r, then b must have the value d, thus being demonstrable. The remaining part $\vdash b \to \urcorner \mathbf{U}b$ is also satisfied: If b is demonstrable, then it has the value d; hence $\mathbf{U}b$ has the value r, thus being refutable. □

Lemma 6.10: $\mathbf{I}a$ *and* $\mathbf{U}a$ *do not collapse in* $\mathcal{L}_1$.

Proof This is guaranteed by the existence of the statement s with truth-value c. The value of $\mathbf{U}s$ is d, and the value of $\mathbf{I}s$ is r. Therefore it cannot be the case that $\mathbf{U}s // \mathbf{I}s$, by Lemma 6.8. □

Lemma 6.11: *From the definitions* (D$\neg_i$)*,* (D$\neg_m$)*,* (D>)*,* (D$\not>$)*,* (D**I**) *and* (D**U**) *we can show that* $\neg_i a // a > \mathbf{I}a$ *and* $\neg_m a // \mathbf{U}a \not> a$.

Proof For the proof of $\neg_i a // a > \mathbf{I}a$ we have to show that $(c)(c \vdash b > \mathbf{I}b \to c, b \vdash)$ and $(c)(c, b \vdash \to c \vdash b > \mathbf{I}b)$, which is just an instance of (D$\neg_i$). The latter is shown as follows:

(1) Assume $c, b \vdash$.
(2) Therefore $c, b \vdash \mathbf{I}b$, by (RW).
(3) Therefore $c \vdash b > \mathbf{I}b$, by (C>).
(4) Therefore $(c)(c, b \vdash \; \to c \vdash b > \mathbf{I}b)$.

The proof of $(c)(c \vdash b > \mathbf{I}b \to c, b \vdash)$ is:

(1) Assume $c \vdash b > \mathbf{I}b$.
(2) Therefore $c, b \vdash \mathbf{I}b$, by (C>).
(3) $\vdash \mathbf{I}b \bigvee \mathbf{I}b \vdash$, by (C**I**). We argue by cases.
(4) Assume $\mathbf{I}b \vdash$.
(5) Therefore $c, b \vdash$, by (2), (4) and (Tg).
(6) Assume $\vdash \mathbf{I}b$.
(7) Therefore $b \vdash$, by (C**I**).
(8) Therefore $c, b \vdash$, by (LW).
(9) Therefore $c, b \vdash$, by (3), (5) and (8).
(10) Therefore $(c)(c \vdash b > \mathbf{I}b \to c, b \vdash)$.

Analogously for the proof of $\neg_m a // \mathbf{U}a \not> a$. □

Lemma 6.12: *In $\mathcal{L}_1$ there is for every statement a an intuitionistic as well as a dual-intuitionistic negation of a. For $\neg_i a$ we have $\neg_i a // \mathbf{I}a$, and for $\neg_m a$ we have $\neg_m a // \mathbf{U}a$.*

Proof We have $a > \mathbf{I}a // \mathbf{I}a$ and $\mathbf{U}a \not> a // \mathbf{U}a$ in $\mathcal{L}_1$. This can be checked by constructing the respective truth-tables. Using Lemma 6.11, we obtain $\mathbf{I}\,a /\!/ \neg_i\, a$ and $\mathbf{U}\,a /\!/ \neg_m\, a$. The modal statements $\mathbf{I}\,a$ and $\mathbf{U}\,a$ exist for any statement in $\mathcal{L}_1$. Therefore intuitionistic and dual-intuitionistic negations exist for any statement in $\mathcal{L}_1$. □

Theorem 6.13 : *If a logic contains for any statement a also its intuitionistic negation $\neg_i\, a$ and its dual-intuitionistic negation $\neg_m\, a$, then these two negations do not (necessarily) collapse, i.e. we do not have $\neg_i\, a /\!/ \neg_m\, a$.*

Proof The logic $\mathcal{L}_1$ is such a logic. □

Popper (1948b) thus showed that there exists a bi-intuitionistic logic. The logic $\mathcal{L}_1$ might not be very interesting in itself. Nevertheless, it is at least interesting from a historical point of view, since it is the first example of a bi-intuitionistic logic to be found in the literature. Moreover, it shows that already Popper had the idea of combining different logics.[68]

6.5. *Six further kinds of negation*

Popper (1948b, §VI) considers three further kinds of negation explicitly, namely $\neg_j$, $\neg_l$ and $\neg_n$. He mentions their duals in a footnote[69] but does not study them. The negation $\neg_j$ coincides with Johansson's minimal negation, as already remarked by Popper.[70] The negation $\neg_n$ coincides with what is now called *subminimal negation*[71].

Definition 6.14: The six negations $\neg_j$, $\neg_{dj}$, $\neg_l$, $\neg_{dl}$, $\neg_n$ and $\neg_{dn}$ are given by the following characterizing rules. We also indicate the respective duals. (The characterizing rules for $\neg_k$, $\neg_i$ and $\neg_m$ are repeated below for comparison.)

Negation	*Characterizing rule*	*Rule name*	*Dual negation*	*Rule in Popper 1948b*
$\neg_j$	$a,b \vdash \neg_j\, c \to a,c \vdash \neg_j\, b$	$(\mathrm{C}\neg_j)$	$\neg_{dj}$	(6.1)
$\neg_{dj}$	$\neg_{dj}\, c \vdash a,b \to \neg_{dj}\, b \vdash a,c$	$(\mathrm{C}\neg_{dj})$	$\neg_j$	–
$\neg_l$	$a,\neg_l\, b \vdash c \to a,\neg_l\, c \vdash b$	$(\mathrm{C}\neg_l)$	$\neg_{dl}$	(6.2)
$\neg_{dl}$	$c \vdash a,\neg_{dl}\, b \to b \vdash a,\neg_{dl}\, c$	$(\mathrm{C}\neg_{dl})$	$\neg_l$	–
$\neg_n$	$a,b \vdash c \to a,\neg_n\, c \vdash \neg_n\, b$	$(\mathrm{C}\neg_n)$	$\neg_{dn}$	(6.3)
$\neg_{dn}$	$c \vdash a,b \to \neg_{dn}\, b \vdash a,\neg_{dn}\, c$	$(\mathrm{C}\neg_{dn})$	$\neg_n$	–
$\neg_k$	$a,\neg_k\, b \vdash c \leftrightarrow a \vdash b,c$	$(\mathrm{C}\neg_k 2)$	$\neg_k$	(4.32)
$\neg_i$	$a \vdash \neg_i\, b \leftrightarrow a,b \vdash$	$(\mathrm{C}\neg_i)$	$\neg_m$	(4.1)
$\neg_m$	$\neg_m\, a \vdash b \leftrightarrow \;\vdash a,b$	$(\mathrm{C}\neg_m)$	$\neg_i$	(4.2)

The two rules $(\mathrm{C}\neg_i)$ and $(\mathrm{C}\neg_m)$ differ slightly from the other rules in that they have only two instead of three statements occurring in each relation of relative demonstrability. However, they can also be given the same form with three such statements, as the following lemma shows.

[68] Cf. Schroeder-Heister 2006 (p. 31, fn 16).

[69] Popper (1948b, p. 328, fn 16) writes: 'There are, of course, dual rules of 6.1 $[\neg_j]$, 6.2 $[\neg_l]$ and 6.3 $[\neg_n]$, two of which are satisfied by $[\neg_m]$, just as 6.1 $[\neg_j]$ and 6.3 $[\neg_n]$ are satisfied by $[\neg_i]$.'
In Curry 1948d, $\neg_l$ is mistakenly considered to be the dual of $\neg_j$, which it is not in the sense used by Popper.

[70] Popper (1948b, p. 328) explains his names for these negations as follows: 'In view of 6.2 [see Definition 6.14], we may call $[\neg_l\, a]$ the "left-hand side negation of a" (in contradistinction to JOHANSSON's $[\neg_j\, a]$ which, in view of 6.1 [see Definition 6.14], is a "right hand side negation"). $[\neg_n\, a]$ may be called the "neutral negation"; it is neutral with respect to right-sidedness and left-sidedness [...].'

[71] See e.g. Dunn 1999 and the references therein.

Lemma 6.15: *The rule* (C$\neg_i$) *is equivalent to*

$$a, c \vdash \neg_i b \leftrightarrow a, b, c \vdash \qquad (\mathrm{C}\neg_i{}')$$

and the rule (C$\neg_m$) *is equivalent to*

$$\neg_m a \vdash b, c \leftrightarrow \ \vdash a, b, c. \qquad (\mathrm{C}\neg_m{}')$$

Proof Rule (C$\neg_i$) follows from (C$\neg_i{}'$) by substituting a for c and applications of (LC). Rule (C$\neg_m$) follows from (C$\neg_m{}'$) by substituting b for c and applications of (RC).

Next we show that (C$\neg_m{}'$) can be obtained from (C$\neg_m$). For the direction from right to left, assume $\neg_m a \vdash b, c$ and use (Cut) with $\vdash a, \neg_m a$, which holds by Lemma 6.3.

For the direction from left to right, we first show that $\neg_m a \vdash a, b \rightarrow \neg_m a \vdash b$ holds:

(1) Assume $\neg_m a \vdash a, b$.
(2) Use (Cut) with $\vdash a, \neg_m a$ (Lemma 6.3) to obtain $\vdash a, a, b$, from which $\vdash a, b$ follows.
(3) Applying (C$\neg_m$), $\neg_m a \vdash b$ follows.

We now show that $\vdash a, b, c$ implies $\neg_m a \vdash b, c$.

(1) Assume $\vdash a, b, c$.
(2) By using definition (D$\vdash$3) and by applying universal instantiation twice, we obtain $(a/d \,\&\, b/d \,\&\, c/d) \rightarrow \neg_m a/d$.
(3) Assume c/d to obtain $(a/d \,\&\, b/d) \rightarrow \neg_m a/d$.
(4) Using an instance of $\neg_m a \vdash a, b \rightarrow \neg_m a \vdash b$, we obtain $b/d \rightarrow \neg_m a/d$.
(5) Reintroduce c/d to obtain $(c/d \,\&\, b/d) \rightarrow \neg_m a/d$.
(6) By universal quantification and application of (D$\vdash$3) we get $\neg_m a \vdash b, c$.

To show that (C$\neg_i{}'$) can be obtained from (C$\neg_i$), we first apply the duality function δ to (C$\neg_i$), which yields (C$\neg_m$). An application of δ to the equivalent (C$\neg_m{}'$) yields (C$\neg_i{}'$). □

In the following, we show how all these negations relate to each other.

Theorem 6.16: *The following statements hold for the characterizing rules given in Definition 6.14:*

(1) $\neg_k$ satisfies the rule for $\neg_i$.
(2) $\neg_i$ satisfies the rule for $\neg_j$.
(3) $\neg_j$ satisfies the rule for $\neg_n$.
(4) $\neg_k$ satisfies the rule for $\neg_l$.
(5) $\neg_l$ satisfies the rule for $\neg_n$.
(6) $\neg_k$ satisfies the rule for $\neg_m$.
(7) $\neg_m$ satisfies the rule for $\neg_{dj}$.
(8) $\neg_{dj}$ satisfies the rule for $\neg_{dn}$.
(9) $\neg_k$ satisfies the rule for $\neg_{dl}$.
(10) $\neg_{dl}$ satisfies the rule for $\neg_{dn}$.

Proof We show (7), (8), (9) and (10). The structural rules (LE) and (RE) will be used tacitly.

(7) We have to show $\neg_m c \vdash a, b \rightarrow \neg_m b \vdash a, c$, presupposing (C$\neg_m$). By Lemma 6.15 we can use (C$\neg_m{}'$) equivalently.
   a) Assume $\neg_m c \vdash a, b$.
   b) Therefore $\vdash c, a, b$, by (C$\neg_m{}'$).
   c) Therefore $\neg_m b \vdash a, c$, again by (C$\neg_m{}'$).

(8) We have to show $c \vdash a, b \rightarrow \neg_{dj} b \vdash a, \neg_{dj} c$, presupposing (C$\neg_{dj}$).
   a) Assume $c \vdash a, b$.
   b) We have $\neg_{dj} c \vdash a, \neg_{dj} c \rightarrow \neg_{dj} \neg_{dj} c \vdash a, c$, as an instance of (C$\neg_{dj}$).
   c) We have $\neg_{dj} c \vdash a, \neg_{dj} c$, by (Rg) and (RW).
   d) Therefore $\neg_{dj} \neg_{dj} c \vdash a, c$, from b) and c) by modus ponens.
   e) Therefore $\neg_{dj} \neg_{dj} c \vdash a, b$, from a) and d) by (Cut) and (RC).

f) We have $\neg_{dj}\neg_{dj}c \vdash a, b \to \neg_{dj}b \vdash a, \neg_{dj}c$, as an instance of (C$\neg_{dj}$).
g) Therefore $\neg_{dj}b \vdash a, \neg_{dj}c$, from e) and f) by modus ponens.

(9) We have to show $c \vdash a, \neg_k b \to b \vdash a \neg_k c$, presupposing (C$\neg_k$).

a) Assume $c \vdash a, \neg_k b$.
b) We have $\neg_k b, b \vdash$ and $\vdash c, \neg_k c$, by (C$\neg_k$).
c) Therefore $b, c \vdash a$, from a) and b) by (Cut).
d) Therefore $\vdash c, \neg_k c$, from b) and c) by (Cut).

(10) We have to show $c \vdash a, b \to \neg_{dl}b \vdash a, \neg_{dl}c$, presupposing (C$\neg_{dl}$).

a) Assume $c \vdash a, b$.
b) We have $\neg_{dl}b \vdash a, \neg_{dl}b \to b \vdash a, \neg_{dl}\neg_{dl}b$, as an instance of (C$\neg_{dl}$).
c) We have $\neg_{dl}b \vdash a, \neg_{dl}b$, by (Rg) and (RW).
d) Therefore $b \vdash a, \neg_{dl}\neg_{dl}b$, from b) and c) by modus ponens.
e) Therefore $c \vdash a, \neg_{dl}\neg_{dl}b$, from a) and d) by (Cut) and (RC).
f) We have $c \vdash a, \neg_{dl}\neg_{dl}b \to \neg_{dl}b \vdash a, \neg_{dl}c$, as an instance of (C$\neg_{dl}$).
g) Therefore $\neg_{dl}b \vdash a, \neg_{dl}c$, from e) and f) by modus ponens.

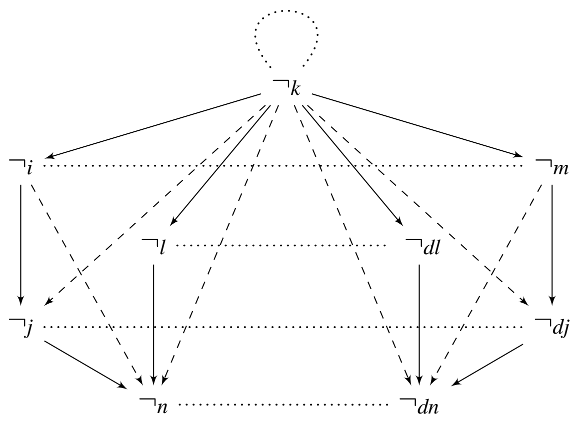
Statements (2)–(5) can be shown analogously, and were already presented by *Popper* (*1948b*, p. 328) without proof. For statements (1) and (6) see Theorem 6.5. □

Corollary 6.17: *By transitivity we have in addition that $\neg_k$ satisfies the rules for $\neg_j$, $\neg_n$, $\neg_{dn}$ and $\neg_{dj}$; $\neg_i$ satisfies the rule for $\neg_n$, and $\neg_m$ satisfies the rule for $\neg_{dn}$.*

Theorem 6.18: *We have the following dualities: $(\neg_k, \neg_k)$, $(\neg_i, \neg_m)$, $(\neg_j, \neg_{dj})$, $(\neg_l, \neg_{dl})$ and $(\neg_n, \neg_{dn})$.*

Proof The self-duality of $\neg_k$ is given by Lemma 6.2. For the duality between $\neg_i$ and $\neg_m$ see Section 6.2. The dualities $(\neg_j, \neg_{dj})$, $(\neg_l, \neg_{dl})$ and $(\neg_n, \neg_{dn})$ can be shown by applications of the duality function δ. □

The following diagram summarizes our results. The dotted lines connect those negations that are dual in the sense of Popper, and the arrows show which negations satisfy the rules of which other negations; the solid arrows are due to Theorem 6.16, and the dashed arrows are due to Corollary 6.17. The diagram is complete; no further (non-trivial) relations hold.[72]



Which negations are logical constants? In distinction to classical, intuitionistic and dual-intuitionistic negation, the six negations $\neg_j$, $\neg_{dj}$, $\neg_l$, $\neg_{dl}$, $\neg_n$ and $\neg_{dn}$ are not given by fully

[72] That $\neg_m$ does not satisfy $\neg_n$ was already observed by *Popper* (*1948b*, p. 328).

characterizing rules. Hence, these negations cannot be considered as logical constants.[73] This is especially interesting in the case of minimal negation $\neg_j$. In order to show this, Popper first introduces the two logical connectives t and f:

$$a//t(b) \leftrightarrow (c)(b/a \leftrightarrow c/a) \tag{Dt}$$

$$(c)(b/t(b) \leftrightarrow c/t(b)) \tag{Ct}$$

$$a//f(b) \leftrightarrow (c)(a/b \leftrightarrow a/c) \tag{Df}$$

$$(c)(f(b)/b \leftrightarrow f(b)/c) \tag{Cf}$$

For t and f, the following lemma holds.

Lemma 6.19: *For all statements b:* $\vdash t(b)$ *and* $f(b) \vdash$.

Proof In (Ct) let c be $t(b)$ in order to obtain $(b/t(b) \leftrightarrow t(b)/t(b))$, from which one can obtain $b \vdash t(b)$. From $b/t(b)$ and (Ct) one immediately obtains $(c)(c/t(b))$, and thus $\vdash t(b)$. Similarly for $f(b) \vdash$. □

This shows that t is a unary verum and that f is a unary falsum.

Theorem 6.20: *None of the rules for* $\neg_j$, $\neg_{dj}$, $\neg_l$, $\neg_{dl}$, $\neg_n$ *and* $\neg_{dn}$ *are fully characterizing.*

Proof The verum t satisfies the rules for $\neg_j$, $\neg_n$, $\neg_{dl}$ and $\neg_{dn}$. The falsum f satisfies the rules for $\neg_l$, $\neg_n$, $\neg_{dj}$ and $\neg_{dn}$. Classical negation $\neg_k$ satisfies the rules for $\neg_j$, $\neg_{dj}$, $\neg_l$, $\neg_{dl}$, $\neg_n$ and $\neg_{dn}$. If the rules for the six latter negations were each fully characterizing, then we would have for all a that $\neg_k a//t(a)$ or $\neg_k a//f(a)$, depending on the negation considered. But both $\neg_k a//t(a)$ and $\neg_k a//f(a)$ can only hold for contradictory object languages. □

7. Conclusion

Popper already developed ideas and had results in areas of philosophical logic that were in some cases only rediscovered later. Examples are his thoughts on a dual-intuitionistic logic, which were then further developed by Cohen (1953) into a sequent calculus for dual-intuitionistic logic, and his proof that there exists a bi-intuitionistic logic. Popper also saw certain logical and philosophical problems that are still discussed today. We mention the problem of trivializing (*tonk*-like) connectives in inferential settings, questions concerning non-conservative language extensions, and the problem of logicality. We saw that Popper developed a theory of modalities and a theory of negations, and we pointed out how his conception of duality plays an important part in both of them. Moreover, Popper also made several minor discoveries; he was, for example, one of the first to consider Peirce's rule. Finally, we would like to emphasize the importance of Popper's use of a purely structural basis for deducibility, which is central in his approach on logic. It allowed him to abstract from object languages that already presuppose the availability of specific logical constants, and enabled him to discuss logical constants in structural terms. His theory of negations is an interesting example of this approach, and our results underline its fruitfulness.

[73] The idea that certain negations could be too weak to be still considered as logical constants is already present in Popper 1943 (p. 50): 'Systems containing the operation of *negation* may be so much weakened that contradictoriness [i.e. the derivability of any two formulas such that one is the negation of the other] only implies n-embracingness [*i.e.* we can prove that any *negation of any formula whatsoever* can be derived (ibid., p. 49)]. It appears, however, that we cannot weaken them further without depriving *negation* of the character of a logical operation.'

Appendix A: The development of Popper's formulation of a basis

In the main part of this paper we selected one version of Popper's basis for the deducibility relation (see Section 2.4). This approach allowed us to quickly progress to the definitions of the logical constants, which were our main concern. It should be noted, however, that in the succession of Popper's articles he presents several formulations to capture what he understands by a basis. We trace this development in the following, taking the order of his publications as a lead.[74] We restrict ourselves again to propositional logic, and do not comment on additional constraints regarding substitution that have to be made for the treatment of quantification.

There are two decisive points in this development. The first is the impasse that Popper found himself in while trying to get rid of the generalized transitivity principle (Tg), which led him to develop Basis II. But *Curry* (*1948a*) showed that Basis II is not, contrary to what Popper claims, equivalent to Basis I. We will point out Popper's error and will show how it may be corrected. The second decisive point was when Popper rediscovered a solution of how to replace (Tg) by a simpler rule, something which had been done similarly before by Gentzen for a system of Hertz.

Popper 1947a, 'Logic without assumptions'

The notion of *absolute validity* is developed, which justifies the rules of a basis:

> There are inferences [...] which can be shown, on our definition of validity, to be valid *whatever the logical form of the statements involved*. [...] We shall say of these inferences that they are *absolutely valid*. (Ibid., p. 274)

The basis that he considers consists of the generalized reflexivity principle (Rg) and the rule (Tg),[75] which he claims, without giving a proof, to be complete with respect to his notion of absolute validity:

> It can be shown that all absolutely valid rules of inference [...] can be reduced to two [namely (Rg) and (Tg)]. By 'reduced', I mean here: every inference which is an observance of some of the rules in question can be shown to be an observance of these two rules [...]. (Ibid., p. 277)

This claim is plausible, especially in view of *Lejewski's* (*1974*) proof of the equivalence of Popper's Basis I and the systems developed in *Tarski 1930a,b, 1935, 1936b*.

Popper 1947b, 'New foundations for logic'

Two different approaches to axiomatizing the deducibility relation are developed, which are summed up on p. 211, ibid. Approach I (developed in §2) consists of one of several possible variants of rules equivalent to (Rg) and (Tg). Approach II (developed in §3) takes a simpler transitivity principle than (Tg) and postulates the availability of conjunction in the object language. These two approaches are exemplified by Basis I and Basis II, respectively.

Basis I. Popper considers each of the combinations (Tg) + (Rg), (Tg) + (2.1) + (2.2) + (2.3) and (Tg) + (2.41) + (2.7) as a possible basis. The components are:

$$a/a \tag{2.1}$$

$$a_1,\ldots,a_n/b \to a_1,\ldots,a_n,a_{n+1}/b \tag{2.2}$$

$$a_1,\ldots,a_n/b \to a_n,\ldots,a_1/b \tag{2.3}$$

[74] The exact order of *Popper 1947a* and *Popper 1947b* is not completely clear; both reference the other as forthcoming and were probably written at around the same time.

[75] Cf. Section 2.4. In *Popper 1947a* these rules are labeled (6.1g) and (6.2g), respectively.

$$a_1,\ldots,a_n/a_i \quad (\text{for } 1 \le i \le n) \tag{(Rg) = (2.4)}$$

$$a_1,\ldots,a_n/a_1 \tag{2.41}$$

$$a_1,\ldots,a_{n+m}/b \to a_n,\ldots,a_1,a_{n+1},\ldots,a_{n+m}/b \tag{2.7}$$

$$\left\{\begin{array}{l} \quad a_1,\ldots,a_n/b_1 \\ \&\ a_1,\ldots,a_n/b_2 \\ \vdots \qquad\quad \vdots \\ \&\ a_1,\ldots,a_n/b_m \end{array}\right\} \to (b_1,\ldots,b_m/c \to a_1,\ldots,a_n/c) \tag{(Tg) = (2.5g)}$$

The combination (Tg) + (Rg) is called Basis I by Popper.

All these combinations have the downside that they have to include the complicated transitivity principle (Tg) instead of a simpler one. Popper considers the rule (Tg) to be problematic and wants to get rid of it. He explains the exact problem later, in *Popper 1948a* (p. 177, fn 6):

> The objection against [(Tg)] is that it makes use of an unspecified number of conjunctive components in its antecedent; this may be considered as introducing a new metalinguistic concept — something like an infinite product.

This leads him to Approach II, which contains Basis II.

Basis II. In §3 the alternative Basis II is proposed, which is supposed to axiomatize the deducibility relation by using the simpler transitivity principle (Te)

$$a_1,\ldots,a_n/b \to (b/c \to a_1,\ldots,a_n/c) \tag{(Te) = (2.5e)}$$

and by additionally requiring that for any two statements a and b of the object language $\mathcal{L}$ their conjunction $a \wedge b$ is also contained in $\mathcal{L}$. *Popper* (*1947b*, p. 206) expresses the motivating idea behind Basis II as follows:

> But the main point of using conjunction is that it should permit us to *link up any number of statements into one*. If it does this, then we can always replace the m conclusions of m different inferences from the same premises by one inference.

One problem with this approach is that in the absence of (Tg), the rules for conjunction have to be sufficiently strengthened in order to work. Popper also noticed this and proposed the following slightly tweaked rules for conjunction:[76]

$$a_1,\ldots,a_n/b \wedge c \to (a_1,\ldots,a_n/b \;\&\; a_1,\ldots,a_n/c) \tag{Cg$_1$}$$

$$(a_1,\ldots,a_n/b \;\&\; a_1,\ldots,a_n/c) \to a_1,\ldots,a_n/b \wedge c \tag{Cg$_2$}$$

Basis II thus consists of (Rg), (Te), (Cg_1), (Cg_2) and the existence postulate for conjunctions.

Popper uses these additional rules for conjunction to prove that (Tg) can be obtained from Basis II. That his proof contains an error was pointed out by *Curry* (*1948a*), who provided the following arithmetical counterexample: Let the object language consist of the natural numbers. Let $a \wedge b$ be the minimum of a and b. Let $a_1,\ldots,a_n/b$ hold if, and only if, $\min(a_1,\ldots,a_n) \le b+n-1$. Under this interpretation (Te), (Rg) and (Cg) of Basis II are satisfied but the rule (Tg) of Basis I is not. To see this let, in (Tg), $n = 1$, $m = 2$, $a = b_1 = b_2 = 1$ and $c = 0$. Curry's counterexample shows that Popper's proof must contain an error. However, it does not tell us which step of Popper's proof actually fails.

[76] In Popper's formulation, (Cg_1) and (Cg_2) appear as one rule (Cg), which is here split up into two parts.

Popper's proof and its error. The crucial step in the proof that Basis I and Basis II are equivalent is to show that (Tg) follows from the rules of Basis II. Popper's attempted proof (ibid, p. 210f.) proceeds as follows. He assumes $a_1,\ldots,a_n/b_i$, for $1 \leq i \leq m$, as well as $b_1,\ldots,b_m/c$, and has to show that $a_1,\ldots,a_n/c$ follows. He attempts to do this in the following way:

(1) From $a_1,\ldots,a_n/b_i$ (for all $1 \leq i \leq m$) obtain $a_1,\ldots,a_n/b_1 \wedge \ldots \wedge b_m$.
(2) From $b_1,\ldots,b_m/c$ obtain $b_1 \wedge \ldots \wedge b_m/c$.
(3) From $a_1,\ldots,a_n/b_1 \wedge \ldots \wedge b_m$ and $b_1 \wedge \ldots \wedge b_m/c$ obtain $a_1,\ldots,a_n/c$.

Step (1) can be proved by induction on m and iterated applications of (Cg_2); Popper gives this part of the proof explicitly. Step (3) is just an instance of his transitivity principle (Te). It is therefore step (2) which must be responsible for the failure of Popper's attempted proof, and it is indeed this inference that fails to be satisfied by Curry's counterexample: If $m = 3$, $b_1 = b_2 = b_3 = 1$ and $c = 0$, then $b_1 \wedge \ldots \wedge b_m/c$ does not follow from $b_1,\ldots,b_m/c$. This derivation would be possible if Popper's rule (3.4g), i.e.

$$a_1,\ldots,a_n,b \wedge c/d \leftrightarrow a_1,\ldots,a_n,b,c/d \tag{3.4g}$$

were secondary to Basis II.[77] Popper (1947b, p. 210) thinks he has already proven this:

> But this situation changes completely if we drop the generalised transitivity principle (Tg) and replace it by the simpler form (Te). In this case, all the rules 3.1 to 3.5 and 3.1g to 3.4g still follow from 3.5g [= (Cg)]; but the opposite is not the case.

If rule (3.4g) were indeed secondary to Basis II, then step (2) could be proved by a simple induction on m. Basis II might therefore be salvageable if we added to the rules (Cg_1) and (Cg_2) the rule (3.4g).[78]

Popper 1947c, 'Functional logic without axioms or primitive rules of inference'

This article starts with a characterization of '[t]he customary systems of modern lower functional logic, such as *Principia Mathematica*, or the systems of HILBERT–ACKERMANN, HILBERT–BERNAYS, or HEYTING, etc.' (ibid., p. 1214). The fourth point of this characterization is:

> (d) Some further very general primitive rules of inference (such as some principles stating that the inference relation is transitive and reflexive) which do not refer to formative signs are assumed, either explicitly or, more often, tacitly.

This corresponds to what Popper uses his basis for. But in this article he proposes to also get rid of the basis:

> The inferential definitions of the conjunction [...] can be reformulated in such a way as to incorporate *all* the rules of inference mentioned. In this way, we can get rid of even the few trivial primitive rules (d) which were left in the previous approach; in other words, we obtain the whole formal structure of logic from metalinguistic inferential definitions alone. (Ibid., p. 1215)

Popper provides the following inferential definition (DB2) for conjunction, which contains (Cg), (Te), (2.1) and (2.2):

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[77] See footnote 53 for Popper's use of the term 'secondary'.

[78] What is actually only needed of (3.4g) is the implication from right to left.

$$a//b \wedge c \leftrightarrow (a_1)\dots(a_n)((a_1,\dots,a_n/a \leftrightarrow (a_1,\dots,a_n/b \,\&\, a_1,\dots,a_n/c)) \\ \&\ (b/c \to (a_n,\dots,a_1/b \to a_1,\dots,a_n/c)) \\ \&\ (a_1,\dots,a_n/c \to a_1,\dots,a_n,b/c) \,\&\, a_1/a_1)) \tag{DB2}$$

As this approach is based on the rules of Basis II, it suffers from the same problems. Hence (Tg) does in general not hold for the deducibility relation.

Popper 1948a, 'On the theory of deduction, part I'

A new notation for the deducibility relation is introduced: $D(a_1,a_2,\dots,a_n)$ stands for $a_2,\dots,a_n/a_1$, and Popper from now on explicitly allows the number of premises to be 0, i.e. he allows $D(a)$, which stands for $/a$. The new basis that Popper considers consists of the following two rules (translated from D-notation into $/$-notation):[79]

$$a_2/a_1 \leftrightarrow a_2,a_2/a_1 \tag{BI. 1}$$

$$a_2,\dots,a_n/a_1 \leftrightarrow (a_{n+1})\dots(a_{n+r})(a_1,\dots,a_n/a_{n+r} \to a_{n+r-1},\dots,a_{n+1},a_n,\dots,a_2/a_{n+r}) \tag{BI. 2}$$

From these two rules Popper obtains (Rg) and the rules

$$a_1,\dots,a_n/b \to a_n,\dots,a_1/b \tag{1.44$'$}$$

$$a_1,\dots,a_n/b \to a_1,\dots,a_{n+r}/b \tag{1.45$'$}$$

$$a_1,\dots,a_n/b \to (b,a_1,\dots,a_n/c \to a_1,\dots,a_n/c) \tag{1.46$'$}$$

Popper then shows how to derive (Tg) from $(1.44')$, $(1.45')$ and $(1.46')$ by an induction on the number m in (Tg). He thus achieves the replacement of (Tg) by a rule which can, in contrast to (Tg), be stated in the metalanguage. He comments:

> The problem of avoiding [(Tg)] was discussed, but not solved, in [*Popper 1947b*]. The lack of a solution led me there to construct Basis II, the need for which, as it were, has now disappeared. (Ibid., p. 177, fn 6)

This result also allows him to modify the implementation of the program of *Popper 1947c*. Instead of using his definition (DB2), he can use this new formulation of Basis I. He mentions this possibility in *Popper 1948a* (p. 177, fn 6).

Already *Curry* (*1948c*) pointed out that this part of the proof is similar to what *Gentzen* (*1933*, esp. §3) had shown, namely that the rule of 'Syllogismus' of the system of *Hertz* (*1929*) can be replaced by his cut rule, while Popper showed that his rule (Tg) can be replaced by the abovementioned rule (1.46′). The 'Syllogismus' rule is practically identical with Popper's rule (Tg), and Gentzen's cut rule is identical to (1.46′). Gentzen's and Popper's proofs proceed in a very similar fashion. The question is how much Popper knew of the systems of Hertz and Gentzen. *Popper* (*1947b*, p. 204) remarks in a footnote that Bernays pointed him to the articles of *Hertz* (*1923, 1931*), and he mentions *Gentzen 1935* in order to compare his concept of relative demonstrability with Gentzen's sequent arrow. There are no further references to either of these authors in Popper's articles.

Basis III. At the very end of *Popper 1948a*, a new possibility for a basis is introduced. Popper takes two-place deducibility (or 'two-termed derivability', as he says), i.e. a/b, characterized by

[79] For a discussion of these axioms cf. also *Schroeder-Heister 2006* (app. 1).

transitivity and reflexivity as a starting point, together with one of the two rules (the second is here translated into /-notation)

$$b/a \leftrightarrow (c)(c/b \rightarrow c/a) \tag{BIII}$$

$$b/a \leftrightarrow (c)(a/c \rightarrow b/c) \tag{2.1}$$

and the following definition of relative demonstrability:

$$a_1, \ldots, a_n \vdash b_1, \ldots, b_m \leftrightarrow (c)(d)((b_1/d \,\&\, \ldots \,\&\, b_m/d) \rightarrow ((c/a_1 \,\&\, \ldots \,\&\, c/a_n) \rightarrow c/d)) \tag{D3.3}$$

The resulting basis is equivalent to Basis I.

Popper 1949, 'The trivialization of mathematical logic'

This is the last of Popper's articles on logic. It does not contain a lot of technical development but gives an overview of his general intentions instead. Popper now seems to have adopted the approach with Basis III. He writes:

> As the sole definiens of our definitions, the idea of deducibility or derivability will be used. We can restrict ourselves to using deducibility from *one* premise. We write '$D(a,b)$' for 'a is deducible from b'. [...] We do not need, for the derivation of mathematical logic, to assume more about deducibility than that it is transitive and reflexive. [...] With the help of '$D(a,b)$' it is easy to define deducibility from n premises. (Ibid., p. 723)

He (ibid., p. 723) also considers to define a generalized deducibility relation $a_1, \ldots, a_n/b_1, \ldots, b_m$ in terms of $D(a,b)$ alone, with the intended meaning that if each of the premises $a_1, \ldots, a_n$ is true, then at least one of the statements $b_1, \ldots, b_m$ must be true. This corresponds to relative demonstrability.

Concluding remarks

While most of the technical development in Popper's articles is done using Basis I, a tendency can be observed towards his using two-place deducibility a/b as the underlying concept, together with derivative n-place notions like relative demonstrability that are defined in terms of two-place deducibility.

Approach II seems ill-chosen to us, even if the suggested corrections were to be carried out. It requires conjunction in the object language in its formulation of the basis, and it confounds structural properties of the deducibility relation with the definition of a logical constant.

Our choice of (a version of) Basis I, on the other hand, allows us to stay as close as possible to Popper's technical development, which proceeds mostly by using Basis I with $n+1$-place deducibility $a_1, \ldots, a_n/b$. This choice also makes it possible to avoid a certain awkwardness in the formulation of proofs, which is already present in several proofs involving relative demonstrability; in order to show that some statement involving relative demonstrability holds, one often has to unfold the definition of relative demonstrability first, then work with the definiens, and finally reintroduce relative demonstrability. Besides its conceptual superiority, Basis I has thus also practical advantages.

Appendix B: Relative demonstrability and cut

Popper's definition (D⊢2) of relative demonstrability

$$a_1, \ldots, a_n \vdash b_1, \ldots, b_m \leftrightarrow (c)((b_1/c \,\&\, \ldots \,\&\, b_m/c) \rightarrow a_1, \ldots, a_n/c) \tag{D⊢2}$$

is not wholly satisfactory from a technical point of view, since it does not allow to show that

$$a_1,\ldots,a_n \vdash b_1,\ldots,b_m,c \to (c,a_1,\ldots,a_n \vdash b_1,\ldots,b_m \to a_1,\ldots,a_n \vdash b_1,\ldots,b_m) \qquad \text{(Cut)}$$

holds for any object language.[80]

Theorem B.1 : *The rule* (Cut) *does not follow from* (D⊢2) *and Basis I.*

Proof We show this by means of a counterexample for the instance $(a \vdash b,c \,\&\, c,a \vdash b) \to a \vdash b$. The negation of this claim is metalinguistically equivalent to $(d)((b/d \,\&\, c/d) \to a/d) \,\&\, c,a/b \,\&\, \neg(a/b)$. Now consider an object language that contains only the three statements a, b and c. Then this is equivalent (also using the rules (Rg) and (Tg) of the Basis I) to the statement $(c/b \to a/b) \,\&\, (b/c \to a/c) \,\&\, c,a/b \,\&\, \neg(a/b)$.The model under which the deducibility relation is only true for $c,a/b$ (and the instances required by (Rg)) satisfies (Tg) and is the desired countermodel. □

However, if one presupposes that the object language contains conjunction, disjunction and implication, then (Cut) can be shown to hold.

Theorem B.2 : *In the presence of conjunction, disjunction and implication,* (Cut) *does follow from* (D⊢2) *and Basis I.*

Proof In the presence of conjunction and disjunction, and in view of Lemma 5.3, it is sufficient to show that $(a \vdash b,c \,\&\, c,a \vdash b) \to a \vdash b$ holds, which is done as follows:

(1) From the instance $b,a/b$ of (Rg) and (C>) we get $b \vdash a > b$.
(2) From the assumption $c,a \vdash b$ and (C>) we get $c \vdash a > b$.
(3) From the assumption $a \vdash b,c$ and (D⊢2) we then get $a \vdash a > b$.
(4) From (C>) and (LC) we get $a \vdash b$. □

Since we do not want to presuppose the existence of any specific logical constants in object languages, we use the following more general definition of relative demonstrability, which is formulated with additional context statements $d_1,\ldots,d_k$ (for $0 \leq l \leq k$) occurring as additional premises in each of the deducibility relations in the definiens (cf. Section 3.4):

$$a_1,\ldots,a_n \vdash b_1,\ldots,b_m \leftrightarrow$$
$$(c)(d_1)\ldots(d_k)((b_1,d_1,\ldots,d_k/c \,\&\, \ldots \,\&\, b_m,d_1,\ldots,d_k/c) \to a_1,\ldots,a_n,d_1,\ldots,d_k/c) \qquad \text{(D⊢3)}$$

Given this definition, (Cut) follows from the rules of Basis I alone, without presupposing the existence of any logical constants.

Theorem B.3 : *If relative demonstrability is defined by* (D⊢3) *instead of* (D⊢2)*, then* (Cut) *follows from the rules of Basis I.*

Proof We have to show

$$\underbrace{a_1,\ldots,a_n \vdash b_1,\ldots,b_m,e}_{A} \to (\underbrace{e,a_1,\ldots,a_n \vdash b_1,\ldots,b_m}_{B} \to \underbrace{a_1,\ldots,a_n \vdash b_1,\ldots,b_m}_{C})$$

[80] That relative demonstrability could be defined in a more general way than by (D⊢2) was already observed in *Schroeder-Heister 2006* (app. 2), without mentioning, however, that the lack of generality of (D⊢2) leads to the failure of (Cut).

(where e is the cut formula). We assume A and B, and have to show C. In order to show C, we further assume $b_1, d_1, \ldots, d_k/c$ & … & $b_m, d_1, \ldots, d_k/c$, and have to show $a_1, \ldots, a_n, d_1, \ldots, d_k/c$.

(1) From B we know that $a_1, \ldots, a_n, d_1, \ldots, d_k, e/c$ holds.
(2) For each i with $1 \leq i \leq m$ we get from $b_i, d_1, \ldots, d_k/c$ by weakening on the left the corresponding relation $a_1, \ldots, a_n, b_i, d_1, \ldots, d_k/c$.
(3) Using (D⊢3), the following is an instance of A:

$$(b_1, a_1, \ldots, a_n, d_1, \ldots, d_k/c \;\&\; \ldots \;\&\; b_m, a_1, \ldots, a_n, d_1, \ldots, d_k/c \;\&\; e, a_1, \ldots, a_n, d_1, \ldots, d_k/c)$$
$$\to a_1, \ldots, a_n, a_1, \ldots, a_n, d_1, \ldots, d_k/c$$

(1), (2) and (3) together imply $a_1, \ldots, a_n, a_1, \ldots, a_n, d_1, \ldots, d_k/c$, and by contracting the premises $a_1, \ldots, a_n, a_1, \ldots, a_n$ we get $a_1, \ldots, a_n, d_1, \ldots, d_k/c$. □

A disadvantage of definition (D⊢3) is that it is not an explicit definition, since $a_1, \ldots, a_n \vdash b_1, \ldots, b_m$ is defined through an unspecified number k of context statements $d_1, \ldots, d_k$. The relation $\vdash$ is thus not eliminable in general. Nonetheless, it is always eliminable in a given logical argument, because the number k can then be specified.

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